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THEORY OF STRUCTURES

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THEORY OF STRUCTURES

BY

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FOURTH EDITION

LONDON
SIR ISAAC PITMAN & SONS, LTD.

First published 1925 *Reprinted* 1944
Second edition 1938 *Reprinted* 1946
Reprinted 1940 *Third edition* 1950
Reprinted 1942 *Fourth edition* 1954
Reprinted 1943 *Reprinted* 1957

SIR ISAAC PITMAN & SONS, LTD.
PITMAN HOUSE, PARKER STREET, KINGSWAY, LONDON,
THE PITMAN PRESS, BATH
PITMAN HOUSE, BOUVERIE STREET, CARLTON, MELBOURNE
27 BECKETTS BUILDINGS, PRESIDENT STREET, JOHANNESBURG
ASSOCIATED COMPANIES
PITMAN MEDICAL PUBLISHING COMPANY, LTD.
45 NEW OXFORD STREET, LONDON, W.C.1
PITMAN PUBLISHING CORPORATION
2 WEST 45TH STREET, NEW YORK
SIR ISAAC PITMAN & SONS (CANADA), LTD.
(INCORPORATING THE COMMERCIAL TEXT BOOK COMPANY)
PITMAN HOUSE, 381-383 CHURCH STREET, TORONTO

MADE IN GREAT BRITAIN AT THE PITMAN PRESS, BATH
E7—(T.5430)

PREFACE

TO THE FOURTH EDITION

During the last few years, progress and development have been made in the theory, design, and fabrication of constructions which have been designated as Pre-stressed Concrete Structures.

At the same time a great deal of research and experimental work has been undertaken in order to determine the physical characteristics and mechanical properties of high-quality concrete, and of the manufacture of this concrete for use in pre-stressed concrete constructions.

This book includes the various structural theories in steel and concrete necessary for the student of Civil and Structural Engineering; in this edition a new chapter has been added dealing with the fundamental principles of pre-stressed concrete design (Chapter XVII).

H. W. COULTAS

PREFACE

TO THE THIRD EDITION

During the last few years great progress has been made in the field of continuous structures, both in construction and analysis. Advance in constructional methods has brought out the importance of design. For a satisfactory design there must be a clear idea in the mind of the designer of the elastic behaviour of a rigidly connected frame. The analysis of built-in and continuous beams is given in Chapter IV, and the principle of east work applied to beams and frames is stated in Chapter IX.

Further methods of analysis have been developed and two of these, the *slope-deflection* and the *method of successive approximations*, are discussed in an additional Chapter XV. The uses of these methods have been recognized by structural engineers and designers: they have been introduced into courses at universities and technical colleges: problems depending f their solution on these new methods are set in many examin tion papers. They are fully stated and discussed in the n chapter, which begins with a development of the slope-deflecti equations for prismatic beams and a discussion of their application to the analysis of statically indeterminate beams, continuous beams, and simple frames. Frames both with an without lateral restraint are considered, and it is shown ho the equations for calculating redundant end moments can b written. Further discussion considers the solution of these equations by the method of successive approximations and illustrates its application to continuous beams and frames a before. There are a large number of examples to illustrate th solution of various problems by these methods.

A further Chapter XVI has been written in which the laws o Maxwell and Betti have been developed, and the application o these laws to the mechanical solution of statically indeterminat structures has been stated.

The author is indebted to friends for reading the text an specially to Mr. J. Heaton, A.M.I.C.E., A.M.I.Struct.E., f his valuable assistance.

H. W. COULTAS

BIRMINGHAM

PREFACE

TO THE FIRST EDITION

IN this book the fundamental principles in the Theory Structures have been presented in as simple and as logical form as possible. Intended primarily for students taking degree course, or the courses for the National Certificates the Ministry of Education and of the Engineering Institutions, is hoped it will be found useful by others engaged in engineeri practice. It has only been possible to present the groundwor of the subject, but references have been put at the end of eac chapter to guide the student to other theoretical methods an to a more complete and advanced study of definite branch of structural engineering. As theory is being constant vised, and as new theories and methods are being brought rward, it is therefore necessary that the student, to keep up to te, must consult the engineering journals and the transactions the Engineering Institutions of this and other countries. he necessary specifications for structural work should also e consulted, such as Specification 153, Parts 1 and 2, 1922, arts 3, 4 and 5, 1923, for Girder Bridges, issued by the British Engineering Standards Association. This specification deals with Materials, Workmanship, Loads and Stresses, Details of Construction and Erection.

The mathematics in this book are fairly simple: partial differentiation has been introduced in Chapter IX, but reference to it can be made in any good book on practical mathematics such as Usherwood and Trimble's, Part II. It is, however, essential that all students of engineering should have a good mathematical foundation.

The examples at the end of the chapters are all taken from recent examination papers, and I wish to thank the Councils of the University of Birmingham, of the Institutions of Civil, Mechanical and Structural Engineers, and also the University London Press for permission to reprint them.

No attempt has been made to deal with design.

In preparing this work, existing textbooks on the subject have been consulted, including those of Morley, Andrews, Husband and Harby, Hunter, Hool and Kinne, and others. These works also contain much useful information on design. I wish to make acknowledgment of my indebtedness to these books. My thanks are also due to Messrs. Longmans, Green & Co. for permission to use examples on inertia (pp. 28-30) rom Mann's *Practical Mathematics*; to Professor F. C. Lea, D.Sc., for the diagram (Fig. 129) of typical moving loads; to Mr. T. H. P. Veal, B.Sc., for the use of notes on reinforced concrete; to Dr. H. P. Budgen for the drawings from which lates I and II were made, and to a few friends for working ut solutions to the examples.

H. W. COULTAS

BIRMINGHAM
Sept., 1925

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CHAPTER I

BEAMS WITH DEAD LOADS

1. **Beams and Bending.** A bar of material acted upon by external forces (including loads and reactions) oblique to its longitudinal axis, is called a beam: the components of the forces perpendicular to the axis cause the straining, called flexure or bending. A beam will always be looked upon as being in a horizontal position, and the external forces as vertical.*

2. **Definitions.** Cantilever, a beam having one end fixed and the other free.

A beam freely supported at both ends is a simply-supported beam, or a simple beam.

An encastré beam is one built in or fixed at both ends.

A beam supported at a number of points is a continuous beam: this type may, of course, have one or both ends fixed or built in. (See para. 40, Chap. IV, for notes on supports.)

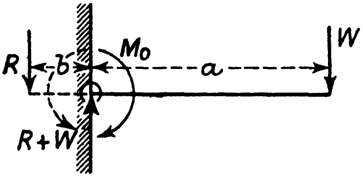


FIG. 1

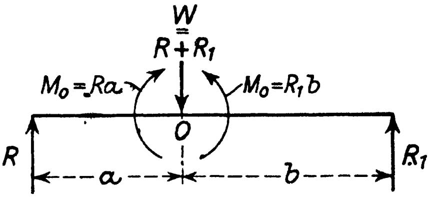


FIG. 2

3. **A Moment** (M) is defined as the product of a force (F) multiplied by its perpendicular distance (x) from a point considered; or

$M = Fx =$ the magnitude of a couple.

Cantilever (Fig. 1).

Moment at $O =$ Magnitude of the couple Wa

$= $ A fixed couple Rb

---

* This will cause simple bending, as it occurs when there is no resultant pull or push along the beam due to external forces; also the external forces must be all applied in the plane in which the beam bends: and the beam section must be symmetrical about a vertical axis through its centroid.

Simply-supported Beam (Fig. 2).

Moment at O = Magnitude of the couple R_1b

= ,, ,, Ra

From ordinary considerations of statics, $W = R + R_1$

4. When a beam is subjected to some system of loading, it is slightly bent out of its horizontal position.

As the beam is in equilibrium under the system of loading, the perpendicular reactions must be equal to the sum of the perpendicular components of the loads

If A is the section of a beam (Fig. 3) situated at a distance

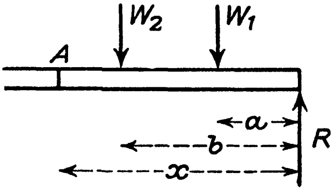


FIG. 3

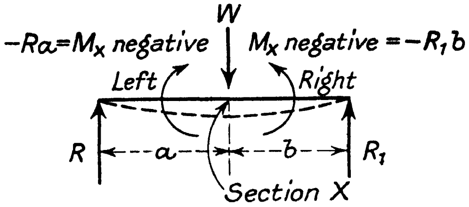


FIG. 4

x from the support point, and R is the reaction of the support, then all the forces to the right of A help to produce the bending action at A; or the beam is subjected to a series of couples. Now, R would tend to bend the beam in an anti-clockwise direction about A: W_2 and W_1 in a clockwise direction about A. Therefore, the resultant effective bending will be the difference of the two effects. Calling the anti-clockwise direction negative and the clockwise direction positive* to the right of a section, the effective bending action about A will be

$$-Rx + W_1(x-a) + W_2(x-b)$$

which is the algebraic sum of the moments, and is defined as the bending moment at A. In general, the bending moment at any section of a beam may be defined as the algebraic sum of the moments of all the external forces acting on that part of the beam to the right or to the left of the section.

5. **Signs for Bending Moments.** Clockwise and anti-clockwise moments to the *right* of a section will be called positive and negative moments respectively. Clockwise and anti-clockwise moments to the *left* of a section will be called negative and positive moments respectively.

---

* *See* Chapter III.

The two equal couples acting in opposite directions to the right and left of the section (Fig. 4) tend to bend the beam concave upwards; or as it will be seen later, the top side of the beam comes into compression.

In the case of Fig. 5, the two equal and opposite couples acting to the right and left of the section bend the beam convex upwards, or a positive moment tends to induce tensile stresses in the top side of the beam at the section considered.

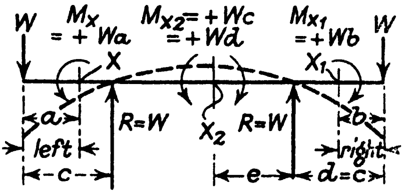


FIG. 5

6. Shearing Forces. In Fig. 6, let X be any section of a beam carrying some system of loads. At this section there is acting a vertical external force which is the resultant of all the forces acting on that part of the beam to the right, or to the left of the section. This force is spoken of as the "shearing force" (S) at the section, and it is balanced by the internal force in the particles of the material.

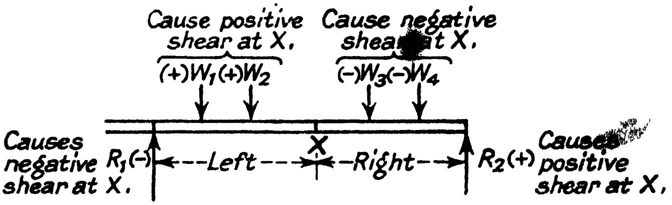

Cause positive shear at X.
Cause negative shear at X.
(+)W_1(+)W_2
(-)W_3(-)W_4
Causes negative shear at X.
R_1(-)
Left
X
Right
R_2(+)
Causes positive shear at X.

FIG. 6

By ordinary statics,

$$R_1 + R_2 = W_1 + W_2 + W_3 + W_4$$

Shearing force at X,

$$S_X = -R_1 + W_1 + W_2$$

(considering the portion of the beam to the left of the section),

or $$S_X = R_2 - W_3 - W_4$$

(considering the portion of the beam to the right of the section).

7. **Signs for Shearing Forces.** External forces acting upwards to the left or right of a section cause negative and positive shearing force respectively. External forces acting downwards to the left and right of a section cause positive and negative shearing force respectively. (*See* Fig. 6.)

8. **Diagrams of Shearing Force and Bending Moment.** Both shearing force and bending moment will generally vary in magnitude from point to point along the length of a loaded beam : their values at any cross-section can often be calculated arithmetically, or general algebraic expressions may give the bending moment and shearing force for any section along the beam. The variation may also be shown graphically by plotting curves, the bases of which represent to scale the length of the beam ; and the vertical ordinates, the bending moments or shearing forces, as the case may be, due regard being paid to the " sign " of the effect.

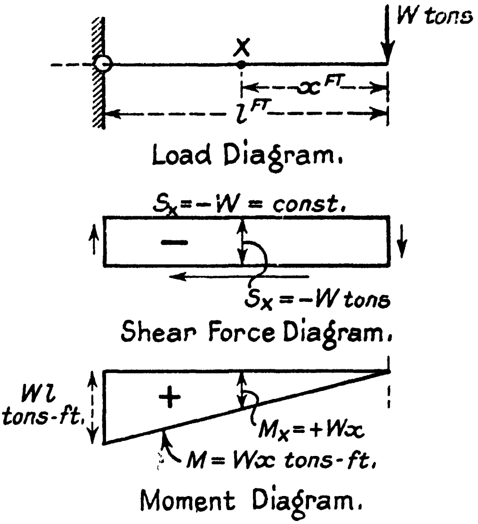


FIG. 7

Some typical examples of bending moment and shearing force diagrams are now given.

9. **Cantilever.** (*a*) Concentrated load at the free end. (Fig. 7.)

$$M_0 = + Wl$$
$$M_X = + Wx$$

M = moment ; S = shear force

NOTE —The moment M at a section is equal to the area of the S.F. diagram to the right of the section.

(*b*) With a uniformly distributed load along the whole beam. (Fig. 8.)

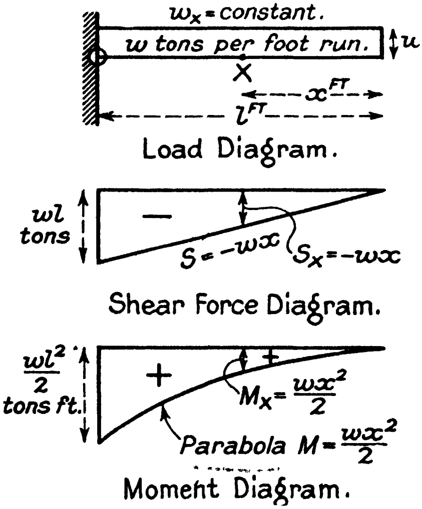


FIG. 8

Consider any section X.

Shear force to the right of $X = wx$

= area of load diagram to the right of X

i.e. $S_X = -wx$ and $S_0 = -wl$

For bending moments to the right of the section, the resultant moment is given by the area of the S.F. diagram to the right of the section.

With several loads on a cantilever, the shearing force and moment diagrams can be drawn by considering diagrams for each load separately and adding the ordinates to make one complete diagram.

Problem 1. A cantilever of length 20 ft. carries a load which decreases uniformly from 2 tons at the fixed end to 0 tons at the free end. Draw the bending moment and shear force diagrams.

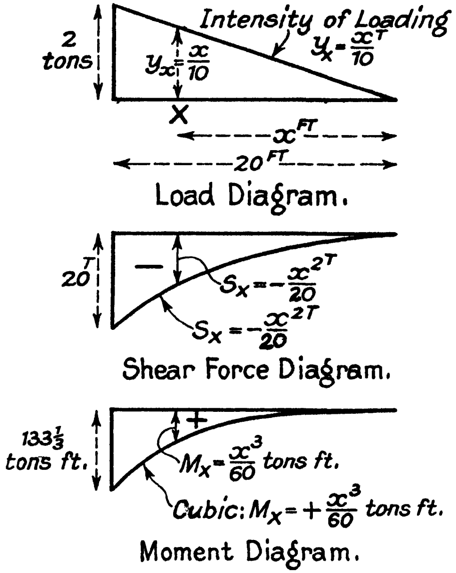
Load rate at $X = y = \frac{2}{20} \times x = \frac{x}{10}$ tons per foot run (Fig. 9.

$S_x =$ Area of the load diagram from the free end to section X

$S_x = \frac{x}{10} \times \frac{x}{2} = \frac{x^2}{20}$ tons (equation of a parabola)

$= \text{S.F. at } X,$

and it acts at the centroid of the load area considered.


Intensity of Loading
$y = \frac{x}{10}^{T}$
2 tons
$y_x = \frac{x}{10}$
X
x^{FT}
20^{FT}
20^{T}
—
$S_x = -\frac{x^2}{20}^{T}$
$S_x = -\frac{x^2}{20}^{T}$
$133\frac{1}{3}$ tons ft.
+
$M_x = \frac{x^3}{60}$ tons ft.
Cubic: $M_x = +\frac{x^3}{60}$ tons ft.

Load Diagram.

Shear Force Diagram.

Moment Diagram.

FIG. 9

Sign of S.F. is negative, as the forces act downwards to the right of the section.

$M_x =$ area of the S.F. diagram (which is a parabola) to the right of $X =$ moment at X

$$= \frac{x^2}{20} \times \frac{x}{3} = \frac{x^3}{60} \text{ tons-ft.}$$

when $x = 20$, $S_x = S_o$, and $M_x = M_o$.

$$\therefore\ S_o = -\frac{20^2}{20} = -20 \text{ tons}$$

$$M_o = \frac{20^3}{60} = 133\tfrac{1}{3} \text{ tons-ft.}$$

10. Simply-supported Beams. (*a*) Concentrated load at the centre of the beam (Fig. 10).

$$R_A = R_B = \frac{W}{2}$$

For all sections between A and C, the shearing force $S_x = R_A$ and is of the negative sense.

For all sections between C and B, the S.F. $S_x' = R_B$ and is of the positive sense.

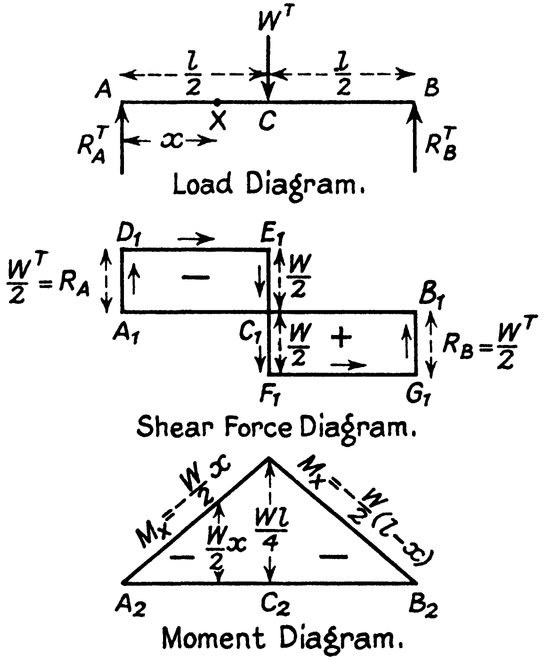


FIG. 10

NOTE. To construct the shear diagram. Take the base $A_1B_1 = AB$ to scale. Start always from the left-hand side of the beam. At A_1, erect a vertical upwards equal to R_A to scale $= A_1D_1$. From D_1, draw D_1E_1 parallel to A_1B_1, as far as the next load, which in this case is W. The direction of W is vertical downwards. From E_1, draw $E_1C_1F_1 = W$ to scale downwards.

$C_1E_1 = -\frac{W}{2}$. $C_1F_1 = +\frac{W}{2}$;

i.e. C_1F_1 is below the axis A_1B_1.

From F_1, draw F_1G_1 to meet the line of action of R_B: from G_1, draw G_1B_1 upwards to meet B_1. This construction can be used for all beams; always start from the left hand of all beams, and the shear diagram is easily drawn, as R_A acts upwards, W downwards, and R_B upwards. The only thing is to remember the sense of the resulting shear force.

Considering moments, take the origin at the left-hand support of the beam. For sections between A and C, $M_x = \frac{W}{2}x$ and is of the negative sense: i.e. a couple of magnitude $\frac{W}{2}x$ acts in a clockwise direction to the left of the section. For sections between C and B, take forces to the right of the sections: and the only force is R_B, causing an anti-clockwise couple to the right of a section $= R_B(l - x)$ $= \frac{W}{2}(l - x)$, and is, therefore, of the negative sense.

The moment is a maximum at C, and

$$= \frac{W}{2}\frac{l}{2} = \frac{Wl}{4} \text{ and of negative sense.}$$

NOTE.—The moment at a section is equal to the area of the S.F. diagram to the right or left of the section. At the section where the shear changes sign, the moment is a maximum.

Problem 2. A simply-supported beam is loaded, as in Fig. 11. Construct the shear force and bending moment diagrams.

$$R_A + R_B = (8 + 12) \text{ tons}$$

As the beam is in equilibrium, the sum of the moments about A = zero,

$$\text{i.e. } R_B \times 20 = 12 \times 12 + 8 \times 5$$

$$R_B = 9\tfrac{1}{5} \text{ tons}$$

$$R_A = 10\tfrac{4}{5} \text{ tons}$$

The S.F. diagram is therefore as shown.

$$M_C = -R_A \times 5 = -54 \text{ tons-ft.}$$

$$M_D = -R_B \times 8 = -73\tfrac{3}{5} \text{ tons-ft.}$$

and both of the negative sense, where M_C and M_D are the moments at C and D and the moment diagram is as constructed.

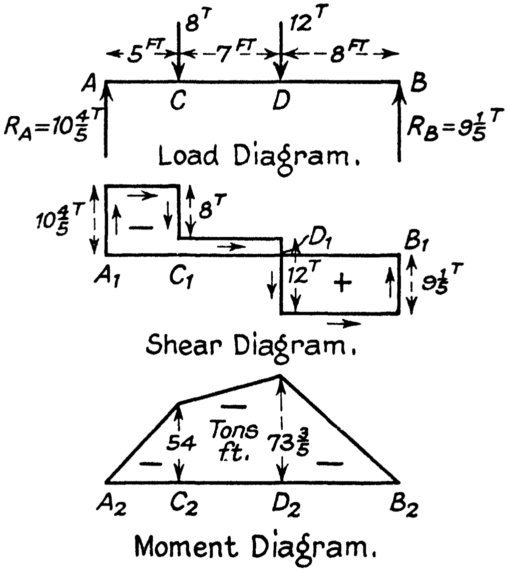


FIG. 11

The negative signs are introduced as the moments are negative.

The shear changes sign at D, and the moment is a maximum negative moment at this point.

(*b*) Simply-supported beam wholly covered with a uniformly distributed load of w tons per foot run. Let A (Fig. 12) be the origin. Let w tons per foot run be the rate of loading. Total load $= wl$ tons.

$$R_A = R_B = \frac{wl}{2} \text{ tons}$$

At any section X, distant x ft. from A, the shear force

$$S_X = -R_A + wx$$

At the centre, C,

$$S_C = -R_A + \frac{wl}{2} = 0$$

The negative sign is introduced, as R_A causes negative shear to the left of a section.

At any section X, distant x ft. from A, the moment

$$M_X = -R_A x + wx \times \frac{x}{2}$$
$$= -\frac{wlx}{2} + \frac{wx^2}{2}$$

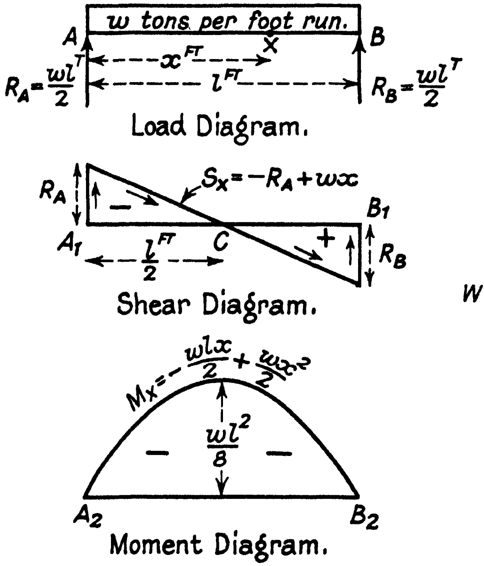


FIG. 12

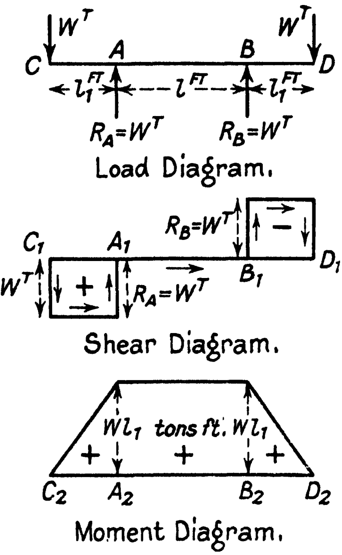


FIG. 13

The moment at C is a maximum,

$$= -\frac{wl^2}{4} + \frac{wl^2}{8} = -\frac{wl^2}{8}$$

NOTE.—The shear force at a section X distant x ft. from A is equal to R_A minus the area of the load diagram from A to X, due regard being paid to the sign. The moment at the section X is equal to the area of the shear force diagram between A and X; and the moment is a maximum at the section where the shear force is zero, i.e. changes sign.

11. **Overhanging Beams.** (*a*) Equal overhangs, with equal concentrated loads at the ends of the overhangs. (Fig. 13.)

$$R_A = R_B = W \text{ tons}$$

Shear Force Diagram. Between A and C, the shear force at any section is W tons and is positive. Between B and A it is zero, and between B and D it is W tons for all sections and is negative. The diagram is constructed by starting at C, and following the forces in the direction they act.

The moments at B and A are both equal to Wl_1 and are both positive, according to the notation of signs adopted. Between A and C, and B and D, the moments at a section are proportional to the distance the section is from C or D. Hence for the overhangs, the moment diagrams are triangles, being zero at C and D. Between B and A, the moment is the same for all sections and equal to $+ Wl_1$ units.

Note.—Between B and A, the moment is a constant, whilst the shear force is zero. It will be shown that the portion of a beam over which the moment is a constant, bends to the arc of a circle. This fact is made use of in experimental work for finding the value of Young's Modulus of Elasticity of a material in flexure when any effects due to shear forces are to be eliminated. In this case, no shear forces are present. (*See* further chapter on "Deflection of Beams.")

Similar shear force and moment diagrams are obtained for a simply-supported beam having two symmetrically-placed concentrated loads of equal value.

(*b*) Uniformly-distributed load over the whole of a beam, having equal overhangs. (Fig. 14.)

$$R_A = R_B = \frac{w(2l_1 + l)}{2}$$

$$= wl_1 + \frac{wl}{2}$$

The shear force and moment diagrams will be as shown.

If $\frac{wl^2}{8} < \frac{wl_1^2}{2}$, that is, $l < 2l_1$,

then the moment at the centre of the span l will be a minimum positive moment.

If $l > 2l_1$, then the moment at the centre of the span l will be a maximum negative moment, and there will be two sections on the span l at which the moment is zero.*

If $l = 2l_1$, the moment at the centre of the span l will be zero.

---

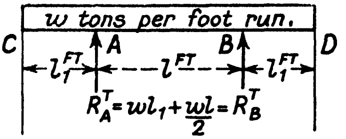
* *See* para. 49, Chapter IV.

In all the three cases there will be maximum positive moments at the supports.

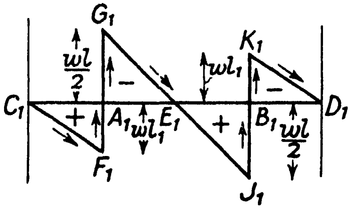
Problem 3. Construct the shear and moment diagrams for a simple beam loaded as in Fig. 15.

The total dead load is 8 tons, and it acts at a distance of 8 ft. from the left-hand support (0).

$$R_0 + R_{20} = 8 \text{ tons}$$



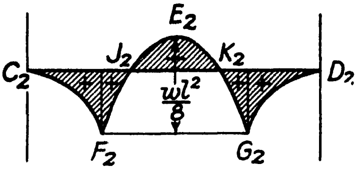
Load Diagram.



$C_1F_1A_1E_1B_1K_1D_1$ is the diagram for a beam with only the overhangs loaded.

$A_1G_1E_1J_1B_1$ is the diagram for a simple beam (AB only loaded).

Shear Diagram.



The shaded area is the Resultant.

$C_2F_2G_2D_2$ is the moment diagram for the beam with the overhangs only loaded.

$F_2E_2G_2$ is the diagram for a simple beam AB.
Moments at the supports $= +\frac{wl_1^2}{2}$

Moment Diagram.

FIG. 14

Moments about 0,

$$R_{20} \times 20 = 8 \times 8$$
$$R_{20} = 3\tfrac{1}{5} \text{ tons}$$
$$R_0 = 8 - 3\tfrac{1}{5} = 4\tfrac{4}{5} \text{ tons}$$

TO CONSTRUCT THE SHEAR DIAGRAM. At 0, erect a perpendicular upwards to scale equal to $4\frac{4}{5}$ tons : this is constant to the section 6 ft. from 0. This shear is of negative sign. Between section 10 ft. and 20 ft. the shear is constant and positive and equal to $R_{20} = 3\frac{1}{5}$ tons. At section 10, draw a perpendicular downwards of $3\frac{1}{5}$ tons to scale, and continue a shear line to the 20 ft. section.

Complete the diagram by joining the negative diagram to the positive one, as shown in the shear diagram. This is true, as at any section on the loaded portion of the beam the shear is equal to R_0 minus the load up to this section. It can be shown that the shear is zero at a section 8·4 ft. from 0 : at this section the moment will be a maximum.

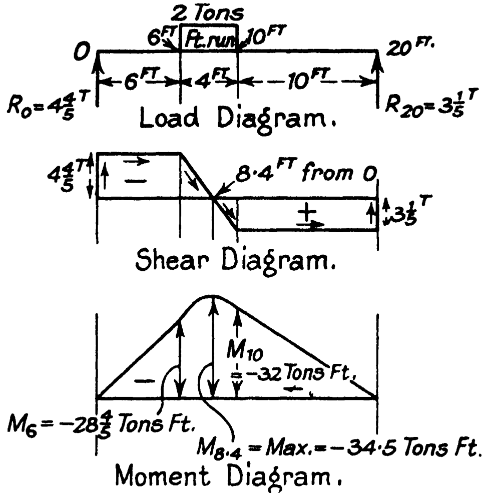


FIG. 15

The moment at section 6 is

$$M_6 = -6 \times 4\tfrac{4}{5} = -28\tfrac{4}{5} \text{ tons-ft.}$$

At section 6 erect an ordinate to scale of $28\frac{4}{5}$ ft.-tons. The B.M. line between 0 and 6 will be a straight line, as it is also between sections 10 and 20, where

$$M_{10} = -10 \times 3\tfrac{1}{5} = -32 \text{ tons-ft.}$$
$$\text{and } M_{20} = 0$$

Between sections 6 ft. and 10 ft. the moment line will be a curve, and the maximum negative moment will be at the section 8·4 ft. from 0 and

$$M_{8\cdot4} = -8\cdot4 \times 4\tfrac{4}{5} + 2 \times \frac{2\cdot4^2}{2}$$
$$= -34\cdot5 \text{ tons-ft.}$$

Second Method. Let the maximum moment be at a distance z ft. from 0.

$$\text{Then } M_z = -4{\cdot}8z + (z-6)^2$$

$$\frac{dM_z}{dz} = -4{\cdot}8 + 2(z-6) = 0$$

$$\text{or } 2z = (4{\cdot}8 + 12) \text{ ft.}$$

$$z = 8{\cdot}4 \text{ ft.}$$

The maximum moment is calculated as shown in the previous paragraph.

Problem 4. A simple beam 30 ft. long carries a load which increases uniformly from zero at one end to 2 tons per ft. at the other end. (Fig. 16.) Find the expression for the bending moment at any section, and draw the shear and moment diagrams. What is the greatest bending moment and where does it occur?

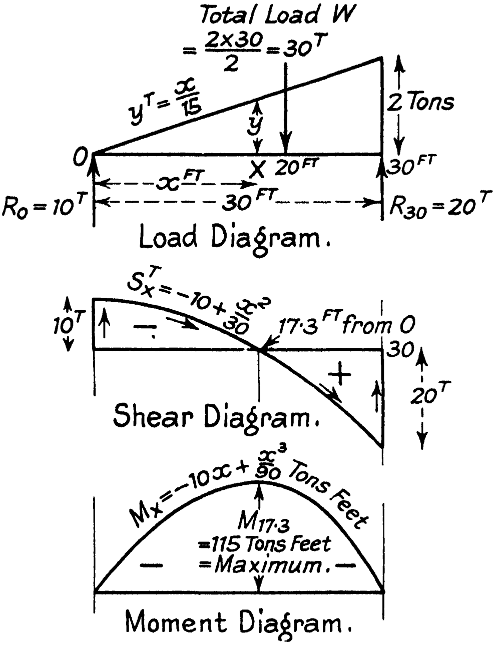


FIG. 16

$$\text{For equilibrium,} \quad R_0 + R_{30} = \frac{2 \times 30}{2} = 30 \text{ tons}$$

Moments about 30,

$$R_0 \times 30 = 30 \times 10$$
$$R_0 = 10 \text{ tons}$$
$$R_{30} = 20 \text{ ,,}$$

Working from 0 as origin and taking a section X to the right distant x ft. from 0.

The rate of loading at X will be

$$y \text{ tons} = \frac{x}{15}$$

Shear at X,

$$S_X = -R_0 + y \times \frac{x}{2} = -R_0 + \frac{x^2}{15 \times 2}$$
$$= -10 + \frac{x^2}{30} \text{ tons}$$
$$= 0, \text{ when } x^2 = 300$$
$$\text{or } x = 17{\cdot}3 \text{ ft. from } 0$$

The shear diagram will therefore be as shown in Fig. 16.

The moment at a section X distant x from 0

$$= M_X = -R_0x + \frac{xy}{2} \times \frac{x}{3}$$
$$= -10x + \frac{x}{15} \times \frac{x}{2} \times \frac{x}{3}$$
$$= \left(-10x + \frac{x^3}{90}\right) \text{ tons-ft.}$$

For maximum moment,

$$\frac{dM_X}{dx} = -10 + \frac{x^2}{30} = 0$$
$$\text{that is, } x = 17{\cdot}3 \text{ ft.}$$
$$\text{which confirms } S_{17\cdot3} = 0$$

At section 17·3 ft. from 0,

$$M_{17\cdot3} = -10 \times 17{\cdot}3 + \frac{17{\cdot}3^3}{90}$$
$$= -173 + 58$$
$$= -115 \text{ tons-ft.}$$

The moment diagram is as shown in Fig. 16.

12. Relation between Loads, Shearing Forces, and Bending Moments. For Simple Beams and Cantilevers.

CONCENTRATED LOADS. S_X = Shear at any section

= Σ forces to the right or left of the section.

DISTRIBUTED LOADS. A small change in the shear force along a length dx = small area of the load diagram,

i.e. $dS = w\,.\,dx$, where dx is a small length of the beam and w the rate of loading along dx assumed constant.*

$\therefore\ S = \int w\,.\,dx$ = total change between the required limits

= (area of load diagram between these limits)

If S_X is the shear force at a section X distant x from the origin and S_Y is the shear at a section Y distant y from the origin, then $S_X = S_Y + \int_y^x w\,.\,dx$ taking appropriate signs for each term.

MOMENT AND SHEAR. From the examples considered, a small change in the moment is equal to a small area of the shear force diagram,

i.e. $dM = S\,.\,dx$ where S = average shear force over dx, then $\frac{dM}{dx}$ = shear force S† = rate of change of the moment diagram,

and $M = \int S\,.\,dx$ = total change of bending moment between the required limits.

In both the simple beam and the cantilever uniform loaded over the whole length

M_X = moment at a section = area of the shear force diagram between the section and the origin,‡ due regard being paid to sign. Summarizing,

$$\frac{dS}{dx} = w; \quad S = \int w\,.\,dx = \frac{dM}{dx}; \quad M = \int S\,.\,dx$$

---

[*] w is not necessarily constant along a beam.

[†] This relation indicates that where a shearing force passes through a zero value and changes sign the value of the moment is a mathematical maximum or minimum. See Note, para. 10, page 10.

[‡] For a simple beam, origin at the supports. For a cantilever the origin is at the free end. However, if the origin for the cantilever is at the support, then M_X = Moment at the support less the area of the shearing force diagram between the support and the section.

13. A graphical method of finding the shear and moment for beam sections depends upon the above relations. Shear forces are found by the graphical integration of the load diagram: moments are calculated by the graphical integration of the shear force diagram.

REFERENCES

For further examples of moments, etc.

1. *Theory of Structures*, Morley. (Longmans, Green & Co.)
2. *Structural Engineering*, Husband and Harby. (Longmans, Green & Co.)
3. *Structural Steelwork*, Black. (Pitman's Technical Primers.)
4. *Strength of Materials*, Part I, S. Timoshenko.
5. *Materials and Structures*, Vol. I, E. H. Salmon. (Longmans, Green & Co.)

EXAMPLES

1. A girder 30 ft. long, supported at the ends, has a uniformly distributed load of 2 tons per lineal foot extending from 5 ft. from one end to within 10 ft. of the other end, and there is a concentrated load of 15 tons at the centre of the uniformly distributed load. Draw the bending moment and shearing force diagrams, giving the maximum and minimum values in each case. (I.C.E.)

2. An overhanging beam AB, 25 ft. long, rests on two supports which are at distances of 5 and 19 ft. respectively from the end A. The beam carries a load of 2 tons at A, 1 ton at B, and 2 tons at the centre of the beam. Draw the shearing force and bending moment diagrams for the beam.

3. Explain clearly the relation between load, shear, and bending moment diagrams. (U. of B.)

4. A beam 25 ft. long is supported at one end and on a pier at a distance of 5 ft. from the other end. The beam is uniformly loaded from end to end with a load of 1 ton per lineal foot, and a concentrated load of 5 tons is hung at the extremity of the overhanging portion. Draw the bending moment and shearing force diagrams. (I.C.E., Oct. 1922.)

5. A girder, 55 ft. long, is supported on two piers—one, 5 ft. from one end; the second, 1 ft. from the other end. It carries a uniformly distributed load of 1 ton per lineal foot. Find the bending moment at the piers and draw the bending moment and shearing force diagrams. (I.C.E., April 1922.)

6. A girder, 50 ft. long, is supported 10 ft. from its left and 15 ft. from its right extremities, the overhanging ends being free. It is loaded with a uniformly distributed load of 1 ton per lineal foot, and there is a concentrated load of 10 tons midway between the two supports. Draw the bending moment and shearing force diagrams, giving the maximum value in each case. (I.C.E., April 1923.)

7. A girder, 30 ft. long, supported at the ends, has a uniformly distributed load of 1·5 tons per lineal foot extending from 5 ft. from one end to 10 ft. from the other end, and there is a concentrated load of 10 tons at the

centre of the uniformly distributed load. Draw the bending moment and shearing force diagrams, giving the maximum and minimum values in each case.

8. A single skin coffer-dam is subjected to a maximum head of water of 20 ft. If the sheeting is held in a vertical position by two horizontal frames, one at the same level as the surface of the water and the other 20 ft. below at the bottom, find the position and amount of the maximum bending moment in a strip of the dam 1 ft. wide. Neglect any fixing moment at the supports and assume the weight of water per cubic foot to be 64 lb. (I.C.E., July 1923.)

9. Draw the shear and moment diagrams for the following beams—

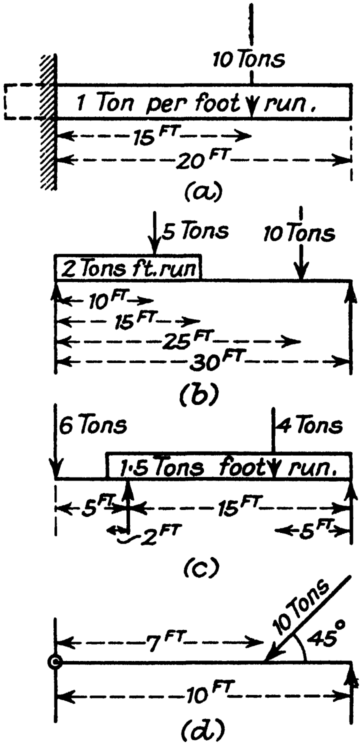


FIG. 17

10. A beam AB is 20 ft. long. It is supported at the end A and at a point 12 ft. from A. There is a load of $\frac{1}{2}$ ton at the end B and a load of 1 ton at a point 8 ft. from A, and a uniformly distributed load of 2 cwt. per foot run along the whole beam. Draw (1) the shear diagram; (2) the bending moment diagram for the beam. Write down the maximum shearing force and the maximum bending moment. (L.U., 1922.)

CHAPTER II

THEORY OF SIMPLE BENDING AND MOMENTS OF INERTIA

15. **Notes on Stresses and Strains.** STRESS. If a body is subjected to external forces and it is cut by a plane section, an internal force will be transmitted across this section tending to hold the body in equilibrium. This force is called stress, and the material of the body is said to be stressed. The stress may or may not be uniformly distributed over the area of the section. The intensity of stress, or as it is often called stress, is the force per unit of area.

If over a small area a sq. in. the total internal force is P tons, then the stress is $\frac{P}{a}$ tons per square inch.

STRAIN. The body which is stressed under the action of the external forces will suffer a change of shape, and it is said to be strained or deformed. If a body of original length l in. suffered a change of length δl, then the unit strain, or as it is sometimes called strain, is $\frac{\delta l}{l} = e$. If l is in inches, then the strain is the amount of deformation per unit length of 1 in.

KINDS OF STRESS. Several kinds of stress may be produced in a body: they depend on the arrangement of the external loads. These stresses are tensile, compressive, and shear: the first two are direct stresses, because they are perpendicular to the plane section under consideration; the last is tangential to the plane. Tensile and compressive stresses may be produced by direct external pulls or thrusts on a body, such external forces being at right angles to the considered planes or, as it will be seen, they may be caused by bending the body, when stresses are produced which are normal to a plane section of the beam. A shear stress is produced when a body is subjected to torsion or a twisting action.

DIRECT STRESSES. When a body is subjected to a pull or tensile force, it is elongated in the direction of the pull, and the body is said to be in a state of tension. When the body is subjected to an external thrust or compressive force, it is shortened in the direction of the force, and is said to be in a state of compression.*

The properties of materials in tension, compression, or torsion may be ascertained by mechanical tests; and by plotting stresses against the corresponding strains, curves known as stress-strain curves are obtained.

MODULI OF ELASTICITY—

Tension or Compression. The modulus of elasticity is denoted by E (load per unit area).

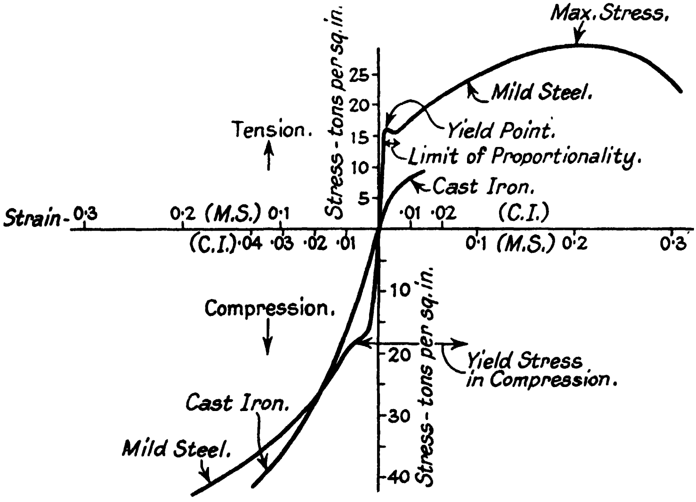


FIG. 18

Let f represent the normal stress and e the corresponding strain (extension or compression per unit length); then $E = \frac{f}{e}$. For ductile materials, such as mild steel, E in compression is taken the same as for tension, an elastic tensile test being more satisfactorily carried out than an elastic compression test.

Torsion. The modulus of elasticity is denoted by G, sometimes N (load per unit area), and is called the modulus of rigidity.

Let q be the shear stress and θ radians the shear strain. Then $G = \frac{q}{\theta}$ (*see* Chap. V).

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* See Chapter V for the definitions of shearing stress and strain; and Chapter XVII for definitions of permanent set, proof stress, or field strength.

For ductile materials, such as mild steel, for a range of stress from zero to a critical stress, the stress is proportional to the strain : i.e. E or G is a constant during this range of stress. The critical stress above which the strain is not proportional to the stress is called the limit of proportionality (often known as the elastic limit). Above this stress, E or G is a variable. With brittle materials, such as cast-iron and concrete, E and G vary with the stress from zero load,* consequently for design work it is necessary to know the value of E or G for the working stress used.

Strength and Elasticity coefficients for materials : Tons/square inch.

Material.	Limit Proportionality.		Ultimate Strength.		E Tension or Compression.
	Tension.	Compression.	Tension.	Compression.	
Wrought-iron . .	12–15		21–24		11,000–13,000
Mild Steel .	17–18	16–17	30–32	(Yield) 22	13,000–14,000
Cast-iron . .	(No definite limit)		7–11	35–60	6,000–10,000
Duralumin . .			22–24		4,300–4,500
Oak . .			4–6	2 to 5 { with grain }	500–700
Soft Woods . .			1–3	1–3	450–500

FIG. 19

Stress-strain diagrams for mild steel and cast-iron are given in Fig. 18. It will be noticed that cast-iron is much stronger in compression than in tension.†

16. **Theory of Bending.** Let X (Fig. 20) be any section of a beam carrying a system of loads. The portion of the beam to the right of the section is in equilibrium against vertical translation, but it would have the tendency to rotate anti-clockwise, and the magnitude and direction of this tendency to rotation is determined by the bending moment at the section. At this portion of the beam there are internal forces induced by the external loading, and these forces produce at the section a *couple* whose magnitude is equal to that of the external bending moment, but acting in the opposite direction. Usually the internal forces will be elastic ones, as the beam will not be stressed above the limit of proportionality of the material. These elastic forces consist of pulls decreasing uniformly to zero and thrusts increasing uniformly from zero. These longitudinal

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* *See* Batson and Hyde (*Mechanical Testing*, Vol. I). Publishers, Chapman & Hall. (*See* paragraph on " Modulus of Direct Elasticity")

† For further work on strengths of materials, the student is referred to textbooks on the subject. (*See* references at the end of the chapter.)

forces form a couple which must at any section, since the beam is in equilibrium, be equal and opposite to the bending moment at the section. This couple is called the ***moment of resistance.***

The axis about which the moment of resistance is taken is called the ***neutral axis*** of the section, and is that axis or that fibre of the beam which is in an unstrained or unstressed condition. It will be denoted by N.A.

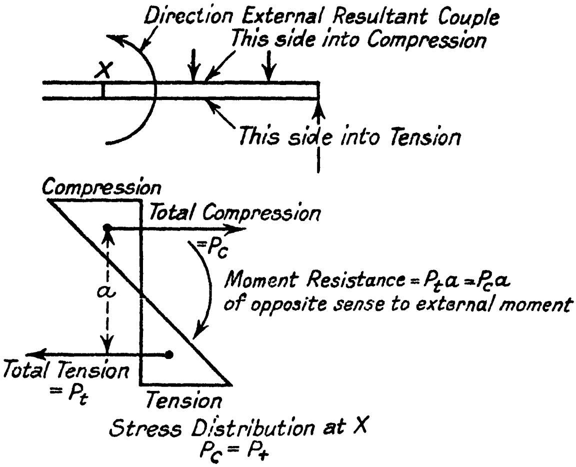
Direction External Resultant Couple
This side into Compression
X
This side into Tension
Compression
Total Compression
$=P_c$
Moment Resistance $= P_t a = P_c a$
of opposite sense to external moment
a
Total Tension
$= P_t$
Tension
Stress Distribution at X
$P_c = P_t$

FIG. 20

The moment of resistance = the algebraic sum of the moments of all the induced tensile and compressive forces taken about the neutral axis.

17. Assumptions Made in the Theory of Simple Bending.*

(1) The beam is stressed within the limit of proportionality of the material.

(2) Young's modulus (E) is the same for tension and compression.

(3) A plane cross-section at right angles to the plane of bending always remains plane.

(4) There is no resultant pull or push on the cross-section of the beam.

(5) The fibres are free to expand or contract laterally.

---

* For simple bending to occur the external forces must be all applied in the plane in which the beam bends. It does not follow necessarily that a beam carrying vertical loads will bend in a vertical plane. Side or horizontal bending will occur if the beam be not symmetrical about a vertical plane passing through the centroid of the section.

18. Let Fig. 21 represent a small portion of a bent beam, so taken that the form assumed is a circular arc of radius equal to the radius of curvature R. This is equivalent to pure bending, the moment being constant over the length considered. It occurs when a bar is bent under equal and opposite couples at its ends.

O is the centre of curvature, db is parallel to ac, ab lies in the plane of the neutral axis.*

Let ce be a fibre situated at a distance y in this case above the neutral surface; in the figure it is in tension, so that it is greater by de than its original length $cd = ab$.

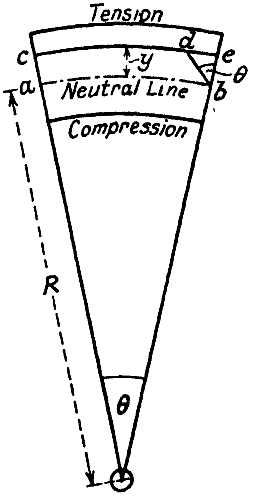

FIG. 21

$$\text{Strain in } ce = \frac{\text{increase in length}}{\text{original length}}$$

$$= \frac{de}{cd} = \frac{de}{ab}$$

but as figures deb and abO are similar,

$$\frac{de}{ab} = \frac{eb}{Ob} = \frac{y}{R}$$

But stress = $E \times$ strain

Let f_t = tensile stress in $ce = \frac{Ey}{R}$

$$\text{then } \frac{f_t}{E} = \frac{y}{R}$$

Similarly for a fibre situated at a distance y_1 from the N.A. on the compression side.

Let f_c = induced compressive stress

$$\text{then } \frac{f_c}{y_1} = \frac{E}{R} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

That is, the intensity of the direct longitudinal stress at any point in the cross-section is proportional to the distance of that point from the N.A., reaching a maximum at the boundaries farthest from the N.A.

Let PQ (Fig. 22) be any cross-section of a beam. Consider a thin horizontal strip of that section parallel to the neutral axis, the breadth of the strip being dy, the length x, and y the height above the neutral axis.

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* The line ab is a trace of the surface in which fibres do not undergo strain during bending. This surface is called the *neutral surface*, and its intersection with any cross-section is called the *neutral axis*.

$$\text{Stress at height } y = \frac{Ey}{R}$$

$$\begin{aligned}\text{Total force on the strip} &= \text{stress} \times \text{area}\\ &= \frac{E}{R} \cdot y \cdot x dy\\ &= \frac{E}{R} y \cdot dA\end{aligned}$$

where dA is the area of the strip.

Total force acting on the whole cross-section is equal to the sum of all the small forces between the limits y_t and y_c.

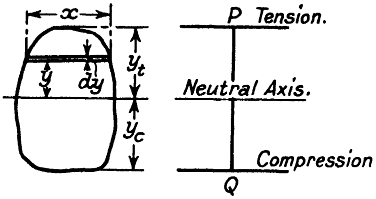

FIG. 22

$$\text{Total force on the whole section} = \frac{E}{R} \Sigma y \cdot dA = \frac{E}{R} A \bar{y}$$

where A = total area of the cross-section and $\bar{y}$ is the height of the centroid of the cross-section above or below the N.A.

But since it is assumed a plane section remains plane after bending, the total force on that section must be zero, or the total tensile forces are equal to the total thrusts.

$$\text{Therefore } \frac{E}{R} A \bar{y} = 0$$

that is, $\bar{y} = 0$, as E, R, and A have definite values. Therefore, the neutral surface must pass through the centroid of the section.

$$\text{Total force on the strip } x \cdot dy = \frac{E}{R} \cdot y \cdot dA$$

$$\text{Moment of this force about the N.A.} = \frac{E}{R} \cdot y \cdot dA \cdot y$$

Moment of resistance for the whole section is equal to the sum of the moments for all similar strips

$$= \frac{E}{R}\Sigma y^2 \cdot dA = \frac{E}{R}I$$

where $I = \Sigma y^2 \cdot dA$ and is called the moment of inertia of the section about the neutral axis.

Dimensions of I are [length units]$^4 = [L]^4$ where $[L]$ represents a unit of length.

Moment of resistance = external bending moment.

$$\text{External moment} = M = \frac{E}{R}I$$

$$\text{and } \frac{M}{I} = \frac{E}{R} \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{Also } \frac{E}{R} = \frac{f}{y}$$

$$\text{therefore } M = f\frac{I}{y} \quad . \quad . \quad . \quad . \quad (3)$$

The general relation can be expressed as,

$$\frac{M}{I} = \frac{E}{R} = \frac{f}{y} \quad . \quad . \quad . \quad (4) \text{ and } (5)$$

where f may be a tensile or a compressive stress and y is the distance of the fibre from the neutral axis.

The maximum intensities of stress are at the outer boundaries, and if these are f_t and f_c respectively,

$$f_t = M\frac{y_t}{I}: \quad f_c = M\frac{y_c}{I}$$

If a section is symmetrical about the neutral axis, i.e.

$$y_c = y_t$$

$$\text{then } f_t = f_c$$

The quantity $\frac{I}{y}$ is called the "Modulus of the section," and is usually denoted by Z,

$$\text{so that } M = fZ \quad . \quad . \quad . \quad (6)$$

There are two moduli for every section which is not symmetrical about the neutral axis: one $= I/y_t$ and the other I/y_c.

Dimensions of Z are [length units]^3 $= [L]^3$

Problem 5. To what radius of curvature may a beam of mild steel be bent so that its maximum tensile stress will not exceed 8 tons per square inch?

$$E = 13{,}000 \text{ tons per square inch}$$

Depth of beam 10 in., and the beam is symmetrical about the N.A.

$$\frac{f_t}{y} = \frac{E}{R} \qquad f_t = 8 \text{ tons/sq. in.}$$

$$y = 5 \text{ in.}$$

$$\begin{aligned} R = \frac{Ey}{f_t} &= \frac{13{,}000 \times 5}{8} \text{ in.} \\ &= 8125 \text{ in.} \\ &= 677 \text{ ft.} \end{aligned}$$

19. **Notes on Moments of Inertia.** (*a*) Let a thin lamina of area A consist of a number of small areas $a_1 . a_2 . a_3$, etc., situated at distances $r_1 . r_2 . r_3$, etc., from some axis RR

Then the moment of inertia of the lamina or total area about the axis RR

$$\begin{aligned} = I_{RR} &= a_1r_1^2 + a_2r_2^2 + a_3r_3^2, \text{ etc.} \qquad . \quad (7) \\ &= \Sigma ar^2 \\ &= \Sigma \text{ moment of a moment.} \end{aligned}$$

Imagine the whole area A concentrated at a distance k from RR, so that

$$Ak^2 = I_{RR} \quad . \quad . \quad . \quad . \quad (8)$$

then k is called the radius of gyration.

(*b*) Let XX be an axis through the centroid of the area A.

$I_{XX} =$ moment of inertia about an axis XX

Let MM be an axis parallel to XX at a distance m from it.

$$\text{Then } I_{MM} = I_{XX} + Am^2 \quad . \quad . \quad . \quad (9)$$

(*c*) Let there be three axes, OX, OY, OZ, mutually perpendicular to one another and meeting at the origin O, which is the centroid of an area A. The axis OZ, being at right angles to the plane of the area,

$$\text{then } I_{ZZ} = I_{XX} + I_{YY} \quad . \quad . \quad . \quad (10)$$

From the relation expressed in equation (10), any number of axes

OX and *OY* may be drawn at right angles to one another, and the sum of their moments of inertia will be equal to I_{zz}, that is, a constant.

Thus, if the moment of inertia about one of these axes is a maximum, then the moment of inertia about the axis at right angles to it must be a minimum. These axes are spoken of as the " Principal Axes of Inertia."

A principal axis can also be an axis of symmetry ; and if an area has one axis of symmetry, this will give one principal axis, and the other principal axis can be determined by drawing it through the centroid and perpendicular to the axis of symmetry.

Referring to Fig. 23, let *OX* and *OY* be the principal axes of inertia for the given area,

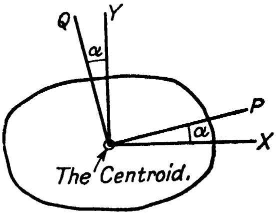

FIG. 23

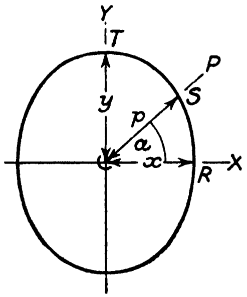

FIG. 24

Let I_{XX} be the maximum moment of inertia
and I_{YY} ,, minimum ,, ,,

OP and *OQ* are another pair of rectangular axes at an angle a to the principal axes,

$$\text{then } I_{PP} + I_{QQ} = I_{XX} + I_{YY} \quad . \quad . \quad . \quad . \quad (11)$$

It can be shown that

$$I_{PP} = I_{XX} \cos^2 a + I_{YY} \sin^2 a \quad . \quad . \quad . \quad (12)$$

$$I_{QQ} = I_{XX} \sin^2 a + I_{YY} \cos^2 a \quad . \quad . \quad . \quad (13)$$

20. **The Momental Ellipse.** *OX* and *OY* are the principal axes of inertia of a plane figure. (Fig. 24.) Let $I_{XX} > I_{YY}$, *OP* be an axis inclined at an angle a to *OX*,

$$\text{then } I_{PP} = I_{XX} \cos^2 a + I_{YY} \sin^2 a$$

NOTE. The theory of simple bending can be used for unsymmetrical beam sections, if the applied bending couple is in an axial plane which contains one of the two "Principal axes of Inertia."

Let the lengths OR, OS, OT be measured to the same scale along the axes OX, OP, OY, such that

$$OR = x = \sqrt{\frac{A}{I_{\text{XX}}}}. = \frac{1}{k_{xx}}$$

$$OS = p = \sqrt{\frac{A}{I_{\text{PP}}}}. = \frac{1}{k_{pp}}$$

$$OT = y = \sqrt{\frac{A}{I_{\text{YY}}}}. = \frac{1}{k_{yy}}$$

A = area of the figure

It can be shown that y and x are the semi-major and minor axes of an ellipse, which is known as the *momental ellipse:*

$\therefore$ the point S lies on the ellipse.

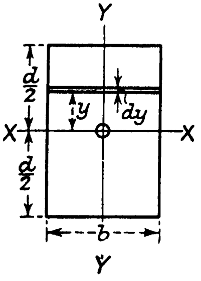

FIG. 25

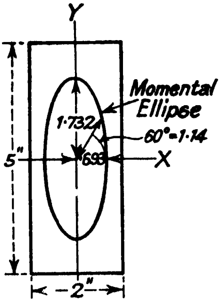

FIG. 26

If the principal moments of inertia are known, then the semi-axes of the momental ellipse can be found and the ellipse drawn to scale. If any radius be drawn and its length r measured to the same scale as x and y, the moment of inertia about that radius $= \frac{A}{r^2}$.

EXAMPLES.

(*a*) **Find the moment of inertia of a rectangle about an axis parallel to the ends and passing through the centroid. Construct the momental ellipse for a rectangle 2" × 5". Also find I about an axis through the centroid at an angle of 60° with the X axis (Fig. 27).**

$$\text{From Fig. 25, } I_{\text{XX}} = \int_{-\frac{d}{2}}^{\frac{d}{2}} \cdot b \cdot dy \cdot y^2 = \frac{bd^3}{12}$$

$$I_{\text{YY}} \text{ by similarity } = \frac{db^3}{12}$$

For the given rectangle, Fig. 26

$$I_{XX} = \frac{2 \times 5^3}{12} = 20{\cdot}83 \text{ in.}^4$$

$$I_{YY} = \frac{5 \times 2^3}{12} = 3{\cdot}33 \text{ in.}^4$$

$$x = \sqrt{\frac{A}{I_{XX}}} = 0{\cdot}693$$

$$y = \sqrt{\frac{A}{I_{YY}}} = 1{\cdot}732$$

x and y are the semi-axes of the momental ellipse.

$$r_{60^\circ} \text{ measures } 1{\cdot}14 \;\therefore\; I_{60^\circ} = \frac{A}{r^2} = \frac{10}{(1{\cdot}14)^2} = 7{\cdot}69 \text{ in.}^4$$

By calculation

$$I_{60^\circ} = 20{\cdot}83 \cos^2 60^\circ + 3{\cdot}33 \sin^2 60^\circ = 7{\cdot}718 \text{ in.}^4$$

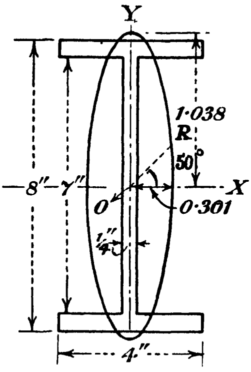

FIG. 27

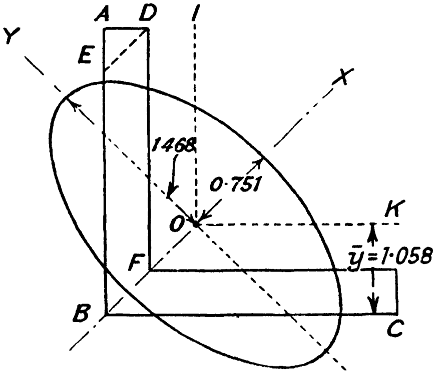

FIG. 28

For a square of side S,

$$I_{XX} = I_{YY} = \frac{S^4}{12}$$

(*b*) For the given section (Fig. 27), find the greatest and least moments of inertia. Draw the momental ellipse for the section, and use it to find the moment of inertia about the axis *OR*.

$$I_{YY} = 2 \times \tfrac{1}{12} \times \tfrac{1}{2} \times 4^3 + \tfrac{1}{12} \times 7 \times (\tfrac{1}{4})^3$$
$$= 5{\cdot}342 \text{ in.}^4$$

$$I_{XX} = \tfrac{1}{12} \times \tfrac{1}{4} \times 7^3 + 2\{\tfrac{1}{12} \times 4 \times (\tfrac{1}{2})^3 + 4 \times \tfrac{1}{2} \times (3\tfrac{1}{2} + \tfrac{1}{4})^2\}$$
$$= 63{\cdot}48 \text{ in.}^4$$

Area of section = 5·75 sq. in.

$$x = \sqrt{\frac{5{\cdot}75}{63{\cdot}48}} = 0{\cdot}301$$

$$y = \sqrt{\frac{5{\cdot}75}{5{\cdot}342}} = 1{\cdot}038$$

r_{50° measures ·445 $\therefore I_{50^\circ} = 29{\cdot}03 \text{ in.}^4$

(c) * For an angle iron $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{1}{2}''$, the momental ellipse is as in Fig. 28. The student is requested to check all the moments of inertia for himself.

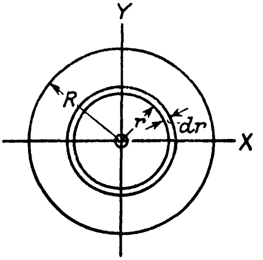

FIG. 29

OX is an axis of symmetry and therefore a principal axis. OY will be the other.

$$I_{AB} = 7{\cdot}27;\ I_{XX} \text{ of } \triangle ADE = {\cdot}628;$$
$$I_{XX} \text{ of } EDFB = 2{\cdot}25$$
$$I_{\text{II}} = I_{xx} = 3{\cdot}632 \text{ in.}^4$$
$$I_{XX} = 2(2{\cdot}25 + {\cdot}628) = 5{\cdot}756 \text{ in.}^4$$
$$I_{YY} = 2 \times 3{\cdot}632 - 5{\cdot}756$$
$$= 1{\cdot}508 \text{ in.}^4$$

The semi-axes of the momental ellipse are $x = {\cdot}751$ and $y = 1{\cdot}468$.

(d) THE CIRCLE. (Fig. 29.) To find the moment of inertia about a diameter, it is necessary to find, first, the moment of inertia about the axis at right angles to the plane of the figure and passing through the centre.

$$I_{zz} = I_{XX} + I_{YY}$$

$$\text{also } I_{zz} = \int_0^R 2\pi r \cdot dr \cdot r^2$$
$$= \tfrac{1}{2}\pi R^4$$
$$\therefore\ I_{XX} = \frac{\pi R^4}{4}$$

(e) FOR A HOLLOW CIRCLE.

$$I_{XX} = \frac{\pi}{4}(R_e^4 - R_i^4) \quad \text{where } R_e = \text{external radius}$$
$$R_i = \text{internal radius}$$

The momental ellipse becomes a circle for the circle, and also for cross-sections which are similarly symmetrical about the two mutual axes: e.g. the square.

---

* Examples (a), (b), and (c) (pp. 28–30) are from Mann's *Practical Mathematics* by permission of Messrs. Longmans, Green & Co.

21. **Routh's Rule.** If a body is symmetrical about three axes which are mutually perpendicular, the (radius of gyration)^2 about one axis is equal to the sum of the squares of the other two semi-axes divided by 3, 4, or 5, according as the body is rectangular, elliptical, or ellipsoidal.

EXAMPLES.

Rectangle—only 2 axes,

then (radius of gyration)^2 about an axis through the centroid and parallel to the ends

$$= k^2 = \frac{\left(\frac{d}{2}\right)^2 + 0}{3} = \frac{d^2}{12}$$

$$I = bd \times \frac{d^2}{12} = \frac{bd^3}{12}$$

Circle $\quad (k_{\text{diameter}})^2 = \frac{\left(\frac{d}{2}\right)^2}{4} = \frac{d^2}{16}$

$$I_{\text{diameter}} = \frac{\pi d^2}{4} \times \frac{d^2}{16} = \frac{\pi d^4}{64} = \frac{\pi R^4}{4}$$

Ellipse $\quad (k_{\text{major axis}})^2 = \frac{(\text{semi-minor axis})^2}{4}$

$$(k_{\text{minor axis}})^2 = \frac{(\text{semi-major axis})^2}{4}$$

Area ellipse $= \pi$ (semi-major axis) (semi-minor axis)

22. **Graphical Methods of Finding Moments of Inertia for Unsymmetrical Figures.** (*See* Fig. 30.) It is required to find:

(1) The position of the centroid with relation to some axis.

(2) The moment of inertia about this same axis, and from which I for an axis through the centroid and parallel to the axis of reference can be obtained.

Enclose the irregular figure in a rectangle, and let two adjacent sides OP and OQ be the axes of reference.

R is a point where the axis OX touches the figure.

Divide the figure into a number of horizontal strips of dy thickness.

Let LM = the width of one strip,

then $LM \cdot dy$ = the area of this strip.

LM is at a distance y from OX.

Let AB be the projection of LM on QS.

Join A and B to R, to cut LM in L_1M_1.

Do the same for other strips and join up corresponding points L_1M_1 to form a new figure called the "first derived figure."

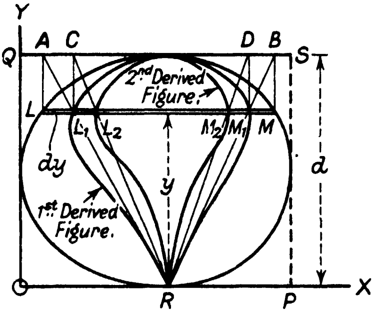

FIG. 30

Let A = area of the original figure

A_1 = area of the first derived figure

d = depth of the figure between the sides of the rectangle QS and OP

Then $\bar{y}$ the perpendicular distance of the centroid from the axis OX

$$= \frac{d \times \text{area of the first derived figure}}{\text{area of the figure}}$$

TO FIND I_{xx} (Moment of Inertia about the axis OX)

Let CD be the projection of L_1M_1 on QS.

Join C and D to R, to cut L_1M_1 in L_2M_2.

As before, continue for other strips and join up all similar points L_2M_2 to form a "second derived figure."

Then $I_{xx} = d^2 \times$ area of the second derived figure.

I about the axis through the centroid and parallel to OX is $I = I_{xx} - A(\bar{y})^2$.

The areas are best found by means of a planimeter.

An example of derived figures is given in Fig. 31.

23. **The Modulus Figure.** It has been shown that the external moment

$$M = \text{stress} \times \text{modulus of section} = f \cdot \left(\frac{I}{y}\right)$$

For non-mathematical sections, the modulus of the section can be obtained graphically from the construction of the modulus figure. By graphical or experimental methods, find the centroid of the section. Through the centroid, draw the axis about which the beam will bend. This will be the neutral axis.

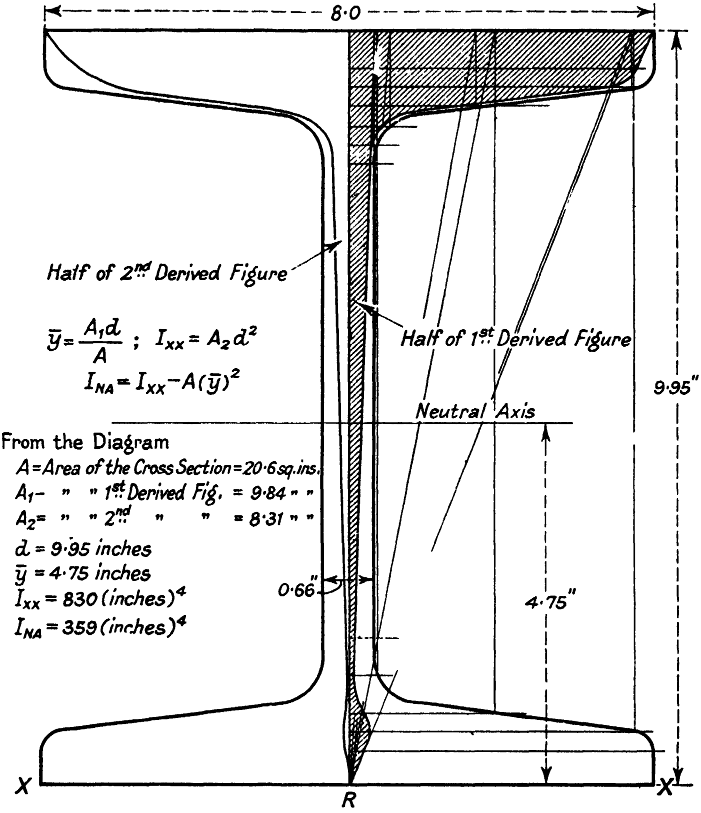
8·0
Half of 2nd Derived Figure
$\bar{y} = \frac{A_1 d}{A}$; $I_{xx} = A_2 d^2$
$I_{NA} = I_{xx} - A(\bar{y})^2$
From the Diagram
A = Area of the Cross Section = 20·6 sq. ins.
A_1 – " " 1st Derived Fig. = 9·84 " "
A_2 = " " 2nd " " = 8·31 " "
d = 9·95 inches
$\bar{y}$ = 4·75 inches
I_{xx} = 830 (inches)^4
I_{NA} = 359 (inches)^4
0·66"
Half of 1st Derived Figure
Neutral Axis
9·95"
4·75"
X
R
X

FIG. 31

With a pole O, in the neutral axis construct the first derived or modulus figures on the tension and compression sides.

(*a*) If a tensile stress is the working criterion, the tensile modulus figure is required. If y_t is not equal to y_c, then on the compression side take a base at a distance y_t from the neutral axis. All strips on the compression side have to be projected on to this base.

All strips have been reduced into terms of the outer tension boundary, and therefore the modulus figure is now an area over which the stress is a constant, being equal to the skin or boundary tensile stress. The total tensile or compressive force will act at the centroids of the respective derived areas above or below the neutral axis. If the area has been reduced to terms of the maximum tensile stress,

$$\text{then } M = f_t \times A_t \times D_t$$

where A_t = area of the derived figure found from the tensile basis above or below the N.A.

D_t = the distance between the centroids of the tensile and compressive derived areas

Shaded Areas = Modulus Figures.

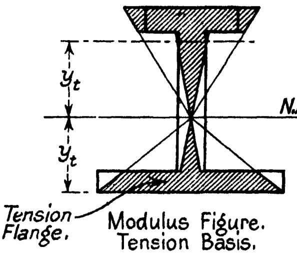

FIG. 32

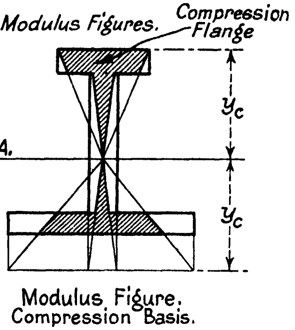

FIG. 33

The positions of the centroids can be found by experiment. An illustration is given in Figs. 32 and 33 and in Fig. 37.

(*b*) If working with compressive stresses, then the compression modulus figure is found by a similar construction.

ILLUSTRATIVE PROBLEMS OF NORMAL STRESSES IN BEAMS.

Problem 6. A rolled steel joist has the following properties—

Depth.	Width of flanges.	Area of section.	I_{XX}	I_{YY}
10 in.	8 in.	20·6 sq. in.	345(in.^{4})	71·6(in.^{4})

Such a joist is to be used as a beam 20 ft. span, loaded in the centre, with the web vertical. Find the safe load that can be carried if the factor of safety is 4. (London Univ., 1923.)

The neutral plane will be XX.

The ultimate stress for good mild steel is about 32 tons/sq. in.

Therefore the safe working stress will be $\frac{32}{4} = 8$ tons/sq. in.

Let W tons be the safe load required.

$$\text{Maximum moment} = \frac{Wl}{4} = \frac{W \times 20}{4} = 5\,W \text{ tons-ft.}$$

$$= 60\,W \text{ tons-in.}$$

$$M_{max} = 8 \times \frac{I_{XX}}{y_t} \qquad y_t = 5 \text{ in.}$$

$$60W = \frac{8 \times 345}{5} = 8 \times 69$$

$$W = \frac{8 \times 69}{60} = 9{\cdot}2 \text{ tons.}$$

Problem 7. A compound beam (Fig. 34A), formed by riveting together two rolled steel joists 16 in. deep, has a span of 25 ft. and carries a uniformly-distributed load of 35 tons.

$$I_{XX} = 726 \text{ in.}^4 \qquad I_{YY} = 27 \text{ in.}^4$$

Take the beam to be so loaded that the flanges are stressed as in Fig. 34A. The joists are illustrated in Fig. 34.

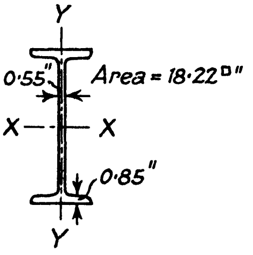

FIG. 34

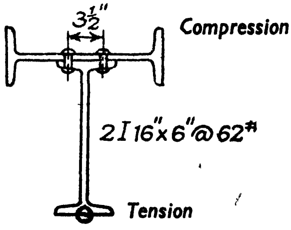

FIG 34A

(1) Find the position of the N.A. above the tension flange of the compound beam. Fig. 34A.

Take moments about ●.

$$2 \times 18{\cdot}22 \times \bar{y} = 18{\cdot}22 \times 8 + 18{\cdot}22 \times \left(16 + \frac{{\cdot}55}{2}\right)$$

From which $\bar{y} = 12{\cdot}1$ in.

I_{NA} = Moment of inertia about the axis through the centroid and parallel to the tension flange.

Neglecting the effect of the rivets and rivet holes, for the compound beam

$$\begin{aligned} I_{NA} &= 726 + 18{\cdot}22 \times (4{\cdot}1)^2 + 27 + 18{\cdot}22 \times (4{\cdot}18)^2 \\ &= 1380 \text{ in.}^4 \end{aligned}$$

The compression boundary is $4{\cdot}18 + 3 \overset{/1\cdot}{=} 7{\cdot}18$ in. from the N.A.
,, tension ,, = 12·1 in. ,,

(1) Neglecting the weight of the beam, the moment at the centre, which is a maximum, is

$$\frac{wl^2}{8} = \frac{Wl}{8} \text{ where } W \text{ is the total load in tons}$$

$$\text{Maximum moment} = \frac{35 \times 25 \times 12}{8} \text{ tons-in.}$$

Let f_c = maximum compressive stress in tons/sq. in.
f_t = ,, tensile ,, ,,

$$\text{Then } \frac{35 \times 25 \times 12}{8} = f_c \times \frac{1380}{7{\cdot}18}$$

$$f_c = 6{\cdot}8 \text{ tons/sq. in.}$$

$$\text{Also } \frac{35 \times 25 \times 12}{8} = f_t \times \frac{1380}{12{\cdot}1}$$

$$f_t = 11{\cdot}5 \text{ tons/sq. in.}$$

(2) Allowing for the weight of the beam, to find the additional stresses due to this weight.

$$\begin{aligned} \text{Total weight of beam} &= 2 \times 62 \times 25 \text{ lb.} \\ &= 3100 \text{ lb.} = 1{\cdot}38 \text{ tons} \end{aligned}$$

Now 35 tons distributed load cause stresses of 6·8(f_c) and 11·5(f_t)
∴ 1·38 tons distributed load cause stresses of

$$\frac{6{\cdot}8 \times 1{\cdot}38}{35} \text{ (compression) and } \frac{11{\cdot}5 \times 1{\cdot}38}{35} \text{ (tension)}$$

$$= \underset{\text{compression}}{0{\cdot}27 \text{ tons/sq. in.}} \text{ and } \underset{\text{tension}}{0{\cdot}45 \text{ tons/sq. in.}}$$

additional stresses which are small.

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See also References, Chapter I.

EXAMPLES

1. Three wooden planks, *a*, *b*, and *c*, of the same material are laid side by side across a span of 7 ft., and a load of $\frac{1}{4}$ ton is laid across them at the centre of the span, so that they all bend to the same radius of curvature. Each plank is 6 in. wide, the depth of the two planks *a* and *c* is 3 in., and of *b* (the central plank) 6 in. Determine—

(*a*) The load carried by each plank.

(*b*) The maximum intensity of stress in each plank.

2. A bar of steel, originally straight, is bent to a radius of 600 in.; the bar is 2 in. wide and 1 in. deep in the plane of bending. Find the bending moment and the greatest intensity of stress induced in the bar. Prove the formula employed. ($E = 12{,}500$ tons per square inch.) (I.C.E.)

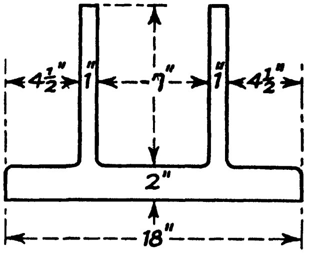

FIG. 35

3. In a supported *T* beam of span 20 ft. and 10 in. deep, the sectional areas of the web and flange are equal. The beam has to carry a uniformly-distributed load of 4 tons, and the allowed working stresses are 3 and 5 tons per square inch in compression and tension respectively. What is the sectional area of the beam? (I.C.E.)

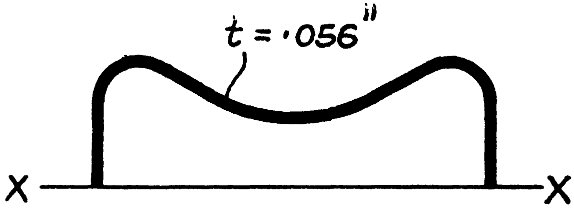
t = ·056"
X
X

FIG. 36

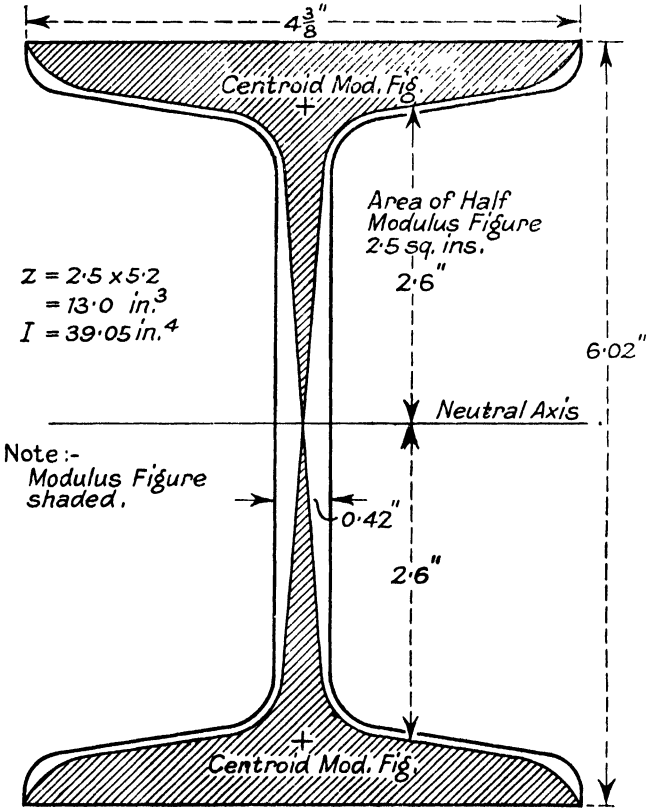
4⅜"
Centroid Mod. Fig.
Area of Half Modulus Figure 2·5 sq. ins.
2·6"
z = 2·5 x 5·2
= 13·0 in.³
I = 39·05 in.⁴
6·02"
Neutral Axis
Note :- Modulus Figure shaded.
0·42"
2·6"
Centroid Mod. Fig.

FIG. 37

4. A cast-iron lintel beam spans a window opening 14 ft. clear in width and carries a load from the brickwork above, assumed to be that included in an equilateral triangle of which the lintel forms the base. The brickwork is 18 in. thick. If the cross-section of the lintel is as shown in the sketch (Fig. 35, page 37), what is (*a*) the maximum compressive stress; (*b*) the maximum tensile stress, produced in the cast-iron? The weight of 1 cubic ft. of brickwork = 100 lb.

5. A steel angle 4″ × 4″ × ½″ is 10 ft. 6 in. long, and built vertically into a concrete foundation. The upper end is quite free except for a wire rope inclined at 45° to the floor. This rope passes through a hole (P) in one side of the angle 2 in. from the corner and 10 ft. from the floor. The rope is in the same vertical plane as the face of the angle through which it passes, and carries a load of 200 lb. The moment of inertia about the neutral axis of the cross-section is 1·93 in.^{4} The neutral axis is distant x = 1·17 in. from the corner. Find approximately the maximum stress produced in the angle. (U. of B.)

6. To what radius may a wooden beam 12 in. deep be bent if the skin stress may not exceed 1000 lb. per square inch, assuming $E = 2 \times 10^6$ lb. per square inch? If the beam is of rectangular shape, 6 in. wide, what is then its amount of resistance?

7. A rolled steel joist, 12 in. deep, has a span of 20 ft. and carries a load of 10 tons uniformly distributed, and a concentrated load of 3 tons at the centre of the span. The flanges are 7 in. wide and ·875 in. thick, and the web is 0·5 in. thick. Determine the maximum stress due to bending. (L.U., July 1923.)

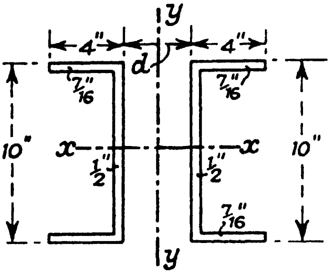

FIG. 38

8. An I section steel joist, 9″ × 4″, M. of I., 81 in. units, has a span of 12 ft. Find the maximum safe distributed load per foot run it will carry with a working stress of 8 tons per square inch.

9. Find by a graphical method the "modulus of the section" of a joist 10″ × 5″ × ½″. Construct a scale for the modulus figure. (U. of B.)

10. A rectangular beam of wood 10 in. deep and 6 in. wide, and having a length of 16 ft., is supported on two supports 10 ft. apart, one support being at the left end of the beam. The beam is loaded with a load of ¼ ton per foot run. Draw the diagram of bending moment, neglecting the weight of the beam, and find the maximum stress per square inch in the beam.

11. Point out the meaning of the moment of inertia of a section of a loaded beam. Find the moment of inertia of an area consisting of a rectangle 10 in. wide and 12 in. deep from which has been taken a smaller concentric rectangle 9 in. wide and 11 in. deep about an axis 6 in. below the top and parallel to it. (I.C.E., April 1923.

12. At what distance d should the two channels be apart in Fig. 38 so that $I_{XX} = I_{YY}$. xx and yy are the 2 axes of symmetry.

13. Find the moment of inertia of the section about the axis xx and also about an axis through the centre of gravity parallel to the axis xx. Fig. 36, page 38.

Method. Divide the length of the section into small lengths of, say, ·1 in. Then the area of the whole section $= n \times \cdot 1 \times t$, where n = number of small lengths. Let y = arm of each small length from the axis xx, then neglecting the moment of inertia of each small length about an axis through its own centroid,

$$I_{XX} = \Sigma a y^2$$

$$\text{which, if } a \text{ is constant} = a\Sigma y^2$$

$$\text{where } a = (\cdot 1 \times t) \text{ sq. in.}$$

14. A piece of steel has to be bent round a drum 5 ft. in diameter. Determine the maximum thickness the steel may have, if the stress is not to exceed the limit of proportionality of the steel which is 14 tons per square inch.

E = 12,500 tons/sq. in. (U. of L.)

15. Prove the formulae for $\bar{y}$, I_{XX}, and I, given in para. 22, p. 31.

CHAPTER III

DEFLECTION OF SIMPLE BEAMS

THE determination of deflections will be ascertained from the differential equation of the deflection curve and also by the use of the bending moment diagram.

24. In the chapter on "Bending," the relations between moment, stress developed, and curvature were found.

$$\frac{f}{y}=\frac{M}{I}=\frac{E^{*}}{R}$$

For very flat curves and for which the radii of curvature are very big,

$$\frac{1}{R}=\frac{d^2y}{dx^2}$$

$$\text{that is, }\frac{d^2y}{dx^2}=\frac{M}{EI}$$

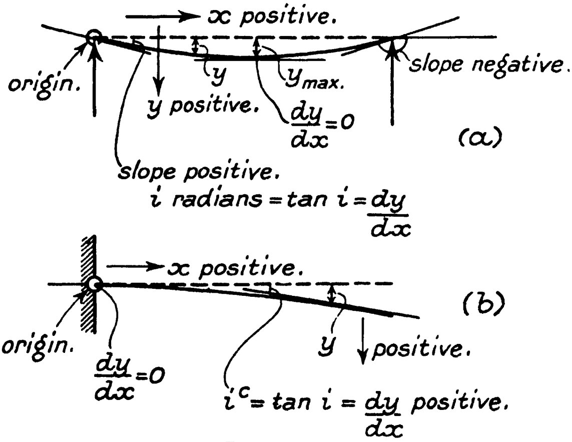

FIG. 39

In Fig. 39, (*a*) is a simply supported beam: under any system of loading the beam will bend concave upwards; that is, the displacements from the horizontal position will be in a downwards direction. The amount of displacement at any point

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[*] This is the expression derived from the case of so called "pure bending." The same equation may be used for the bending of prismatical bars by transverse loads, if the effect of the shearing forces be neglected.

will be the deflection of the beam at that point. Let this displacement be positive when measured in a downwards direction and be equal to $+ y$.

When writing down the fundamental equation of any simple and fixed beam, always start from the left of the beam as origin, and give the moments their correct sign according to the way they tend to bend the beam about the left of a section considered, and which were indicated in Chapter I. These signs will give, on solution, the correct sign for the displacement y.

In the case of the cantilever, the fixed point is taken as origin, Fig. 39, (*b*). For overhanging and continuous beams the left-hand support is considered as the origin.

For symmetrical loadings, deflections can be most elegantly found by mathematical methods : for systems of irregular loadings, it had been found necessary to use graphical methods, but a mathematical form developed by W. H. Macaulay* can be used in all cases and is much simpler. It can be used for all kinds of beams having EI constant.

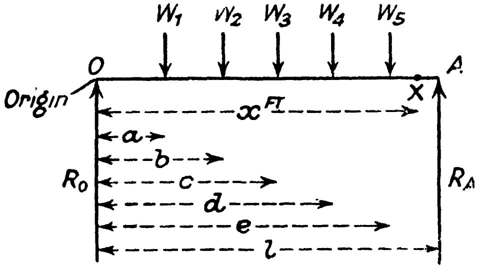

FIG. 40

25. The slope of the tangent at any point $= \dfrac{dy}{dx} = \tan i$ $= i$ radians, where $i =$ angle in radians which the tangent makes with the x axis, and it is always very small, so that $\tan i$ is nearly equal to i.

THE DIFFERENTIAL EQUATION OF THE DEFLECTION CURVE

26. **General Mathematical Method** (Fig. 40). (Due to Mr. W. H. Macaulay.*) OA is a simple beam loaded in an irregular

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* *Messenger of Mathematics*, No. 573, xlviii, Jan. 1919. Also refer to "The Elastic Equation for Beams" by W. D. Womersley, *Concrete and Constructional Engineering*, Vol. XX, 1925.

manner and of length l. Take the origin at O (the left-hand support), and the axis of x to the right and positive. Take any section X distant x from O and beyond the last applied load. R_o and R_A are calculated by the usual methods.

Moment at X

$$= M_x = -R_o x + W_1(x-a) + W_2(x-b) + \ldots W_n(x-n)$$

This is an expression for the moment for any section X, if the terms inside the brackets are omitted for values of x, which make them negative.

$$EI\frac{d^2y}{dx^2} = M_x = -R_o x + W_1(x \quad a) + W_2(x-b) + \ldots W_n(x-n)$$

Integrate twice (taking EI constant),

$$EIy = -R_o\frac{x^3}{6} + \frac{W_1}{6}(x-a)^3 + \frac{W_2}{6}(x-b)^3 + \ldots \frac{W_n}{6}(x-n)^3 + Ax + B \ldots \text{(eqn. } C\text{)}$$

This expression is true for all values of x between 0 and l, omitting terms which become negative for particular values of x.

For simple beams, when $x = 0, y = 0$;

Then $B = 0$

To find A, put $x = l$, then $y = 0$ (generally); Then

$$0 = -\frac{R_o}{6}l^3 + \frac{W_1}{6}(l-a)^3 + \frac{W_2}{6}(l-b)^3 + \ldots \frac{W_n}{6}(l-n)^3 + Al$$

From this latter equation, A may be found: substitute it in equation C to give the general value of y.

The slope at any section will be found by differentiating y with respect to x when the value of A is known, and the resulting equation will hold good for any value of x, omitting terms in the brackets which become negative for particular values of x.

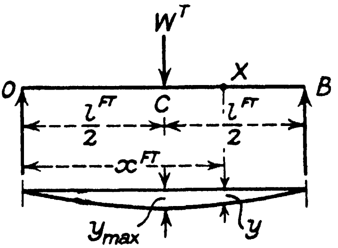

FIG. 41

EXAMPLES.

(*a*) *Simple Beams.* Load at the centre (Fig. 41.) Supports same height.

$$R_o = R_B = \frac{W}{2}.$$

$$EI\frac{d^2y}{dx^2} = M_x = -\frac{Wx}{2} + W\left(x - \frac{l}{2}\right)$$

$$EIy = -\frac{W}{2 \times 6}x^3 + \frac{W}{6}\left(x - \frac{l}{2}\right)^3 + Ax + B$$

when $x = 0 \qquad y = 0 \;\therefore\; B = 0$

When $x = l$, $y = 0$,

$$\text{then } 0 = -\frac{W}{12}l^3 + \frac{W}{6}\left(\frac{l}{2}\right)^3 + Al$$

$$Al = \frac{3Wl^3}{48}$$

$$A = \frac{Wl^2}{16}$$

$$EIy = -\frac{Wx^3}{12} + \frac{W}{6}\left(x - \frac{l}{2}\right)^3 + \frac{Wl^2x}{16}$$

which holds good for sections between C and B.

Obviously for all sections between O and C, the middle term will be negative, as $x < \frac{l}{2}$.

Between O and C,

$$EIy = -\frac{Wx^3}{12} + \frac{Wl^2x}{16}$$

y is a maximum when $x = \frac{l}{2}$

i.e. the deflection is a maximum,

$$\text{when } EIy_{max} = -\frac{Wl^3}{96} + \frac{Wl^3}{32} = \frac{Wl^3}{48} \qquad . \quad . \quad . \quad (1)$$

Between C and B, considering slopes

$$EI\frac{dy}{dx} = EIi_x = -\frac{Wx^2}{4} + \frac{W}{2}\left(x - \frac{l}{2}\right)^2 + \frac{Wl^2}{16}$$

$$= \frac{W}{16}(4x^2 - 8xl + 3l^2)$$

Between O and C,

$$EI\frac{dy}{dx} = EIi_x = -\frac{Wx^2}{4} + \frac{Wl^2}{16}$$

$$\text{when } x = 0, \quad i_0 = \frac{Wl^2}{16EI} \qquad . \qquad . \quad (2)$$

$$\text{,,} \quad x = \frac{l}{2} \quad i_{\frac{l}{2}} = 0$$

(*b*) *Simple Beam, with Uniformly-distributed Load the Whole Length of the Beam.* (Fig. 42) Supports same height

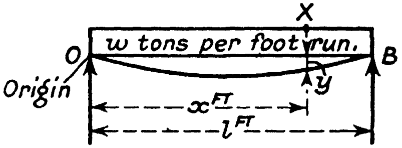

FIG. 42

$$R_0 = R_B = \frac{wl}{2}$$

$$M_x = -\frac{wl}{2}x + wx \times \frac{x}{2} = EI\frac{d^2y}{dx^2}$$

$$EIy = -\frac{wl}{12}x^3 + \frac{w}{24}x^4 + Ax + B$$

When $x = 0$, $y = 0$, $B = 0$,

and $x = l$, $y = 0$,

$$\text{then } 0 = -\frac{wl^4}{12} + \frac{wl^4}{24} + Al$$

$$A = \frac{wl^3}{24}$$

$$EIy = -\frac{wl}{12}x^3 + \frac{wx^4}{24} + \frac{wl^3}{24}x$$

The maximum value of y (giving the maximum deflection), when $x = \frac{l}{2}$.

$$EIy_{max} = -\frac{wl}{12} \times \frac{l^3}{8} + \frac{wl^4}{24 \times 16} + \frac{wl^3}{24} \times \frac{l}{2}$$

$$EIy_{max} = \frac{5wl^4}{384}$$

$$y_{max} = \frac{5wl^4}{384EI} = \frac{wl^4}{76{\cdot}8EI} = \frac{Wl^3}{76{\cdot}8EI} \quad . \quad (3)$$

To find the slopes,

$$EI\frac{dy}{dx} = EIi_x = -\frac{wlx^2}{4} + \frac{wx^3}{6} + \frac{wl^3}{24}$$

$$\text{when } x = 0, \quad i_0 = \frac{wl^3}{24EI} = \frac{A}{EI} \quad . \quad . \quad . \quad (4)$$

Ⓒ *Simple Beam Symmetrically Loaded with Equal Concentrated Loads.* **(Fig. 43.) Supports same height.**

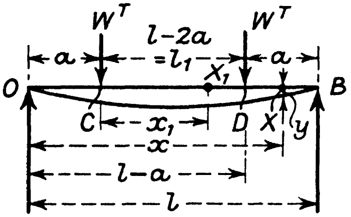

FIG. 43

OB is a beam of length l, having loads of W at distances a from O and B respectively.

$$R_0 = R_B = W$$

The moment at any section X_1 between C and D ($CX_1 = x_1$),

$$= -R_0(a + x_1) + Wx_1 ;$$

$$(R_0 = W)$$

$$= -R_0a = -Wa = \text{constant}$$

$$\frac{1}{R} = \frac{M}{EI} = \frac{-Wa}{EI} = \text{constant}$$

Therefore, between C and D, the beam bends to the arc of a circle.

Take a section X between D and B,

$$M_x = EI\frac{d^2y}{dx^2} = -Wx + W(x-a) + W(x-l_1-a)$$

$$EIy = -\frac{Wx^3}{6} + \frac{W}{6}(x-a)^3 + \frac{W}{6}(x-l_1-a)^3 + Ax + B$$

When $x = 0$, $y = 0$, then $B = 0$

When $x = l$, $y = 0$

$$\text{then } 0 = -\frac{Wl^3}{6} + \frac{W}{6}(l-a)^3 + \frac{W}{6}(l-l_1-a)^3 + Al$$

$$= -\frac{Wl^3}{6} + \frac{W}{6}(l-a)^3 + \frac{W}{6}a^3 + Al$$

$$A = \frac{Wal}{2} - \frac{Wa^2}{2} \checkmark$$

$$EIy = -\frac{Wx^3}{6} + \frac{W}{6}(x-a)^3 + \frac{W}{6}(x-l_1-a)^3 + \frac{Walx}{2} - \frac{Wa^2x}{2} \quad \ldots \quad (4a)$$

At the load points C and D, $x = a$ and $(l-a)$

so that $y_c = y_d$

$$EIy_c = -\frac{Wa^3}{6} + \frac{W}{6}(a-l_1-a)^3 + \frac{Wa^2l}{2} - \frac{Wa^3}{2}$$

$a - l_1 - a$ is neglected because it is negative.

$$EIy_c = \frac{Wa^2l}{2} - \frac{2}{3}Wa^3 = \frac{Wa^2l_1}{2} + \frac{Wa^3}{3} \quad . \quad . \quad (5)$$

The deflection is a maximum at the centre of the beam :

i.e. at $x = \frac{l}{2}$

$$EIy_{max} = -\frac{Wl^3}{48} + \frac{W}{6}\left(\frac{l}{2}-a\right)^3 + \frac{Wal^2}{4} - \frac{Wa^2l}{4} \checkmark$$

$$= \frac{Wal^2}{8} - \frac{Wa^3}{6} \checkmark \quad . \quad . \quad . \quad . \quad (6)$$

$$EI(y_{max} - y_c) = \frac{Wal^2}{8} - \frac{Wa^3}{6} - \frac{Wa^2l}{2} + \frac{2}{3}Wa^3$$

$$= Wa\left(\frac{l^2}{8} - \frac{al}{2} + \frac{a^2}{2}\right)$$

$$= \frac{Wa}{8}(l^2 - 4al + 4a^2)$$

$$= \frac{Wa}{8}(l-2a)^2 = \frac{Wal_1^2}{8}$$

The relative deflections at the centre of the beam and at the load points

$$= \frac{\text{constant moment } Wa}{8EI}\,(\text{distance between loads})^2.$$

Differentiating equation (4*a*),

$$EI\frac{dy}{dx} = EIi_x = -\frac{Wx^2}{2} + \frac{W}{2}(x-a)^2 + \frac{W}{2}(x-l_1-a)^2 + \frac{Wal}{2} - \frac{Wa^2}{2}$$

At the origin, $x = 0$

$$EIi_0 = \frac{Wal}{2} - \frac{Wa^2}{2} = A$$

At the centre of the beam, $i_{\frac{l}{2}} = 0$

At the load point C, $x = a$

$$EIi_c = -\frac{Wa^2}{2} + \frac{Wal}{2} - \frac{Wa^2}{2}$$

$$= \frac{Wa}{2}(l-2a) = \frac{Wa}{2}l_1$$

$$i_c = \frac{\text{constant moment} \times (\text{distance between loads})}{2EI} \quad . \quad (7)$$

27. **Note.** The relative deflection at the centre of the portion of a beam of length l_1 bending to the arc of a circle and a load point can be proved from the properties of a circle. (Fig. 44.)

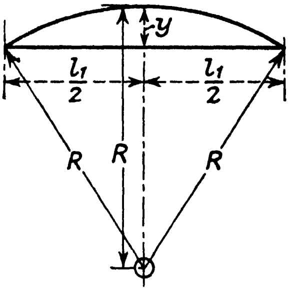

FIG. 44

$$R^2 = (R-y)^2 + \left(\frac{l_1}{2}\right)^2;$$

$$R^2 = R^2 - 2Ry + y^2 + \frac{l_1^2}{4}$$

$$y(2R-y) = \left(\frac{l_1}{2}\right)^2,$$

$$\text{or} \quad y \,.\, 2R = \left(\frac{l_1}{2}\right)^2$$

neglecting y^2 as being of the second degree of smallness.

$$y = \frac{l_1^2}{8R}$$

$$\frac{1}{R} = \frac{M}{EI}\ (\text{constant}) : \text{ then } y = \frac{Ml_1^2}{8EI} \quad . \quad (8)$$

where M is a constant.

28. Overhanging Beams. Equal concentrated loads at the ends of equal overhangs. (Fig. 45.) Supports O, B, at the same level.

$$R_o = R_B = W$$

This case is exactly the reverse of the last example considered (Art. 26c). The beam between O and B will bend to the arc of a circle.

Let OB be the base line, and let $l = l_1 + 2a$.

At the centre of the beam, the deflection above OB will be

$$y_c = -\frac{Wal_1^2}{8EI}$$

being negative because the deflection is upwards.

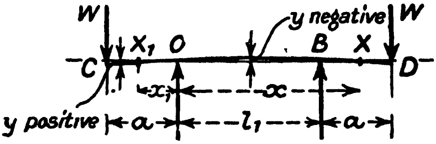

FIG. 45

The loaded ends will deflect below the base line by amounts y_c and y_d (and $y_c = y_d$).

$$\text{where } EIy_d = \frac{Wa^2l}{2} - \frac{2Wa^3}{3} \qquad \text{(eqn. 5)}$$

$$= \frac{Wa^2l_1}{2} + \frac{Wa^3}{3} \qquad \text{(eqn. 5)}$$

The slope of the beam at the support O

$$i_o = -\frac{Wal_1}{2EI}$$

These results will be obtained by taking O as the origin and obtaining an expression for the moment at a section X distance x from O and between B and D. The deflection of C is the same as for D; for any section between B and D

$$M_x = W(a + x) - Wx - W(x - l_1)$$

$$EIy = \frac{Wax^2}{2} - \frac{W}{6}(x - l_1)^3 - \frac{Wal_1x}{2}$$

Illustrative Problem 8. A beam of length 20 ft. is loaded as in Fig. 46.

Find—

(*a*) the maximum deflection and the section where it occurs ;
(*b*) the deflection and slope at the load point ;
(*c*) the slopes at the ends of the beam.

$$\text{Take } E = 13{,}000 \text{ tons/sq. in.}$$

$$I = 300 \text{ (in.)}^4$$

$$R_o = 3 \text{ tons} \qquad R_B = 9 \text{ tons}$$

At any section X between C (the load point) and B,

$$M_x = -3x + 12(x-15)$$

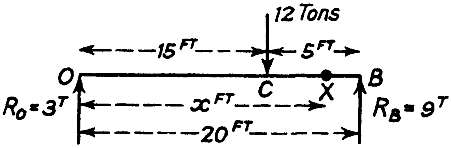

FIG. 46

$$EIy = -3\frac{x^3}{6} + \frac{12}{6}(x-15)^3 + Ax + B$$

when $x = 0$, $\quad y = 0$, $\quad B = 0$

„ $\quad x = l = 20$, $\quad y = 0$

$$\text{then } 0 = -\frac{20^3}{2} + 2 \times 5^3 + 20A$$

Solving : $A = +187{\cdot}5$

$$EIy = -\frac{x^3}{2} + 2(x-15)^3 + 187{\cdot}5x.$$

The maximum deflection will occur at some section between O and C ; the equation for any section between O and C is

$$EIy = -\frac{x^3}{2} + 187{\cdot}5x.$$

In this case, x always $<$ or $=$ 15, and $(x-15)$ therefore always negative or zero, and is eliminated.

For maximum deflection.*

$$\therefore\ EI\frac{dy}{dx} = -\frac{3x^2}{2} + 187{\cdot}5 = 0$$

$$x^2 = 125$$

$$\therefore\ x = 11{\cdot}2 \text{ ft.}$$

The maximum deflection is at a section 11·2 ft. from O.

$$13{,}000 \times 300 \times y = -\frac{11{\cdot}2^3}{2} + 187{\cdot}5 \times 11{\cdot}2$$

E and I are in inches and x in feet; the right-hand side of the equation must be multiplied by 12^3 to make the units correct on both sides. $(x \text{ ft.})^3 = (12x)^3$ in.; y will then be in inches.

$$39 \times 10^5 y = 1400 \times 1728$$

$$y = {\cdot}62 \text{ in.}$$

The deflection under the load point is

$$EIy_c = -\frac{15^3}{2} + 187{\cdot}5 \times 15$$

Solving $y_c = {\cdot}5$ in. nearly.

SLOPES.

$$EI\frac{dy}{dx} = EIi_x$$

$$EIi_x = -\frac{3x^2}{2} + 6(x-15)^2 + 187{\cdot}5 \text{ for sections between } C \text{ and } B$$

$$EIi_x = -\frac{3x^2}{2} + 187{\cdot}5 \text{ for sections between } O \text{ and } C$$

As E and I contain inch units, and x is in feet, multiply the right-hand sides of the equations by 12^2 to make the units agree.

When $x = 0$

$$i_o = \frac{187{\cdot}5 \times 144}{39 \times 10^5} = +{\cdot}00695$$

at $x = 15$, i.e. at C,

$$i_c = \frac{\left(-3 \times \frac{15^2}{2} + 187{\cdot}5\right)}{39 \times 10^5} \times 144 = -{\cdot}0056$$

---

* For simple and fixed beams loaded unsymmetrically with one load, the maximum deflection occurs in the longer portion of the beam between the load and a reaction.

At $x = 20$, the right-hand support,

$$i_{20} = \frac{\left(-3 \times \frac{20^2}{2} + 6 \times 5^2 + 187{\cdot}5\right)}{39 \times 10^5} 144 = -{\cdot}0098$$

29. The general mathematical method can be made to apply to any irregular system of loading of cantilevers. (Fig. 47.)

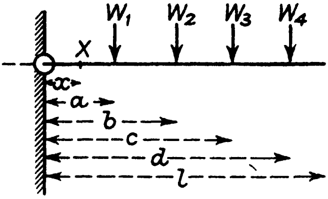

FIG. 47

Take any section X between the origin and the first load, and distant x from O.

Let the loads be W_1, W_2, W_3, etc., at distances a, b, c, d, etc., from O.

The moment at X, taking moments to the right of the section, will be positive; EI = constant.

$$\text{then } EI\frac{d^2y}{dx^2} = W_1(a-x) + W_2(b-x) + W_3(c-x) + \ldots W_n(n-x)$$

n may be $=$ or $< l$

$$EI\frac{dy}{dx} = -\frac{W_1}{2}(a-x)^2 - \frac{W_2}{2}(b-x)^2 \ldots + A$$

$$\text{when } x = 0, \quad \frac{dy}{dx} = 0$$

$$\text{Then } A = +\frac{W_1}{2}a^2 + \frac{W_2}{2}b^2 + \frac{W_3}{2}c^2 + \ldots \frac{W_n}{2}.n^2$$

$$EIy = \frac{W_1}{6}(a-x)^3 + \frac{W_2}{6}(b-x)^3 + \ldots + \frac{W_1a^2x}{2} + \frac{W_2b^2x}{2} \ldots + \frac{W_nn^2x}{2} + B$$

$$\text{when } x = 0, \quad y = 0$$

$$\text{Then } B = -\frac{W_1a^3}{6} - \frac{W_2b^3}{6} \ldots - \frac{W_nn^3}{6}$$

$$\text{and } EIy = \frac{W_1(a-x)^3}{6} + \frac{W_2(b-x)^3}{6} + \dots + \frac{W_n(n-x)^3}{6}$$

$$+ \frac{W_1a^2x}{2} + \frac{W_2b^2x}{2} + \dots + \frac{W_nn^2x}{2}$$

$$- \frac{W_1a^3}{6} - \frac{W_2b^3}{6} - \dots - \frac{W_nn^3}{6} \quad . \quad . \quad . \quad (9)$$

To find the deflection at any section X, values of $(a-x)$, $(b-x)$, etc., which become of negative value, are eliminated.

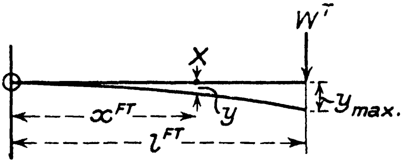

FIG. 48

If $x = l$, all terms $(a-x)$, etc., disappear, even when $n < l$.

With one load W at the end of the beam, $x = n = l$ and $W_n = W$. (Fig. 48.)

$$\left.\begin{array}{l} \text{Then } \dfrac{Wn^2x}{2} = \dfrac{Wl^3}{2} \\[2ex] \text{and } -\dfrac{Wn^3}{6} = -\dfrac{Wl^3}{6} \end{array}\right\rangle \text{Sum} = \frac{Wl^3}{3}$$

Deflection at the end of the cantilever due to W only $= \dfrac{Wl^3}{3EI}$

30. In the general method, if there is a distributed load on a part of the beam, imagine it extended to the section X taken. The part added is neutralized by a load acting upwards on the extended portion.

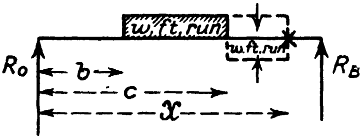

FIG. 49

E.g. (in Fig. 49)

$$M_x = -R_ox + \frac{w}{2}(x-b)^2 - \frac{w}{2}(x-c)^2$$

31. EXAMPLES.*

(*a*) *Cantilever Beams.* A concentrated load of W tons at the end. (Fig. 48.) For any section X, the moment is clockwise to the right of the section and positive —(EI constant)

$$\text{then } \frac{d^2y}{dx^2} = \frac{W(l-x)}{EI}$$

$$\frac{dy}{dx} = i_x = \frac{W}{EI}\int (l-x)dx + C$$

$$= \frac{W}{EI}\left(lx - \frac{x^2}{2}\right) + C \quad . \quad . \quad . \quad (9a)$$

C is a constant of integration.

$$\text{when } x = 0, \quad \frac{dy}{dx} = 0, \quad C = 0$$

$$\text{At } x = l, \quad i_l \text{ is a maximum} = \frac{Wl^2}{2EI} \quad . \quad . \quad . \quad (10)$$

$$\text{Integrating (9a), } y_x = \frac{W}{EI}\int\left(lx - \frac{x^2}{2}\right)dx + C_1$$

$$= \frac{W}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_1 \quad . \quad . \quad . \quad (10a)$$

$$\text{when } x = 0, \quad y = 0, \quad C_1 = 0$$

$$x = l, \quad y = y_{max}$$

$$y_{max} = \frac{W}{EI}\left(\frac{l^3}{2} - \frac{l^3}{6}\right) = \frac{Wl^3}{3EI} \text{ units} \quad . \quad . \quad . \quad (11)$$

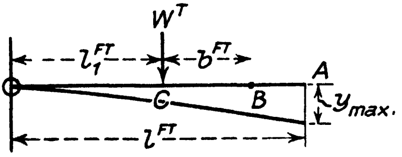

FIG. 50

The slopes and deflections at other sections are found by substituting the necessary values of x in equations $9a$ and $10a$ where C and C_1 are both zero.

(*b*) *Cantilever of Length l, with a Concentrated Load of W tons a Distance l_1 from the Origin* (Fig. 50.)

At the point of loading C,

$$i_c = \frac{Wl_1^2}{2EI} \quad . \quad (12) \qquad y_c = \frac{Wl_1^3}{3EI} \quad . \quad (13)$$

Between C and the end of the beam A, there is no moment, so that the slope of the beam will be a constant $= i_c$.

---

[*] The examples (*a*) and (*b*) for cantilevers are worked out by straightforward integration. The student is asked to work out the problems dealing with irregular concentrated loadings by the method indicated in paragraph 29.

The deflection at any section B, between C and A, and distant b from C, is

$$y_B = \frac{Wl_1^3}{3EI} + b\,(\text{slope from } C \text{ to } B)$$

$$= \frac{Wl_1^3}{3EI} + \frac{bWl_1^2}{2EI} \quad . \quad . \quad . \quad . \quad (14)$$

The deflection at the end of the beam

$$= \frac{Wl_1^3}{3EI} + \frac{W(l-l_1)l_1^2}{2EI}$$

$$= \frac{Wl_1^2}{6EI}(3l - l_1) \quad . \quad (15)$$

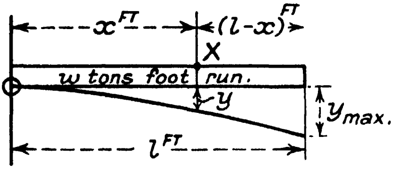

FIG. 51

(c) *Cantilever* with a uniformly-distributed load of w tons per foot run. (Fig. 51.) EI constant.

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = + \frac{w(l-x)^2}{2EI}$$

$$\therefore \frac{dy}{dx} = i_x = \frac{w}{2EI}\int (l-x)^2 dx + C$$

$$= -\frac{w}{2EI}\frac{(l-x)^3}{3} + C$$

$$x = 0, \quad i_0 = 0, \quad C = \frac{wl^3}{6EI} \quad . \quad . \quad . \quad (16)$$

$$x = l, \quad i_l = \text{maximum} = \frac{wl^3}{6EI} \quad . \quad . \quad . \quad (17)$$

$$\therefore \frac{dy}{dx} = i_x = -\frac{w(l-x)^3}{6EI} + \frac{wl^3}{6EI}$$

$$\text{Then } y_x = \frac{w}{6EI}\int [-(l-x)^3 + l^3]\,dx + C_1$$

$$= \frac{w}{6EI}\left[\frac{(l-x)^4}{4} + l^3x\right] + C_1$$

$$x = 0, \quad y_0 = 0, \quad C_1 = -\frac{wl^4}{24EI} \quad . \quad . \quad (18)$$

$$x = l, \quad y_l = \text{maximum}$$

$$y_{max} = \frac{wl^4}{6EI} - \frac{wl^4}{24EI}$$

$$= \frac{wl^4}{8EI} = \frac{Wl^3}{8EI} \quad . \quad . \quad . \quad (19)$$

The slope and deflection at any section are found by giving x the necessary values in the equations, for i_x and y_x, giving C and C_1 their values.

32. **Relations between Slope, Deflection, and Moment;** see also Art. 12 *ante.*

Take EI constant

$$EI\frac{d^2y}{dx^2} = M$$

$$EI\frac{dy}{dx} = EIi_x = \int M \cdot dx; \ \frac{dy}{dx} = i_x = \int \frac{M}{EI}dx.$$

Let $i_{x=x_2} = i_{x_2}$ be the slope at a section X_2 distance x_2 from the origin

„ $i_{x=x_1} = i_{x_1}$ at a section X_1, distance x_1 from the origin.

$$\text{Then } EI(i_{x_2} - i_{x_1}) = \int_{x=x_1}^{x=x_2} M \cdot dx = \Big[A_{BM}\Big]_{x_1}^{x_2} \quad . \quad . \quad (20)$$

= area of the bending moment diagram between the sections x_1 to x_2.

The angle between the two tangents at X_1 and X_2 equals the area of the bending moment diagram between the corresponding verticals divided by EI.

Let $x_1 = 0$, then $i_{x_1} = i_o =$ slope at the origin

$$\text{Then } i_x = i_o + \frac{1}{EI}\int_0^{x_2} M \cdot dx \quad . \quad . \quad (21)$$

Generally for cantilevers and fixed beams, $i_o = 0$, (but regard must be paid to the moment sign) if the fixed end is at the origin.

$$\text{Further } y_x = \int i_x \cdot dx$$

$$\text{Then } (y_{x=x_2} - y_{x=x_1}) = \int_{x=x_1}^{x=x_2} i_x \cdot dx \quad . \quad . \quad . \quad (22)$$

= area of the slope diagram between x_2 and x_1

$$\text{If } x_1 = 0: \quad y_{x=x_1} = 0$$

$$\text{Then } y_{x_2} = \int_0^{x_2} i_x \,.\, dx = \begin{matrix}\text{area of the slope diagram}\\ \text{between } X_2 \text{ and the origin}\end{matrix} \quad . \quad (23)$$

SUMMARY. $S = \int w \,.\, dx$; $M = \int\int w \,.\, dx \,.\, dx$;

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\int\int w \,.\, dx \,.\, dx; \; M = \int S dx.$$

Total change of slope ϕ

$$= \frac{dy}{dx} = \frac{1}{EI}\int M \,.\, dx = \frac{1}{EI}\int\int S \,.\, dx \,.\, dx$$

$$= \frac{1}{EI}\int\int\int w \,.\, dx \,.\, dx \,.\, dx$$

$$\therefore \; y = \int \phi \,.\, dx = \frac{1}{EI}\int\int M \,.\, dx \,.\, dx$$

$$= \frac{1}{EI}\int\int\int S \,.\, dx \,.\, dx \,.\, dx$$

$$= \frac{1}{EI}\int\int\int\int w \,.\, dx \,.\, dx \,.\, dx \,.\, dx.$$

33. Relations between Load, Shear, Moment, Slope, and Deflection.

$$M = EI\frac{d^2y}{dx^2}$$

$$\text{Shear} = S = \frac{dM}{dx} = EI\frac{d^3y}{dx^3}$$

$$\text{Rate of loading} = w = \frac{dS}{dx} = \frac{d^2M}{dx^2} = EI\frac{d^4y}{dx^4}$$

$$\left.\begin{aligned}\text{Slope} = i = \frac{dy}{dx} &= \int\frac{M\cdot dx}{EI}\\ \text{Deflection} = y &= \int i\cdot dx\end{aligned}\right\} \text{where } EI \text{ may be constant or variable.}$$

34. Now, $\frac{d^2y}{dx^2} = \frac{M}{EI}$

Then $x\,.\,\frac{d^2y}{dx^2} = \frac{Mx}{EI}$

*Integrate between $x = x_2$ and $x = x_1$

$$\left(x\,.\,\frac{dy}{dx} - y\right)_{x_1}^{x_2} = \frac{1}{EI}\int_{x_1}^{x_2} Mx\,.\,dx \text{ when } EI = \text{constant}$$

$$\text{Then } x_2 i_{x_2} - y_{x_2} - x_1 i_{x_1} + y_{x1} = \frac{1}{EI}\int_{x_1}^{x_2} Mx\,.\,dx \qquad . \quad (24)$$

35. The Interpretation of Equations (20) and (24).

Equation (20). The angle between the two tangents at the points X_1 and X_2 distant x_1 and x_2 from any origin of the deflection curve equals the area of the bending moment diagram between the verticals through X_1 and X_2 divided by EI.—*Rule* (1).

Equation (24). Imagine a vertical taken through the origin of the beam, and again consider the two points X_1 and X_2 distant x_1 and x_2 from the origin. The left-hand side of Equation (24) represents the distance between the point of intersection of the tangent to the deflected beam at X_1 with the vertical through the origin, and the point of intersection of the tangent to the deflected beam at X_2 and the vertical through the origin. This is equal to the moment about the origin of the bending moment diagram between the verticals through X_1 and X_2 divided by EI, which is the right-hand side of equation (24).

---

* Integrate by "parts method."

If the origin is supposed to be moved to the point X_1, then the displacement of X_1 from the tangent at X_2 is equal to the moment with respect to the vertical through X_1 of the area of the bending moment diagram between the verticals through X_1 and X_2, divided by EI (for beams of uniform section).—*Rule* (2).

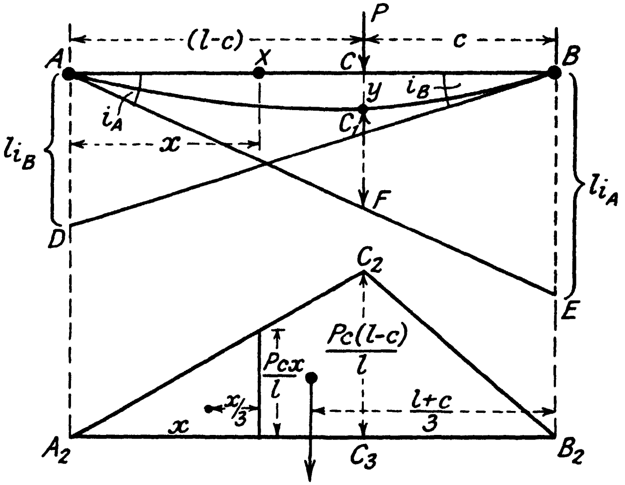

FIG. 52. THE LOWER DIAGRAM IS THE MOMENT DIAGRAM

Further: If the origin is transferred to X_2, then X_2 is displaced from the tangent at X_1 by an amount equal to the moment with respect to the vertical through X_2 of the area of the bending moment diagram between the verticals through X_1 and X_2, divided by EI.

EXAMPLES.

(*a*) *Deflection of a Beam Supported at the Ends.*

Let a beam AB of uniform section (EI is a constant) and length l support a load P at the point C distant $(l-c)$ from the support A. (Fig. 52.) Find (1) the slopes at the ends of the beam; (2) the deflection and slope at the load point; and (3) the maximum deflection, and the section where it occurs.

Let i_A and i_B be the slopes at A and B. From Fig. 52 it is seen that the distance between A and the tangent at B is equal to li_B, and that the distance between B and the tangent at A is equal to li_A.

From Rule (2), li_B is equal to the moment about A, of the moment diagram between the verticals through A and B, divided by EI. This is equal to the area of the moment diagram $A_2C_2B_2$ about A_2. The centroid is $\frac{l+c}{3}$ from B_2 and therefore $\frac{2l-c}{3}$ from A_2

$$\therefore \qquad li_B = \frac{1}{EI}\left[\frac{Pc(l-c)}{l} \times \frac{l}{2} \times \frac{(2l-c)}{3}\right]$$

$$\therefore \qquad i_B = \frac{Pc}{6EIl}[(2l-c)(l-c)]$$

and i_A can be shown $= \frac{Pc(l^2-c^2)}{6lEI}$.

If i_A is positive, then i_B is negative.

To find the deflection and slope at the point C_1 of the bent beam corresponding to the load point C.

Deflection at C_1. The deflection is obviously equal to CC_1.

$CC_1 = CF - C_1F$

$= (l-c)i_A -$ distance of C_1 from the tangent at A

$= (l-c)i_A -$ Moment of area $A_2C_2C_3$ of the Moment Diagram about C_2C_3 divided by EI

$$= \frac{Pc(l-c)(l^2-c^2)}{6lEI} - \frac{Pc(l-c)}{l} \times \frac{(l-c)}{2} \times \frac{(l-c)}{3EI}$$

$$= \frac{Pc(l-c)}{6lEI}[l^2 - c^2 - l^2 + 2lc - c^2]$$

$$= \frac{Pc(l-c)}{6lEI}(2lc - 2c^2)$$

$= \frac{Pc^2(l-c)^2}{3lEI} =$ deflection below the origin A of the load point C.

If $c = \frac{l}{2}$ then $y_{l/2} = \frac{Pl^3}{48EI}$.

The Slope at C_1.

By rule (1), the angle between the tangents at A and C_1 is equal to the area of the Moment Diagram between the verticals through A and C_1 divided by EI

$$= \frac{Pc(l-c)^2}{2lEI}.$$

Then the slope of the tangent at C from equation (20) is therefore

$$i_c = \frac{Pc(l^2-c^2)}{6lEI} - \frac{Pc(l-c)^2}{2lEI} \quad . \quad . \quad . \quad . \qquad \text{(X)}$$

$$= \frac{Pc}{6lEI}(l^2 - c^2 - 3l^2 + 6lc - 3c^2)$$

$$= + \frac{Pc}{3lEI}(-2c^2 + 3lc - l^2)$$

If $c = \frac{l}{2}$ then $i_c = 0$.

For all values of c, i_A is positive: i_c is positive or negative depending upon the sign of $-2c^2 + 3lc - l^2$, thus

If $3lc > l^2 + 2c^2$ then i_c is positive
If $3lc < l^2 + 2c^2$ then i_c is negative
If $c = \frac{l}{2}$ then $3lc = l^2 + 2c^2$.

$\therefore$ if $c > \frac{l}{2}$ then $3lc > l^2 + 2c^2$ and i_c is positive

and if $c < \frac{l}{2}$ then $3lc < l^2 + 2c^2$ and i_c is negative

$$\text{i.e. } c < \frac{l}{3} + \frac{2}{3}\frac{c}{l}$$

and if $c = \frac{l}{3}$ then $\left(-\frac{2l^2}{9} + l^2 - l^2\right)$ is negative.

Thus for all values of $c < \frac{l}{2}$, i_c is negative.

Therefore the maximum displacement of the beam will occur in that portion of the beam $l - c$ or c which is $> \frac{l}{2}$: because it is in the longer portion that the sign of the tangent changes from positive to negative and therefore at some point within this length it is zero.

The Position, and the Value, of the Maximum Deflection.

Let the maximum deflection be at section X distant x from the origin A and between A and C, where $AC = l - c$ is $> l/2$.

$$M_x = \text{Moment at } X = \frac{Pcx}{l}.$$

$$\therefore \quad y_x = xi_A - \frac{Pcx}{l} \times \frac{x}{2} \times \frac{x}{3} \times \frac{1}{EI}$$

$$= \frac{Pc(l^2 - c^2)x}{6lEI} - \frac{Pc}{6lEI} \cdot x^3$$

$$\frac{dy_x}{dx} = \frac{Pc(l^2 - c^2)}{6lEI} - \frac{Pc}{6lEI} \cdot 3x^2$$

$$= 0 \text{ for Max. } y_x$$

$$\therefore \quad 3x^2 = l^2 - c^2$$

$$\therefore \quad x = \sqrt{\frac{l^2 - c^2}{3}} \quad \ldots \quad \text{(A)}$$

If $c = 0$ then $x = \frac{l}{\sqrt{3}} = 0{\cdot}576l$

If $c = \frac{l}{2}$ then $x = \frac{l}{2} = 0{\cdot}500l$

If $c = \frac{l}{4}$ then $x = \frac{\sqrt{5}l}{4} = 0{\cdot}558l$

Equation (A) shows that the maximum deflection is always near the middle of the beam. In the limiting case where $c = 0$ and P is at the support the point of maximum deflection is only a distance of 0·076l from the middle. The deflection at the middle is therefore a close approximation to the maximum.

$$EIy_{max} = \frac{Pc(l^2-c^2)}{6l} \times \sqrt{\frac{l^2-c^2}{3}} - \frac{Pc}{6l}\cdot\left(\sqrt{\frac{l^2-c^2}{3}}\right)^3$$

$$= \frac{Pc}{6l}\left[\frac{(l^2-c^2)^{3/2}}{3^{1/2}} - \frac{(l^2-c^2)^{3/2}}{3^{3/2}}\right]$$

$$= \frac{Pc}{6l} \times \frac{(l^2-c^2)^{3/2}\,[3^{3/2}-3^{1/2}]}{9}$$

$$= \frac{Pc}{6l} \times \left[\frac{l^2-c^2}{3}\right]^{3/2} \times 2 = \frac{Pc}{3l}\left(\frac{l^2-c^2}{3}\right)^{3/2}.$$

$$\therefore \quad y_{max} = \frac{Pc(l^2-c^2)^{3/2}}{9\sqrt{3}EIl}.$$

$$\text{If } c = \frac{l}{2},\ y_{max} = \frac{Pl^3}{48EI}$$

Illustrative Problem 9.

A beam *ADBC* with an overhang *BC* is bent in one case by a force *P* at the end *C*, and in another case by the force *P* applied at the middle *D* of the span *AB*. Prove that the deflection δ_D at the centre point *D* in the first case is equal to the deflection δ_{C_1} at the end *C* in the second case.*

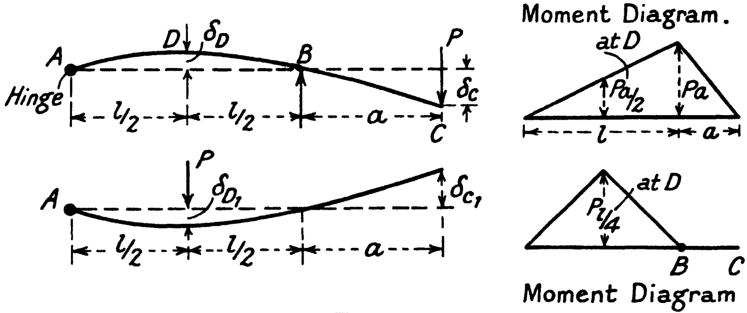

FIG. 53

It is required to prove that δ_D (Case I) $= \delta_{C_1}$ (Case II). (Fig. 53.)

Case I. *D* on the deflected beam is below the tangent at *A* by an amount

$$\frac{Pa}{2} \times \frac{l}{2} \times \frac{1}{2} \times \frac{l}{6} \times \frac{1}{EI} = \frac{Pal^2}{48EI}$$

D on the horizontal between *A* and *B* is below the tangent at *A* by an amount

$$\frac{1}{2} \times Pa\frac{l}{2} \times \frac{l}{3} \times \frac{1}{EI} = \frac{Pal^2}{12EI}$$

---

* Certain writers denote the deflection by the symbol δ, consequently in the problems 9 to 11, the symbol of y previously used is replaced by δ.

$\therefore$ D on the deflected beam is

$$\frac{Pal^2}{12EI} - \frac{Pal^2}{48EI} = \frac{Pal^2}{16EI}$$

above the supports A and B.

Case II. C is above the supports A and B by an amount

$$= a \times \text{the slope of the tangent at } B$$

$$\therefore \qquad \delta_{C_1} = a \times \frac{1}{2} \times \frac{Pl}{4} \times \frac{l}{2} \times \frac{1}{EI} = \frac{Pal^2}{16EI}$$

$\therefore$ δ_D (Case I) $= \delta_{C_1}$ (Case II).

Conclusion. A load P at C causes a deflection X at D which is equal to the deflection X at C when the load P is at D.

This is an example of a Theorem which is known as *Maxwell's Theorem of Reciprocal Deflections*, and is used a great deal in the theory of the solution of Statically Indeterminate Structures.*

Illustrative Problem 10. (Fig. 54.)

A bar ABC is hinged at A and supported at the same level at B. AB is 10 ft. and BC is 5 ft. A concentrated load of 5 tons is carried at the overhanging end C. If $E = 30 \times 10^6$ lb./sq. in. and $I = 2000$ in. units, calculate—

(*a*) the deflection at C;
and (*b*) the slope of the beam at B.

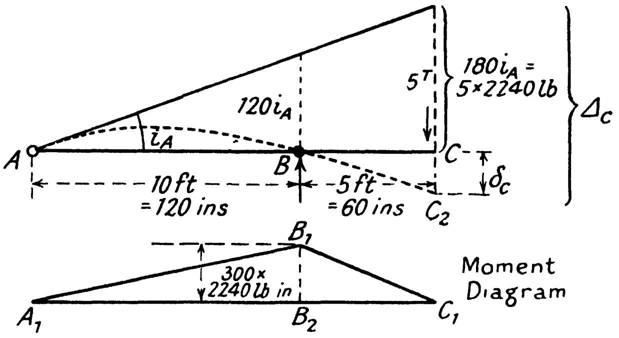

FIG. 54

$$120 i_A = \frac{300 \times 2240 \times 120}{2} \times \frac{1}{3} \times \frac{120}{EI}$$

$$i_A = \frac{300 \times 2240}{6} \times \frac{120}{EI}$$

---

* The proof of this theorem is given in a paper by the author, reference No. 9, page 220. *See* also Chapter XVI, pp. 482, 483.

$$\therefore \quad 180 i_A = \frac{300 \times 2240 \times 120 \times 180}{6EI}$$

C_2 the deflected point of C is below the tangent at A by an amount

$$\Delta_c = \frac{300 \times 2240 \times 120}{2EI} \times \left(\frac{1}{3} \times 120 + 60\right) + \frac{300 \times 2240 \times 60}{2EI} \times 40.$$

$\therefore$ deflection of $C = \delta_c = \Delta_c - 180 i_A$

$$= \frac{300 \times 2240}{6EI}[360 \times (100) + 180 \times 40 - 120 \times 180]$$

$$= \underline{0{\cdot}0403 \text{ inch.}}$$

The slope of the tangent at B

$$= \frac{300 \times 2240 \times 120}{2EI} - \left(i_A = \frac{300 \times 2240 \times 120}{6EI}\right)$$

$$= \underline{448 \times 10^{-6} \text{ radians.}}$$

DEFLECTIONS OF CANTILEVERS.

(*a*) A cantilever of length l with a concentrated load of w tons, a distance l_1 from the origin O. (Fig. 55.)

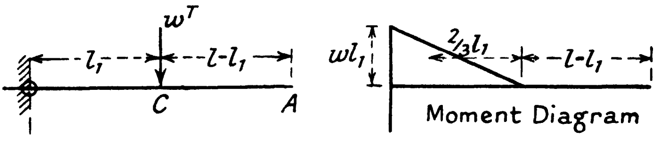

FIG. 55

EI = constant.

A will deflect by an amount δ_A which is equal to the distance of the deflected beam from the tangent at the origin. This tangent is horizontal.

$$EI\delta_A = Wl_1 \times \frac{l_1}{2} \times \left(l - \frac{l_1}{3}\right) = \frac{Wl_1^2}{2}\left(l - \frac{l_1}{3}\right).$$

If $$l_1 = l \text{ then } \delta_A = \frac{Wl^3}{3EI}$$

The slope of the tangent i_c at the load point is found from the equation

$$EIi_c = Wl_1 \times \frac{l_1}{2} = \frac{Wl_1^2}{2}$$

$$\therefore \quad i_c = \frac{Wl_1^2}{2EI}.$$

The deflection $$\delta_c = \frac{Wl_1}{EI} \times \frac{l_1}{2} \times \frac{2l_1}{3}$$

$$= \frac{Wl_1^3}{3EI}.$$

(*b*) Cantilever of length l with a uniformly distributed load w tons per foot run. EI is a constant.

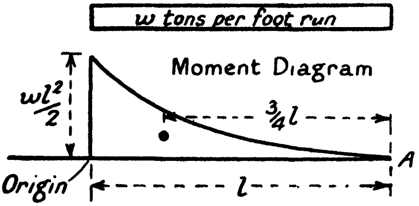

FIG. 56

A is below the origin by an amount equal to δ_A which is

$$= \frac{wl^2}{2} \times \frac{1l}{3} \times \frac{3l}{4} \times \frac{1}{EI}$$

$$= \frac{wl^4}{8EI}.$$

(*c*) A cantilever AB of uniform section is deflected by a couple of magnitude M, applied at a point C distant c from the origin B. Find the deflection of the end A. EI is constant.

The moment M will be constant from C to B

$$\therefore \quad EI\delta_A = Mc(l - c/2)$$

$$\therefore \quad \delta_A = \frac{Mc}{EI}\left(l - \frac{c}{2}\right).$$

If $$c = l$$

$$\delta_A = \frac{Ml^2}{2EI}.$$

Illustrative Problem 11. (Fig. 57.)

Find the deflection of C of the cantilever beam CB of the structure CBA under the load P. Neglect the displacement due to the axial load in AB. EI is the same for the two members CB and BA.

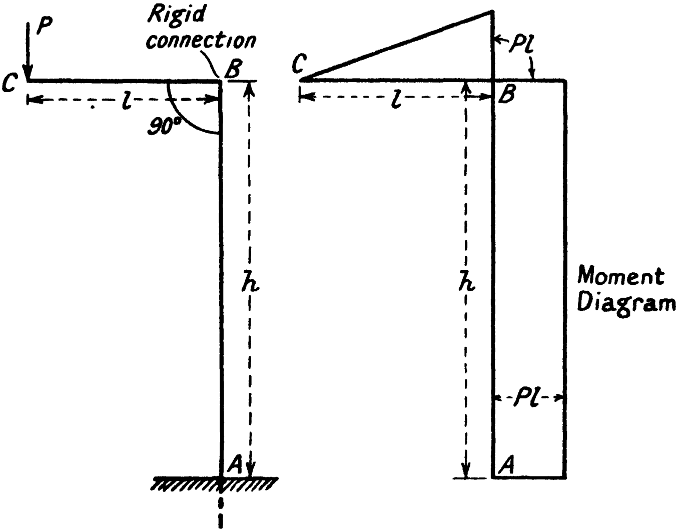

FIG. 57

The displacement of C is made up of two parts: one due to the change of slope of BC due to the displacement of B with respect to A, and the other due to the bending moment on BC. Let α_p be the deflective of C.

$$EI\delta_c = \left(l\,.\,Plh + Pl\,.\,\frac{l}{2} \times \frac{2l}{3}\right)$$

$\left(\frac{Plh}{EI}\right.$ is the slope of the tangent at B for the cantilever AB, and therefore BC also rotates through the angle $\left.\frac{Plh}{EI}\right)$

$$\therefore \qquad \delta_c = \frac{1}{EI}\left(Pl^2\,.\,h + Pl^2\,.\,\frac{l}{3}\right)$$

$$= \frac{Pl^2}{EI}\left(h + \frac{l}{3}\right).$$

36. Beams for which E and I are Variable. (*See* Fig. 58.)

(1) (*a*) If E is constant and I varies, then

$$Ey_x = \int\int \frac{M_x \cdot dx \cdot dx}{I_x}$$

where M_x and I_x are the bending moment and moment of inertia respectively for a section X, and y_x is the deflection of the beam at the section X.

The integral above may be an awkward one, even if I_x varies as x, and consequently it will be better to use a graphical method. (For the case where I_1 the moment of inertia is a constant over a portion of the beam, and I_2 constant over another length and so on, a method is given in paragraph 39.)

(*b*) If both E and I vary, then $y_x = \int\int \frac{M_x \,.\, dx \,.\, dx}{E_x \,.\, I_x}$

GRAPHICAL METHOD

(2) Construct the load diagram from a consideration of the loads and the weight of the beam itself.

(3) Find the reactions by ordinary methods.

(4) Construct the moment diagram, using, say, the method $M_x = -R_0x +$ (area of the load diagram from the origin to the section X) multiplied by (the distance of the centre of gravity of this area from the section X).

(5) Construct an EI diagram on the length of the beam as base.

(6) Draw a $\frac{M}{EI}$ diagram, i.e. divide corresponding ordinates of the M and EI diagrams. M stands for moment.

(7) Use the $\frac{M}{EI}$ diagram as a load diagram. Find the reactions in terms of $\left\{\frac{M \times \text{length}}{EI}\right\}$ units. These represent the slope of the beam at the point at which they act.

(8) Construct the deflection diagram by taking moments of the $\frac{M}{EI}$ diagram about the section considered, and using a similar equation as in (4) or the deflection at a section is the resulting moment of the M/EI diagram about the section, or $y =$ slope × (distance to the section from the support minus the distance of the section with respect to the tangent to the support).*

---

* Conjugate Beam Method. See *Strength of Materials*, Part I, S. Timoshenko.

Dimensions* of $\frac{\text{Moment}}{EI}$ are $\frac{[M]^1[L]^1[L]^1}{[T]^2} \times \frac{[T]^2[L]^2}{[M]^1[L]^1} \times \frac{1}{[L]^4} = \frac{1}{[L]}$

Dimensions of the area of the $\frac{\text{Moment}}{EI}$ diagram are $\frac{[L]}{[L]}$

Dimensions of the reactions are, therefore, nil.

Dimensions of the resultant moment of the area of the $\frac{\text{Moment}}{EI}$ diagram are $\frac{[L]}{[L]}[L] = [L]$

Thus the resultant moment of the $\frac{\text{Moment}}{EI}$ diagram about a section point gives the deflection at the section.

Illustrative Problem 12. A timber beam simply supported is 96 in. long, and varies uniformly in width from 10 in. to 18 in.; it has a constant depth of 2 in. Find the maximum moment and the section at which it occurs. Construct the moment and deflection curves and find the maximum deflection. The weight of a cubic inch of timber is ·032 lb.

The diagrams are shown in Fig. 58.

The total weight of the beam is 86 lb.

To find the reactions, treat as a rectangular plus a triangular beam in plan.

Taking A (the left-hand support as origin), it is found that

$$R_A = 39 \text{ lb. and so } R_B = 47 \text{ lb.}$$

The moment at any section X distant x from A,

$$M_x = -39x + 2 \times 10 \times x \times \cdot032 \times \frac{x}{2} + \frac{x}{2} \times \frac{x}{12} \times \cdot032 \times 2 \times \frac{x}{3}$$

$$= -39x + \cdot32x^2 + \cdot00089x^3$$

For maximum conditions, $\frac{dM_x}{dx} = 0$

---

* [L] represents one dimension in length.
[M] ,, ,, ,, ,, mass.
[T] ,, ,, ,, ,, time.

For the section having the maximum moment,

$$\frac{dM_x}{dx} = -39 + \cdot 64x + \cdot 00267x^2 = 0$$

$$\therefore \quad x = 50\cdot 5 \text{ in.}$$

$$M_x \text{ maximum} = 1029 \text{ lb.-in.}$$

The moment curve is shown in Fig. 58.

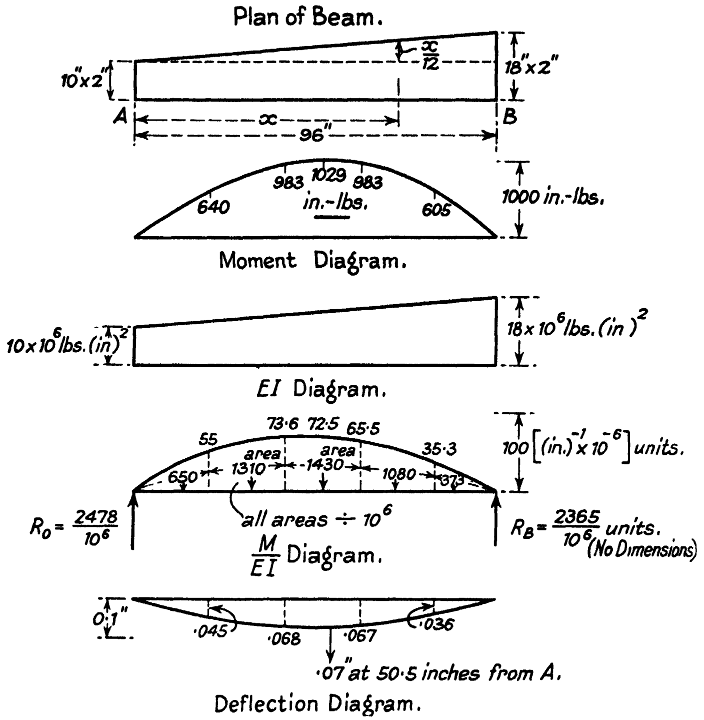

FIG. 58

To Find the Deflection Curve. I_x varies as x, and a diagram of EI lb. (inch)^2 units plotted against x is shown in the figure.

The dimensions of EI* are $[M]^1\,[L]^3\,[T]^{-2}$.

Divide the moment diagram by the EI diagram and so obtain a $\frac{\text{Moment}}{EI}$ (numerical units) diagram.

---

* *See* footnote, page 69.

Treat this diagram as a load diagram and find the reactions in the usual way. Then by section (8) page 68, the deflection curve can be found.

The areas have been worked out by Simpson's Rule.

The deflection curve is shown in Fig. 58, and it is a maximum when x is 50·5 in. from A and equal to ·07 in.

37. **Resilience of Beams Due to Bending.** When a beam is bent within the elastic limit or limit of proportionality, the material is subjected to varying elastic tensile and compressive forces and, therefore, it possesses strain energy, and being within the elastic limit, this energy will be restored when the loads on the beam are removed; this strain energy is the resilience of the beam.

Work done by a couple = magnitude of the couple × the angle through which it turns.

Let M = the final couple and i the angle through which a plane section of the beam turns;

$$\text{then work done} = U = (\tfrac{1}{2}M)i$$

Before loading no moment at all, thus average moment $= \dfrac{M}{2}$

$$\text{A small increment of work} = dU = \tfrac{1}{2}Mdi$$

In a short length, dx of a loaded beam over which the moment is M, the change of slope will be di

or di = small angle through which the internal moment of resistance will move.

The elastic strain energy of this portion is

$$\tfrac{1}{2}M \cdot di$$

Over a given length the resilience is

$$\frac{1}{2}\int M \cdot di$$

$$= \frac{1}{2}\int M\frac{di}{dx} \cdot dx \cdot = \frac{1}{2}\int M\frac{d^2y}{dx^2} \cdot dx \text{ as } \left(di = \frac{dy}{dx}\right)$$

$$= \frac{1}{2}\int\frac{M^2 \cdot dx}{EI} \quad . \quad . \quad (25)$$

If EI is a constant,

$$U = \frac{1}{2EI}\int M^2 \cdot dx \quad . \quad . \quad (26)$$

EXAMPLE.

By means of resilience, find the deflection at the centre of a beam for a simple beam loaded with W tons at the centre.

Let y = deflection at the centre;
then the work done by $W = \frac{1}{2}Wy$ (W being applied gradually
= 2 × strain energy of half the beam

$$U = \frac{1}{2}Wy = 2 \times \frac{1}{2EI}\int_0^{\frac{l}{2}}\left(\frac{W}{2}x\right)^2 \cdot dx = \frac{W^2l^3}{96EI}$$

(For a section X between the origin and the centre,

$$M_x = \frac{W}{2}x \Big)$$

Integrating and substituting the limiting values of x, and solving,

$$y_{centre} = \frac{Wl^3}{48EI}$$

38. **Beam Deflection for Any Loading.**[*] Generally on a beam with any loading at a section where the deflection is required, take any extra load $F = 1$ ton; let m be the additional bending moment at any section due to the unit weight. The deflection at the section is y, due to the original loading. Let M be the moment existing at the section.

It will be shown in Chapter VIII, paragraph 101a, that

$$y = \int\frac{M}{EI}\cdot\left(\frac{dM}{dF}\right).dx \quad . \quad . \quad . \quad . \quad (27)$$

$$\text{where } \frac{dM}{dF} = m.$$

$$\therefore y = \int\frac{Mmdx}{EI} \quad . \quad . \quad . \quad (28)$$

EXAMPLE.

Find the deflection at the centre of a beam with a uniformly-distributed load over the whole length of the beam.

The additional moment at a section between the origin and the centre of the beam due to the reaction of a concentrated load of 1 ton at the centre

$$= -\frac{x}{2}$$

---

[*] *See* also Chapter VIII and "Principle of Least Work," Chapter IX.

Therefore $y = \frac{2}{EI}\int_0^{\frac{l}{2}}\left(\frac{-wlx}{2} + \frac{wx^2}{2}\right)\left(-\frac{x}{2}\right)\cdot dx$

$$y = \frac{2}{EI}\int_0^{\frac{l}{2}}\left(\frac{wlx^2}{4} - \frac{wx^3}{4}\right)dx$$

Integrating and substituting the values of x necessary,

$$y_{max} = \frac{5wl^4}{384EI} = \text{deflection at the centre}$$

See Chapter VIII for other examples.

39. **Beams of Varying Cross-section.** (*See* also paragraph 36.) If the moment of inertia of the cross-section of a beam is not constant through its entire length, the deflection will be

$$y = \frac{1}{E}\int\frac{Mm}{I}\cdot dx \quad . \qquad \text{(eqn. 28)}$$

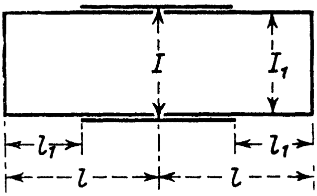

FIG. 59

Such an example arises in the case of a plate girder where the cross-section of the beam varies over different lengths.

EXAMPLE.

For the beam in Fig. 59, where I_1 is a constant over a length l_1 and I a constant over a length $(l - l_1)$, the deflection at the centre of the beam is found from the equation

$$yE = 2\int_{l_1}^{l}\frac{Mm}{I}dx + 2\int_0^{l_1}\frac{Mm}{I_1}\cdot dx$$

Find the deflection at the centre of such a beam which carries w tons per foot run. Imagine a unit load additional at the centre.

Length of beam $= 2l$. $y_c =$ deflection at the centre.

For any section X distant x from the left-hand support,

$$M_x = -wlx + \frac{wx^2}{2}$$

$$m = -\frac{x}{2} \text{ (due to the unit load only)}$$

$$y_cE = 2\int_{l_1}^{l}\frac{w\left(\frac{lx^2}{2} - \frac{x^3}{4}\right)}{I}dx + 2\int_0^{l_1}\frac{w\left(\frac{lx^2}{2} - \frac{x^3}{4}\right)}{I_1}dx$$

$$y_cE = \frac{5wl^4}{24I} + \frac{wl_1^3}{24}\left(\frac{1}{I_1} - \frac{1}{I}\right)(8l - 3l_1)$$

Illustrative Problem 13. A weight of 1 ton is dropped from a height of 5 in. on to the centre of a R.S.J. 16″ × 6″ ($I_{xx} = 726$ in.^{4}) 20 ft. long, and simply supported as a beam.

Taking $E = 13{,}000$ tons/square inch, find the maximum stress developed in the beam.

Let y = maximum deflection of the beam at the centre. Let an equivalent static load of W_1 tons at the centre cause this same deflection y.

Let the falling weight be W tons; and h inches be the height through which it falls.*

The resilience of the beam under the load of W_1 tons

$$\frac{W_1 y}{2} = \frac{2}{2EI}\int_0^{\frac{l}{2}}\left(-\frac{W_1}{2}x\right)^2 \cdot dx$$

$$= \frac{1}{EI}\left[\frac{W_1^2}{4}\cdot\frac{x^3}{3}\right]_0^{\frac{l}{2}} = \frac{W_1^2 l^3}{96EI}$$

$$\therefore y = \frac{W_1 l^3}{48EI}$$

$$\text{Then}\quad W(h + y) = \frac{W_1^2 l^3}{96EI}$$

Neglecting y as being very small,

$$Wh = \frac{W_1^2 l^3}{96EI}$$

For the problem,

$$1 \times 5 = \frac{W_1^2 \times (20 \times 12)^3}{96 \times 13{,}000 \times 726}$$

$$W_1^2 = \frac{5 \times 96 \times 13{,}000 \times 726}{240 \times 240 \times 240}$$

$$= 338$$

$$W_1 = 18{\cdot}3 \text{ tons}$$

Moment = skin stress × modulus of the section

$$M = f\frac{I}{y};\ I = I_{xx}$$

$$f = \frac{My}{I} = \frac{18{\cdot}3 \times \frac{240}{4} \times 8}{726} \text{ tons/sq. in.}$$

---

* Assume no loss of energy at impact: that the resulting displacement curve corresponds to that obtained under static loading and that the beam is stressed within the limit of proportionality of the material.

$$\text{Max. stress} = 12{\cdot}1 \text{ tons/sq. inch}$$

$$y = \frac{18{\cdot}3 \times 240 \times 240 \times 240}{48 \times 13{,}000 \times 726} = {\cdot}56 \text{ in.}$$

therefore y is hardly negligible.

Allowing for the falling weight moving through y,

$$\therefore\ 1\left(5 + \frac{W_1 \times (240)^3}{48EI}\right) = \frac{W_1^2(240)^3}{96EI}$$

$$5 + \frac{W_1}{33{\cdot}8} = \frac{W_1^2}{67{\cdot}6}$$

$$\therefore\ W_1^2 - 2W_1 - 338 = 0$$

$$W_1 = \frac{2 \pm \sqrt{4 + 1352}}{2}$$

$$= \frac{38{\cdot}8}{2} = 19{\cdot}4 \text{ tons}$$

$$\text{Max. stress in this case} = 12{\cdot}1 \times \frac{19{\cdot}4}{18{\cdot}3}$$

$$= 12{\cdot}8 \text{ tons/sq. inch}$$

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Statically Indeterminate Stresses (Chapter I), Parcel and Maney.

Solution of Problems in Strength of Materials, S. A. Urry (Pitman).

EXAMPLES

1. A cantilever 10 ft. long carries 5 tons at its outer end and another 5 tons 6 ft. away from the fixed end. Find its deflection by the graphic method, assuming that $I = 1000$ in.^{4} and $E = 13{,}000$ tons per square inch throughout its length. Check the result by calculation. (U. of B.)

2. A rectangular beam of wood 12 ft. long, 3 in. wide, and 4 in. deep is supported at its ends, and is loaded with loads 300 lb. and 500 lb. placed at points 5 and 8 ft. respectively from one end. Calculate the maximum bending moment produced by the loading and find the deflection produced by bending at the midpoint of the beam. $E = 1{,}400{,}000$ lb. per square inch.

3. A rolled steel joist, 10 in. deep by 5 in. wide, has an effective span of 10 ft., the ends being freely supported. The maximum moment of inertia of the section is 145·6 in inch units. From what height can a weight of

half a ton fall on the middle of the joist without producing a stress greater than 15 tons per square inch? Only 75 per cent of the kinetic energy of the falling weight is transformed into the work of deformation.

E = 12,600 tons per square inch. (I.M.E.)

4. A hollow pole, made of mild steel tube 5 in. outside diameter ½ in thick, is firmly fixed in the ground, the top being 9 ft. above ground level. A horizontal pull of 1,000 pounds is applied to the pole at a height of 5 ft. from the ground. Calculate the deflection of the top of the pole from the vertical.

E = 13,000 tons per square inch.

5. Describe a graphical method of finding the centroid and the moment of inertia of the cross-section of a beam. A beam 5 ft. long deflects 0·0024 in. under a central load of 2·5 tons. $I = 39\cdot05$ in.^{4}. Find the modulus of elasticity for the beam, neglecting shear. (U. of B.)

6. A beam of oak 1 in. square by 3 ft. 6 in. long is given to you. Describe how you would determine the modulus of elasticity of the beam and what value you would expect to get by means of overhanging beams and of central deflections. (U. of B.)

7. A beam of mild steel 4 in. wide, 6 in. deep, simply supported on two rollers 10 ft. apart is loaded with a central weight of 1 ton. Calculate

(*a*) the maximum tensile stress in the material;

(*b*) the central deflection. (U. of B.)

8. A uniform beam 16 ft. long is supported at two points 2 ft. from either end. At the middle of the beam, and also at each extremity, loads of 1 ton are placed. Draw the curves of shearing force and bending moment for the beam.

9. Find, in any way, the deflection of the centre of the beam of Question 9. Give your results in terms of the moment of inertia of the cross-section and of Young's modulus. (U. of B.)

10. A girder of I-section rests on supports 25 ft. apart and carries a load of 7 tons at a distance of 10 ft. from one support. If the moment of inertia of the cross-section is 695 in. units and E is 30,000,000 lb. per square inch, find the deflection of the girder at the load due to bending, and the position and amount of the maximum deflection. The weight of the girder may be neglected. (U. of L.)

11. A uniform rolled-steel joist of 18 ft. span, simply supported at each end, carries a load which increases uniformly from 5 cwt. per foot run at the left support to 17 cwt. per foot run at the right-hand support. Find the position and magnitude of the maximum bending moment, and find the slope and deflection of the girder at the centre of the span. Given: Young's modulus for steel = 30×10^6 lb. per square inch. Moment of inertia of a normal section of joist about neutral axis = 130 (inches)^4. (U. of B.)

12. An I section steel joist, 9 in. by 4 in., moment of inertia 81 in. units, has a span of 12 ft. Find the maximum safe distributed load per foot run it will carry with a working stress of 8 tons per square inch and the central deflection at this load, if E = 13,500 tons per square inch. (I.C.E., Oct. 1922.)

13. The following observations were made in testing a sample of oak by bending, the load being applied in the centre of the span. Width 2·02 in. depth 2·97 in., span 55 in. The loads W and the corresponding deflection scale readings were—

W lb.	100	200	300	400	500	600	700	800	900	1000
R in.	·145	·218	·287	·358	·43	·5	·57	·643	·721	·800

Determine from these observations the modulus of elasticity of the oak and also its limiting elastic stress. (U. of L., 1922.)

14. An I section girder 10 in. × 6 in. with web 0·4 in. and flanges 0·7 in. thick, is firmly built into a wall in a horizontal position, so that it can act as a cantilever 12 ft. long. Neglecting deflection due to shear, calculate what deflection would be produced at the end of the cantilever by a load of 0·5

tons placed on the cantilever 6 ft. from the end. $E = 30 \times 10^6$ lb. per square inch.

15. A horizontal rolled steel joist 10 in. × 6 in. is supported at its ends and has a span of 10 ft. A load of 400 lb. falls from a height of $3\frac{1}{2}$ in. on to the middle of the joist. Neglecting loss of energy at impact, find the maximum instantaneous stress produced in the joist, given that the maximum moment of inertia of the section is 210 in. units and $E = 13{,}250$ tons per square inch.
(U. of L., 1922 : S. of M.)

16. What distributed load will a T 4 in. × 6 in. × $\frac{1}{2}$ in. support over a span of 8 ft. for the working stress not to exceed 7 tons per square inch? What will be the maximum deflection? For the same working conditions of stress, how will the load be altered if a plate 6 in. × $\frac{1}{2}$ in. is riveted on to the T, along the centre 4 ft.? What will be the deflection at the centre and also 2 ft. from a support?

17. A beam 60 ft. long is simply supported. It carries a central load of 2 tons. The moment of inertia of the section for the middle 20 ft. is 600 in. units: for the section for the remaining length it is 450 in. units. Find the central deflection; also the deflections for the sections 15 and 25 ft. from the left-hand support.

$$E = 13{,}000 \text{ tons per square inch.}$$

18. A weight W is dropped from a height h on to the centre of a simply supported beam of rectangular cross-section and length l. I is the moment of inertia and A the area of the cross-section. The maximum deflection of the beam is small compared with h. Show that $Wh = y^2 \cdot \frac{24EI}{l^3}$ and that the resulting maximum stress developed is $f = \sqrt{\frac{Wh \cdot 18E}{lA}}$. Assume no loss of energy at impact and that the stresses developed are within the elastic range of the material.

CHAPTER IV

STATICALLY INDETERMINATE PROBLEMS IN BENDING. BUILT-IN, CONTINUOUS, AND PROPPED BEAMS WITH DEAD LOADS

40. Redundant Constraints. In the previous chapters, three types of beams have been considered: the cantilever, the beam freely supported at the ends, and the beam with overhangs. In all cases, the reactions at the supports can be determined from the fundamental equations of statics: hence the problems are "statically determinate." In the problems on the bending of beams which follow, the equations of statics are not sufficient to determine all the reactive forces at the supports, so that additional equations, based on a consideration of the deflections of the beams, must be derived. These problems are examples of "statically indeterminate structures."* There are three types of supports a beam may have: (*a*) hinged movable support, (*b*) hinged immovable support,† and (*c*) a built-in end.

Type (*a*) support can be imagined as a hinged joint supported on frictionless rollers on a horizontal plane *mm*. It is evident that in this type of joint the reaction must act through the centre of the hinge and vertical to the horizontal plane *mm*. The only unknown element of this reaction is its magnitude.

In connection with the type (*b*) support, it is evident that the reaction must pass through the centre of the hinge, but it may have any direction in the plane of the beam. There are now two unknowns to be determined from the equations of statics, the direction of the reaction and its magnitude, or the vertical and horizontal components of the reaction. In the case of the built-in end (*c*), not only are the direction and magnitude of the reaction unknown, but also the point of application. The reactive forces distributed over the built-in section can, however, be replaced by a vertical and a horizontal force acting through the centroid of the beam section at the commencement of the support, and also by a couple of magnitude M. For beams, loaded by transverse loads in one plane, to determine the reactions at the supports, the three equations of statics are—

---

* *See* also page 162. † *See* "Three-hinged Arch."

Σ Vertical forces = 0.
Σ Horizontal forces = 0.
Σ Moments about any point = 0.

If the beam is supported so that there are only three unknown reactive forces, then they can be found from the above three equations. When the number of reactive elements is larger than three, there are then *redundant constraints,* and the problem is statically indeterminate. In the cantilever there is only one support. The number of reactive elements is three and therefore they can be determined from the equations of statics. For beams supported at the ends it is usually assumed that one of the supports is of type (*a*), and the other of type (*b*). In this case there are only three unknown reactive elements, which can be found from the equations of statics. If a beam has immovable hinges at both ends, then there are two unknown reactive elements at each end, the two components of the corresponding reaction and for determining these four unknowns there are only the three equations of statics. Hence we have one redundant constraint, and a consideration of the deformation of the beam is necessary to determine the reactions.

In the case of the beams built-in at one end and freely supported at the other, there are three reactive elements at one end, and one at the other. Hence the problem is statically indeterminate, with one redundant constraint (*see* paragraphs on Propped Beams for method of solution). The built-in end is assumed to be direction-fixed, i.e. the angle of the tangent to the beam at the support after deformation in zero. For beams, built-in at both ends and direction-fixed at these ends, there are six reactive elements, and therefore there are three redundant constraints. However, for ordinary purposes, the horizontal components of the reactions can be neglected, which reduces the number of statically indeterminate quantities to two. In the examples which follow, the moments at the supports will be taken as the statically indeterminate quantities (paras. 42 to 50). As the beams are direction-fixed, then the solution for the support moments depends upon the fact that the change in direction of the tangents to the two ends of the beam is zero after deformation of the beam occurs. In the case of a beam on three supports, there is one statically indeterminate reactive element which may be taken as the central support.* Here one support is usually considered as an immovable hinge while the other supports are hinges on rollers. In the case of a beam continuous over many supports, one support is again considered as an immovable hinge, while the others are hinges on rollers. In this arrangement every intermediate support has only one unknown reactive element, the magnitude of the vertical reaction: hence the number of statically indeterminate elements is equal to the number of intermediate supports. (If the ends of the beams are built-in, then the number of indeterminate elements is increased.) If the number of supports is large, then the solution of the problem is simplified by taking the bending moments at the intermediate supports as the statically indeterminate quantities and not the reactions. (*See* para. 51 onwards.)

41. **Built-in or Encastré Beams** are beams fixed at each end, so that the supports completely constrain the inclinations of the beams at the ends. The two ends are usually at the same level, and the slope of the beam is then usually zero at each end if the constraint is effectual, i.e. $dy/dx = 0$.†

A built-in beam is in effect an overhanging beam, the overhangs having downward loads which cause positive moments at the supports, which are equivalent to the fixing moments; and due to these moments there will be a positive moment at all sections of the beam, neglecting the ordinary loads for the time being. It was seen that for the overhanging beam, the loads on the beam between the supports cause moments as for a simple beam, and the moment at any section between the supports was equal to the negative moment due to simple beam loading plus a positive moment due to the overhanging loads. For the built-in beam, the moment at a section is the algebraic sum of the moments treating the beam as a simple beam plus the amount due to the end fixing moments.

42. **Built-in Beams with Any Symmetrical Loading.** (Fig. 60.) The fixing couples will evidently be equal. There being equal couples at the ends, the positive moment at any section will be of the value of a fixing couple.

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* *See* example, pages 112 and 208, for solution of this unknown reaction.

† These beams are also called direction fixed ended beams.

In Fig. 60 are shown the positive and negative moment diagram for a fixed beam, the shaded diagram being the resultant moment diagram.

Let M_{sx} and M_{fx} represent the moments at a section X due to the simple beam and fixing moments.

Then resultant moment at $X = M_x$ = algebraic sum of M_{sx} and M_{fx}

$$= -M_{sx} + M_{fx}$$

$$i_x = \int \frac{M_x \cdot dx}{EI}$$

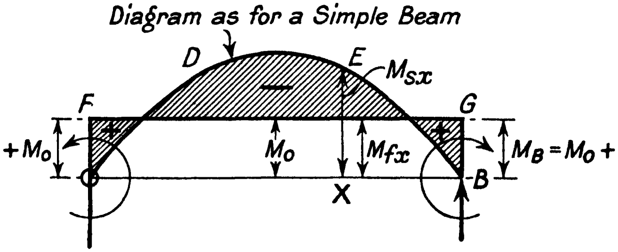


FIG. 60

Taking EI as constant,

$$EI(i_{x=l} - i_{x=0}) = \int_0 M_x \,.\, dx$$

$$= \int_0^l (-M_{sx} + M_{fx}) \cdot dx = 0.$$

since as the beam is horizontal at the ends, the change of slope is zero;

that is, $\int_0^l M_{sx} \cdot dx = +\int_0^l M_{fx} \cdot dx$; $M_{fx} = M_o$ = i.e. constant $= M_B$

so that $\int_0^l M_{sx} \cdot dx = M_o l$, where M_o = fixing couple at O and B

i.e. Area of the simple beam moment diagram $ODEB$ (Fig. 60) = area of the fixed or cantilever moment diagram $OFGB$. (1)

or $A_s = A_f$

A_s of the negative sense, A_f of the positive.

43. Beam Loaded in the Centre. (Fig. 61.)

$$A_f = A_s$$

$$M_o l = \frac{Wl}{4} \times \frac{1}{2} \times l = \frac{Wl^2}{8} \therefore M_o = +\frac{Wl}{8} \text{ units} \quad . \qquad (2)$$

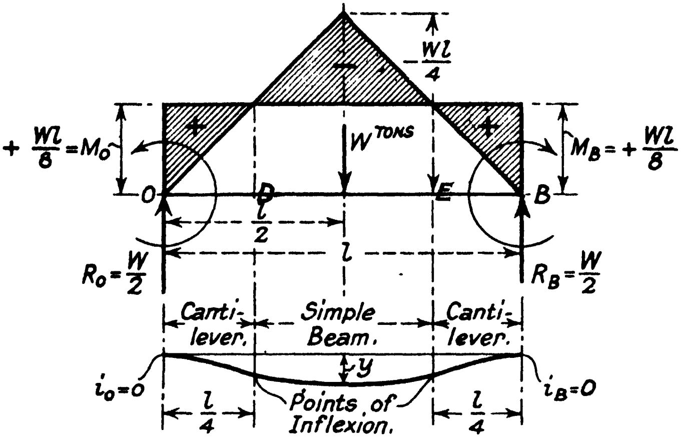


FIG. 61†

44. Beam with a Uniformly-distributed Load Over the Whole Span. (Fig. 62) page 83.

$$M_{s\,max.} = \frac{wl^2}{8} \qquad A_f = A_s$$

$$M_o l = \frac{wl^2}{8} \times l \times \frac{2}{3} = \frac{2wl^3}{24} \therefore M_o = +\frac{wl^2}{12} \text{ units} \quad . \quad . \qquad (3)$$

45. Built-in Beams with any Loading. (Fig. 63, page 84.) As before, the change of slope between the ends is zero, so that

$$\int_0^l (M_s + M_f) \cdot dx = 0^* \quad . \quad . \qquad (4)$$

$$\text{or } A_s + A_f = 0$$

M_o is not necessarily equal to M_B, and so let $M_o < M_B$.

---

* M_s = Simple Moment at any section.
M_f = Fixing " " "

† *Note on Fig.* 61. Bending moments of opposite sign evidently tend to produce bending of opposite curvature. Change of sign involves passing through a zero value of bending moment. This point of zero moment and change of sign is called a point of inflexion or virtual hinge. The fixed beam above is equivalent to a simple beam supported at the ends of two cantilever beams.

The fixing moment diagram will now be a trapezium, and the positive moment at a section X, x from the origin will be

$$M_f = M_o + (M_B - M_o)\frac{x}{l}$$

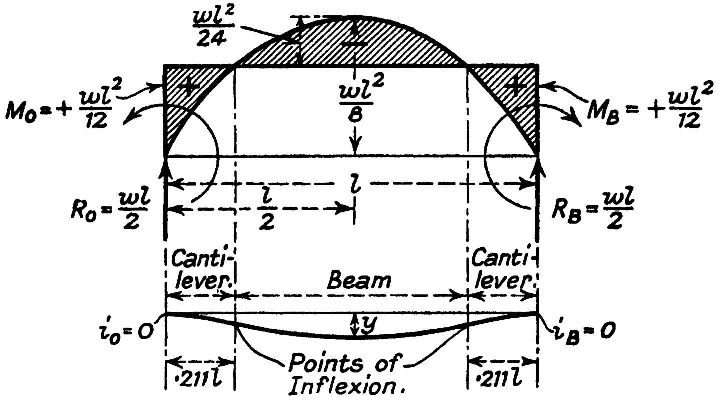


FIG. 62

Equation (4) now becomes

$$0 = \int_0^l \left\{ M_s + M_o + (M_B - M_o)\frac{x}{l} \right\} \cdot dx \quad . \quad (5)$$

The equation of the areas is now not sufficient to determine M_o or M_B.

Taking O as origin,

$$\frac{M}{EI} = \frac{d^2y}{dx^2} = \frac{M_s + M_f}{EI}$$

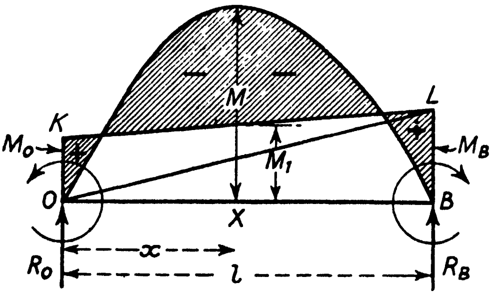
Multiplying both sides by x and integrating by parts (*see* Chap. III),

$$EI\left[x\frac{dy}{dx} - y\right]_0^l = \int_0^l (M_s + M_f)x \cdot dx$$

$$= \left(\int_0^l M_s x dx + \int_0^l M_f \cdot x \cdot dx\right)$$

$$= A_s\bar{x}_s + A_f\bar{x}_f \quad . \quad . \quad . \quad (6)$$

where $\bar{x}_s$ and $\bar{x}_f$ are the distances of the centroids of the negative and positive fixed moment diagrams from the origin O.



If $M_B > M_o$

$$R_o = R_{os} - \left(\frac{M_B - M_o}{l}\right)$$

R_{os} = Reaction as for a Simple Beam.

$$R_B = R_{BS} + \left(\frac{M_B - M_o}{l}\right)$$

R_{BS} = Reaction as for a Simple Beam.

$$x\triangle OKL = M_o \times \frac{l}{2} \times \frac{l}{3}$$

$$x_1\triangle OLB = M_B \times \frac{l}{2} \times \frac{2l}{3}$$

$\bar{x}$ and $\bar{x}_1$ from 0

Fig. 63

Further $\left(x\dfrac{dy}{dx} - y\right)_0^l = 0$. y is the same for values $x = 0$ and $x = l$ and $\dfrac{dy}{dx} = 0$, when $x = l$ and $x = 0$

$$\begin{aligned} \text{then } A_s\bar{x}_x + A_f\bar{x}_f &= 0 \\ \text{or } A_f(\bar{x}_s + \bar{x}_f) &= 0 \\ \text{or } A_s(\bar{x}_s + \bar{x}_f) &= 0 \end{aligned} \Bigg\} \text{ for } A_s = A_f . \quad . \quad . \quad (7)$$

As $A_s = A_f$, then $\bar{x}_s = \bar{x}_f$: so that the centroids lie on the same vertical line.

From Fig. 63,

$$A_f\bar{x}_f = M_o\frac{l}{2} \times \frac{l}{3} + M_B\frac{l}{2} \times \frac{2}{3}l$$

$$= M_o\frac{l^2}{6} + 2M_B\frac{l^2}{6} \quad . \quad . \qquad (8)$$

$$\text{then } A_s\bar{x}_s = M_o\frac{l^2}{6} + 2M_B\frac{l^2}{6}$$

$$\text{or } M_o + 2M_B = -\frac{6A_s\bar{x}_s}{l^2}. \quad . \quad . \quad . \qquad (9)$$

This is a special case of the Theorem of Three Moments (*see* para. 52).

From eqn. (1), $M_o + M_B = -\frac{2A_s}{l}$ . . . . (10)

From equations (9) and (10),

$$M_B = \frac{2A_s}{l} - \frac{6A_s\bar{x}_s}{l^2} \quad . \quad . \quad . \quad . \qquad (11)$$

$$M_o = \frac{6A_s\bar{x}_s}{l^2} - \frac{4A_s}{l} \quad . \quad . \quad . \qquad (12)$$

and A_s will be of negative sign.*

If a load of P tons is at a distance nl from the origin, then

$$M_o = Pl\,.\,n(1-n)^2 \text{ and } M_B = Pl\,.\,n^2(1-n) \quad . \qquad (12a)$$

46. **Relation Between the Moments and the Reactions.** Suppose the two ends O and B of a beam are hinged and at one end B is applied a couple M_B in a clockwise direction. This couple will obviously induce a vertical reaction r downwards, at the other hinge O, and an equal and opposite one at B, to form a couple, such that $r_1 l = M_B$, but of the opposite sign; therefore $r_1 = \frac{M_B}{l}$.

---

* When the fixing moment diagram has been drawn, the difference of ordinates between it and the bending moment diagram for the simple beam gives the bending moment for the built-in beam. The resultant diagrams have been shown for the cases considered.

Now suppose at the hinge O a couple $= M_o$ is also applied, but in the opposite direction, tending counter-clockwise to the one at B, then equal and opposite reactions r_2 will be applied at B and O, the one acting downwards at B and the other upwards at O.

When two fixing couples of equal magnitude are applied at the ends of a beam and acting in opposite directions, they do not affect the reactions at the supports which will be the same for the beam as those for a simple beam for $r_2 = r_1$.

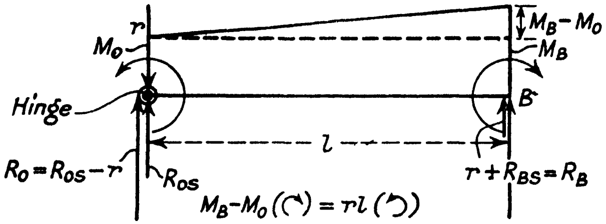


FIG. 64

Now let the fixing couple at O be M_o, and less than M_B; then the induced reactions at O and B, due to M_o, will be less than those due to M_B (see Fig. 64).

Let r_2 be the induced reactions due to M_o and r_1 those due to M_B,

$$r_2 = \frac{M_o}{l} \text{ and } r_1 = \frac{M_B}{l}$$

$$r_1 > r_2 \text{ and } r_1 - r_2 = \frac{M_B - M_o}{l} = r$$

Now at O due to these couples there will be a force r_2 acting downwards and r_1 acting upwards. A resultant force r will act downwards. To balance this force, a force equal to r will act upwards at B.

Therefore, to find the reactions at the supports of a fixed beam, calculate the reactions as for a simple beam: Where $M_B > M_o$, subtract a force $\frac{M_B - M_o}{l}$ from the simple beam reaction at O and add a force $\frac{M_B - M_o}{l}$ to the simple beam reaction at B.

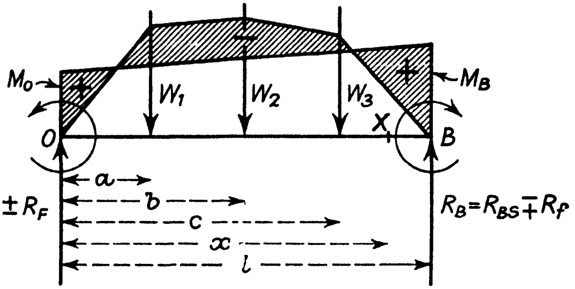
47. **Procedure for Any Loading of Fixed Beams** (1) Find the end fixing moments by formulae (11) and (12).

(2) The reactions for a simple beam.

(3) $\dfrac{M_B - M_o}{l}$

(4) At the support where the fixing moment is the smaller, subtract (3) from (2).

At the support where the fixing moment is the larger, add (3) and (2).



R_{OS} = Reaction as for a Simple Beam.

R_F = Reaction due to Fixed Moments $= \dfrac{M_B - M_O}{l}$

FIG. 65

48. **Deflections of Fixed Beams. General Method.** (Fig. 65.)

R_{os} = value of reaction at O as for a simple beam.

(*a*) Moment at a section X distant x from O and between the last load and B,

$$EI\frac{d^2y}{dx^2} = -R_{os}x + W_1(x-a) + W_2(x-b) + \ldots W_n(x-n) + M_o \pm \frac{(M_B - M_o)x}{l} \quad . \quad . \quad (13)$$

$$EIy = -\frac{R_{os}x^3}{6} + \frac{W_1(x-a)^3}{6} + \frac{W_2(x-b)^3}{6} + \ldots \frac{M_o x^2}{2} \pm \frac{(M_B - M_o)x^3}{6l} + Ax + B \quad . \quad . \quad . \quad (14)$$

$x = 0, \qquad y = 0, \quad \therefore B = 0$

$x = l, \qquad y = 0$

$$\text{and } A = \frac{R_{os}l^2}{6} - \frac{W_1(l-a)^3}{6l} - \ldots - \frac{M_o l}{2} \pm \frac{M_o l}{6} \mp \frac{M_B l}{6} \quad (15)$$

Substitute in equation (14) to give the general equation for y.

(*b*) If using the reactions

$$EI\frac{d^2y}{dx^2} = -R_o x + W_1(x-a) + \ldots + M_o \quad . \quad (16)$$

where $R_o = R_{os} \pm R_{of}$, i.e. the reaction at the origin; and, as before, neglect all terms such as $(x-a)$, etc., which become negative for a particular value of x.

49. **Fixed Beams. Central Load** W. (*See* Fig. 61, page 82.)

$$EI\frac{d^2y}{dx^2} = -\frac{W}{2}x + W\left(x - \frac{l}{2}\right) + M_o; \quad \left(M_o = \frac{Wl}{8}\right)$$

$$EIy = -\frac{Wx^3}{12} + \frac{W}{6}\left(x - \frac{l}{2}\right)^3 + \frac{Wl}{8} \times \frac{x^2}{2} + Ax + B$$

if $x = 0, y = 0 \quad \therefore \quad B = o$; if $x = l, y = 0$, and

$$Al = \frac{Wl^3}{12} - \frac{W}{6}\left(\frac{l}{2}\right)^3 - \frac{Wl^3}{16} \text{ so that } A = 0$$

Between $x = 0$ and $x = \frac{l}{2}$

$$EIy = -\frac{Wx^3}{12} + \frac{Wlx^2}{16}$$

y is a maximum when $x = \frac{l}{2}$

$$EIy_{max} = \frac{-Wl^3}{96} + \frac{Wl^3}{64} = \frac{Wl^3}{192} \quad . \quad . \quad . \quad (17)$$

The deflection at $D = \frac{l}{4}$ from O and $E = \frac{l}{4}$ from B

$$= y_D = \frac{1}{EI}\left(\frac{-Wl^3}{12 \times 64} + \frac{Wl^3}{16 \times 16}\right) = \frac{Wl^3}{384EI} \quad . \quad (18)$$

$$= \frac{y_{max}}{2} = y_E \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Note at the points D and E, the moment changes sign. From O to D it is positive, D to E it is negative.

Therefore O to D will bend as for a cantilever, concave downwards; D to E will bend as a simple beam, concave upwards, because bending moments of opposite sign produce bending of opposite curvature.

Slope at D, $x = \frac{l}{4}$ is $EI\frac{dy}{dx} = \frac{-Wx^2}{4} + \frac{Wlx}{8}$

$$EIi_D = \frac{-Wl^2}{64} + \frac{Wl^2}{32} = \frac{Wl^2}{64} \quad . \quad . \quad . \quad (20)$$

Note, the slope increases from zero at the origin to $\frac{Wl^2}{64EI}$ at D and then decreases to zero at the centre of the beam.

Thus the points D and E are points of inflection. At these points the slope of the tangent to the deflection curve is a maximum.

50. **Fixed Beam with Uniformly-distributed Load.** (Fig. 62, page 83). (Over the whole span.)

$$M_o = M_B = \frac{wl^2}{12}$$

$$R_o = \frac{wl}{2} \text{ as for a simple beam}$$

$$M_x = EI\frac{d^2y}{dx^2} = -\frac{wl}{2}x + \frac{wx^2}{2} + M_o$$

$$EIy = -\frac{wlx^3}{12} + \frac{wx^4}{24} + \frac{M_ox^2}{2} + Ax + B$$

Both A and B are zero ($A = 0$ for $\frac{dy}{dx} = 0$ when $x = 0$)

$$\therefore EIy = -\frac{wlx^3}{12} + \frac{wx^4}{24} + \frac{wl^2}{24}x^2$$

At the centre of the beam, y is a maximum, $x = \frac{l}{2}$

$$EIy_{max} = -\frac{wl}{12}\left(\frac{l}{2}\right)^3 + \frac{w}{24}\cdot\left(\frac{l}{2}\right)^4 + \frac{wl^2}{24}\left(\frac{l}{2}\right)^2$$

$$= \frac{wl^4}{384} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

To find the sections having no moment

$$M_x = 0 = -\frac{wlx}{2} + \frac{wx^2}{2} + \frac{wl^2}{12}$$

$$x^2 - lx + \frac{l^2}{6} = 0$$

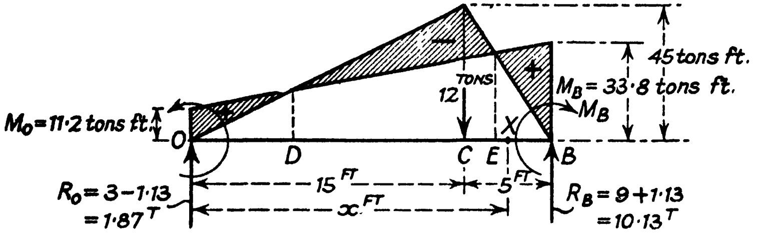
$$x = \frac{l \pm \sqrt{l^2 - \frac{2}{3}l^2}}{2} = \frac{l \pm - \cdot 58l}{2}$$

$$x = \cdot 21l \text{ and } \cdot 79l$$

The points of inflection are at these distances from the origin.

To find the slope at the section distant ·21*l* from *O*,

$$EIi_x = -\frac{wlx^2}{4} + \frac{wx^3}{6} + \frac{wl^2x}{12}$$



45 tons ft.
$M_B = 33 \cdot 8$ tons ft.
M_B
12 TONS
$M_0 = 11 \cdot 2$ tons ft.
O
D
C
E
X
B
15^{FT}
5^{FT}
x^{FT}
$R_0 = 3 - 1 \cdot 13 = 1 \cdot 87^T$
$R_B = 9 + 1 \cdot 13 = 10 \cdot 13^T$

FIG. 66

$$EIi_{\cdot 21l} = -\frac{wl(\cdot 21l)^2}{4} + \frac{w(\cdot 21)^3l^3}{6} + \frac{wl^3 \cdot 21}{12}$$

$$= -\frac{wl^3}{90} + \frac{wl^3}{642} + \frac{wl^3}{57}$$

$$EIi_{\cdot 21l} = \frac{wl^3}{126} \text{ and is a maximum}$$

Illustrative Problem 14.

A built-in beam 20 ft. long carries a load of 12 tons at a distance of 15 ft. from the left-hand end. Find the maximum deflection, the section at which it occurs, and the points of inflection EI = a constant. (Fig. 66.)

$$(1)\ \frac{(M_o + M_B)\,20}{2} = 45 \times \frac{20}{2}$$

Equating areas of positive and negative moments.

$$(2)\ 45 \times \frac{15}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \overset{5}{\cancel{15}} + \frac{45 \times 5}{2} \times \left(15 + \frac{5}{3}\right) = M_o \cdot \frac{\overset{10}{\cancel{20}}}{\cancel{2}} \times \frac{20}{3}$$

$$+ M_B \times \frac{\overset{10}{\cancel{20}}}{\cancel{2}} \times \frac{40}{3}$$

Equating moments of the positive and negative areas about O.

From (1) $M_o + M_B = 45$

„ (2) $M_o + 2M_B = 78{\cdot}75$

$$\left.\begin{aligned} M_B &= 33{\cdot}8, \text{ say, and both} \\ M_o &= 11{\cdot}2 \text{ are positive} \end{aligned}\right\}$$

$$\frac{M_B - M_o}{l} = \frac{22{\cdot}6}{20} = 1{\cdot}13 \text{ tons}$$

The total reaction at the origin $= 3 - 1{\cdot}13 = 1{\cdot}87$ tons

„ „ end $B = 9 + 1{\cdot}13 = 10{\cdot}13$ „

Total . . 12·0 „

Deflection of the Beam. Take a section X distance x from O and between the load point and B.

Then $$EI\frac{d^2y}{dx^2} = -3 \times x + 12(x - 15) + M_o + \frac{M_B - M_o}{l} . x$$

$$EI\frac{d^2y}{dx^2} = -3x + 12(x - 15) + 11{\cdot}2 + 1{\cdot}13x$$

$$EIy = \frac{-3x^3}{6} + \frac{12(x-15)^3}{6} + \frac{11{\cdot}2x^2}{2} + \frac{1{\cdot}13x^3}{6} + Ax + B$$

$$B = 0. \quad A = 0, \text{ because at } x = 0, \ \frac{dy}{dx} = 0$$

Between O and C the term $x - 15$ is negative and is eliminated.

$$EIy = \frac{11{\cdot}2x^2}{2} - \frac{1{\cdot}87x^3}{6}$$

For a maximum and which occurs between O and C, for OC is longer than CB.

$$EI\frac{dy}{dx} = 11{\cdot}2x - {\cdot}94x^2 = 0,$$

$$\text{that is, } 11{\cdot}2 = {\cdot}94x$$

$$x = 11{\cdot}9 \text{ ft.}$$

$$EIy_{max} = \frac{11{\cdot}2}{2} \times 11{\cdot}9^2 - {\cdot}31x^3$$

EI in tons-feet units, y_{max} in feet,

$$y_{max} = + \frac{275}{EI}$$

$$\text{If } E = 13{,}000 \text{ tons/sq. in.}$$

$$\text{,, } I = 300 \text{ (inches)}^4$$

$$y_{max} = \frac{275 \times 12^3}{13{,}000 \times 300} \text{ inch}$$

$$= 0{\cdot}12 \text{ inch.}$$

Cf. y_{max} with that for a simple beam, problem 8, Chapter III. The points of inflection occur at (1), a point between O and C.

$$(1)\ EI\frac{d^2y}{dx^2} = 11{\cdot}2 - 1{\cdot}88x = 0$$

$$x = 5{\cdot}95 \text{ ft.}$$

and (2), a point between C and B,

$$EIy = \frac{11{\cdot}2}{2}x^2 - \frac{1{\cdot}87x^3}{6} + \frac{12(x-15)^3}{6}$$

$$\frac{EI}{dx}dy = 11{\cdot}2x - {\cdot}94x^2 + 6(x-15)^2$$

For a maximum which is at the point of inflection,

$$EI\frac{d^2y}{dx^2} = 11{\cdot}2 - 1{\cdot}88x + 12(x-15) = 0$$

$$10{\cdot}12x = 168{\cdot}8$$

$$x = 16{\cdot}7 \text{ ft.}$$

51. **Continuous Beams.** A beam resting on more than two supports and covering more than one span is called a continuous beam. The ends of the beam may overhang, be simply supported or fixed. The moments of inertia of the section may also vary with the span, i.e. one span may have a moment of inertia I, another I_1 and so on.

There will be fixing or hogging moments at all the intermediate supports and also at the end supports, if they are fixed, or if the end portions overhang.

52. **Theorem of Three Moments.** (Fig. 67.) Take two consecutive spans AB and BC of lengths l_1 and l_2 respectively. For each span, the moment at a section is the algebraic sum of the moments as for a simple beam and that caused by the fixing moments at the supports.

The slope of the beam at the supports may or may not be zero.

Let support B be y_1 below support A

and ,, B ,, y_2 ,, C.

If B is above A or C, then the sign of y_1 or y_2 is negative.

Let EI be the same for both the spans.

(*a*) Consider beam AB and take A as origin; The moment at a section X distance x from A is $M_{sx} + M_{fx}$

M_{sx} moment as for a simple beam,

M_{fx} moment due to end fixing moments.

M_{fx} is generally of the *opposite sign* to M_{sx}; M_{sx} −; M_{fx} +;

$$EI\frac{d^2y}{dx^2} = M_{sx} + M_{fx}$$

$$\left(x\frac{dy}{dx} - y\right)_0^{l_1} = \frac{1}{EI}\int_0^{l_1}(M_{sx} + M_{fx})\,x\,.\,dx$$

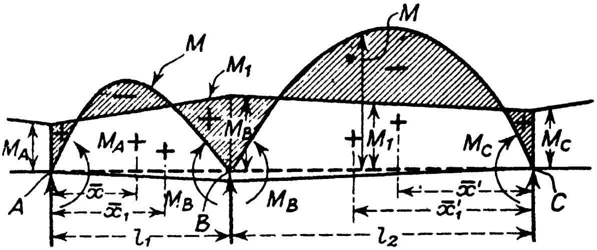


FIG. 67

i_B = slope of beam at B, EI constant

$$l_1 i_B - y_1 = \frac{A_s\bar{x} + A_f\bar{x}_1}{EI} \quad . \quad . \quad (22)$$

A_s = area of simple moment diagram and

A_f = area of fixed moment diagram;

$\bar{x}$ and $\bar{x}_1$ are the distances of the centroids of the simple and fixed moment areas from A.

(*b*) CONSIDER BEAM BC WITH C AS ORIGIN. Take x to the left as positive and y downwards as positive.

$$\left(x\,.\,\frac{dy}{dx} - y\right)_0^{l_2} = l_2 i_B - y_2 = \frac{A_s'\bar{x}' + A_f'\bar{x}_1'}{EI} \quad . \quad . \quad (22a)$$

A_s' = area simple moment diagram;

A_f' = area fixed moment diagram;

$\bar{x}'$ and $\bar{x}_1'$ are the distances of the respective centroids from C

From (22), $$i_B = \frac{A_s\bar{x} + A_f\bar{x}_1}{EIl_1} + \frac{y_1}{l_1} \quad . \quad . \quad . \quad (23)$$

From (22*a*), $$i_B = \frac{A_s'\bar{x}' + A_f'\bar{x}_1'}{EIl_2} + \frac{y_2}{l_2} \quad . \quad . \quad . \quad (23a)$$

but of *opposite sign* to i_B in equation (23) because $\bar{x}'$ and $\bar{x}_1'$ are taken in direction C towards B.

Therefore $$\left(\frac{A_s\bar{x} + A_f\bar{x}_1 + EIy_1}{l_1}\right) = -\left(\frac{A_s'\bar{x}' + A_f'\bar{x}_1' + EIy_2}{l_2}\right)$$

From Equation (8),

$$A_f\bar{x}_1 = \frac{(M_A + 2M_B)}{6}l_1^2$$

$$A_f'\bar{x}_1' = \frac{(M_C + 2M_B)}{6}l_2^2$$

$$\therefore \frac{A_s\bar{x}}{l_1} + \frac{M_Al_1^2}{6l_1} + \frac{2M_Bl_1^2}{6l_1} + \frac{EIy_1}{l_1} = -\left(\frac{A_s'\bar{x}'}{l_2} + \frac{M_Cl_2^2}{6l_2} + \frac{2M_Bl_2^2}{6l_2} + \frac{EIy_2}{l_2}\right)$$

Hence,

$$\frac{A_s\bar{x}}{l_1} + \frac{A_s'\bar{x}'}{l_2} + \frac{M_Al_1}{6} + \frac{M_B}{3}(l_1 + l_2) + \frac{M_Cl_2}{6} + EI\left(\frac{y_1}{l_1} + \frac{y_2}{l_2}\right) = 0$$

So that $$\frac{6A_s\bar{x}}{l_1} + \frac{6A_s'\bar{x}'}{l_2} + M_Al_1 + 2M_B(l_1 + l_2) + M_Cl_2 + 6EI\left(\frac{y_1}{l_1} + \frac{y_2}{l_2}\right) = 0 \quad . \quad (24)$$

and this is the general equation for a beam continuous over three supports and where EI is a constant.

If $y_1 = y_2 = 0$,

that is, supports at the same level, then

$$\frac{6A_s\bar{x}}{l_1} + \frac{6A_s'\bar{x}'}{l_2} + M_Al_1 + 2M_B(l_1 + l_2) + M_Cl_2 = 0 \quad . \quad (25)$$

Let the supports be at the same level and let a load P_1 on the span AB be distant k_1l_1 from A, and let a load P_2 on the span BC be distant k_2l_2 from C.

$$\text{Then } \frac{6A_s\bar{x}}{l_1} = P_1l_1^2(k_1 - k_1^3) \quad . \quad . \quad . \quad . \quad . \quad (25a)$$

$$\text{and } \frac{6A_s'\bar{x}'}{l_2} = P_2l_2^2(k_2 - k_2^3) \quad . \quad . \quad . \quad . \quad . \quad (25b)$$

Let a uniformly distributed load of w per unit length extend from k_nl_1 to k_fl_1 on the span AB. k_nl_1 and k_fl_1 are both measured from A.

$$\text{Then } \frac{6A_s\bar{x}}{l_1} = \int_{k_1=k_n}^{k_1=k_f} w\,.\,d(k_1l_1)(k_1 - k_1^3)l_1^2 \quad \text{where } w\,.\,d(k_1l_1)$$

corresponds to P_1 in equation (25*a*).

$$\therefore \quad \frac{6A_s\bar{x}}{l_1} = wl_1^3\int_{k_1=k_n}^{k_1=k_f}(k_1 - k_1^3)\,.\,dk_1 = wl_1^3\left[\frac{k_1^2}{2} - \frac{k_1^4}{4}\right]_{k_1=k_n}^{k_1=k_f}$$

$$= \frac{wl_1^3}{4}\left[2k_1^2 - k_1^4\right]_{k_1=k_n}^{k_1=k_f} \quad . \quad . \quad . \quad . \quad . \quad (25c)$$

If the load occupies the whole of the span $AB = l_1$

$$k_f = 1, \; k_n = 0$$

$$\text{then } \frac{6A_s\bar{x}}{l_1} = \frac{wl_1^3}{4}[2-1] = \frac{wl_1^3}{4} \quad . \quad . \quad . \quad . \quad . \quad (25d)$$

Similarly for span BC if the uniformly distributed load occupies a part or the whole of the span BC: in this case working from C.

If the type of loading on the continuous beam be known, then the terms dependent on the simple moment diagram can be easily calculated in terms corresponding to those given in equations (25*a*) to (25*b*).

If the loads are uniformly distributed over the spans, values of A_s and A_s' can be obtained (in terms of these loads and the spans) $= \frac{wl^2}{8} \times \frac{2l}{3}$.

The reactions are found by the method previously indicated in Art. 46.

Total reaction at B is

$$R_B = R_{B_{AB}} + \frac{M_B - M_A}{l_1} + R_{B_{BC}} + \frac{M_B - M_C}{l_2} \quad . \quad . \quad (26)$$

$R_{B_{AB}} =$ Reaction at B considering AB as a simple beam

$R_{B_{BC}} =$ „ „ BC „

and similarly for A and C making allowance for the fixing moments necessary.

If the moments of inertia are different and I_1 is the moment of inertia of beam of length l_1 and I_2 is the moment of inertia of beam of length l_2 and E is the same for both beams, then eqn. (24) will read—

$$\frac{6A_s\bar{x}}{l_1I_1} + \frac{6A'_s\bar{x}'}{l_2I_2} + \frac{M_Al_1}{I_1} + 2M_B\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right) + \frac{M_Cl_2}{I_2} + 6E\left(\frac{y_1}{l_1} + \frac{y_2}{l_2}\right) = 0$$

53. **Deflections of Continuous Beams of Uniform Section.** (Fig. 68.) The mathematical method of Mr. W. H. Macaulay is easily applied to the case of continuous beams.

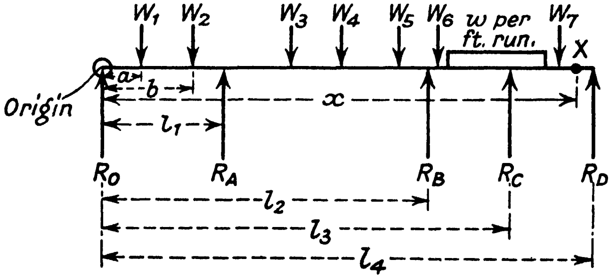


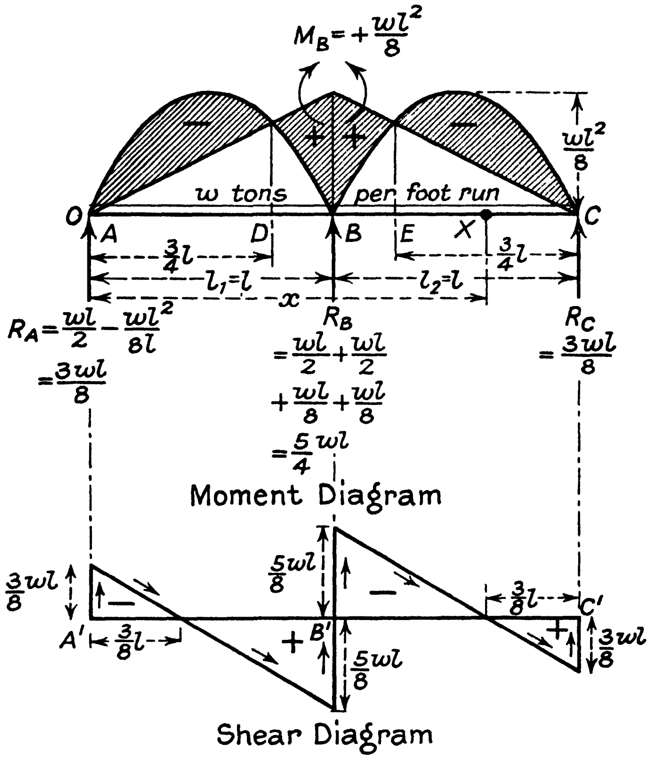
FIG. 68

Take the left-hand support as the origin.

The true reactions R_o, R_A, R_B, R_C, R_D . . . R_n and R_{n+1} have all been obtained by some method, and are, therefore, known: *they include the fixing moment effect.*

Take a section X in the space between the last load and the last support. The spans are l_1, l_2, l_3 . . . l_n

$$EI\frac{d^2y}{dx^2} = -R_ox + W_1(x-a) + \ldots W_n(x-n)$$
$$-R_A(x-l_1) - R_B(x-l_2) - \ldots - R_n(x-l_{n-1})$$



Note - *Shear changes sign at 3 places, indicating two maximum negative moments (one for each span) and one maximum positive moment*

FIG. 69

$$EIy = \frac{-R_o x^3}{6} - R_A\frac{(x - l_1)^3}{6} - R_B\frac{(x - l_2)^3}{6} - \dots - R_n\frac{(x - l_{n-1})^3}{6}$$

$$+ \frac{W_1}{6}(x - a)^3 + \dots + \dots + \frac{W_n}{6}(x - n)^3 + \dots$$

$$+ Ax + B \qquad \dots\dots\dots \qquad (26a)$$

$$x = 0, \qquad y = 0, \qquad B = 0$$

To find A, take the conditions regarding y as for the first support distant l_1 from the origin.

Substitute the value of A in equation (26*a*) and the general equation for y at a section for an irregularly-loaded continuous beam is found. If there is a uniformly-distributed load on a portion of the beam, then it must be treated as indicated in paragraph 30, Chapter III. If at the support at the origin there is a fixing moment (M_o), then this term will appear in the fundamental equation

$$EI\frac{d^2y}{dx^2} = -R_o x - \ldots - R_n(x - l_{n-1}) + \ldots + W_n(x - n) + M_o.$$

The foregoing is a good method when dealing with concentrated loads only when working with uniformly-distributed loads, which vary for different spans, take each span separately and work as shown in Problem (17) by the theorem of three moments.

Illustrative Problem 15.

(1) Beam continuous over two spans of length l and loaded with a uniformly-distributed load of w tons per unit length. All supports at the same level. Find the maximum moment and where it occurs, the point of inflection for each span, and the maximum deflection. (Fig. 69.)

The origin will be at A.

$$M_A = M_C = 0$$

$$l_1 = l_2 = l$$

Using Equation (25),

$$2M_B(l + l) = 2 \times \frac{6}{l}\left(\frac{wl^2}{8} \times \frac{2}{3}l \times \frac{l}{2}\right)$$

$$= +\frac{wl^3}{2}$$

$$\therefore M_B = \frac{wl^2}{8} \quad . \quad . \quad . \quad . \quad (27)$$

$$R_A = \frac{wl}{2} - \frac{wl}{8} = \frac{3wl}{8}$$

$$R_C = \frac{3wl}{8}$$

$$R_B = \frac{wl}{2} + \frac{wl}{8} + \frac{wl}{2} + \frac{wl}{8} = \frac{5}{4}wl$$

The maximum positive moment is $M_B = +\frac{wl^2}{8}$

There are also two maximum negative moments, one for each span, occurring at the points of zero shear shown, which are at distances of $\frac{3}{8}l$. from the end supports.

Take a section X in the second span, distant x from A

$$EI\frac{d^2y}{dx^2} = -\frac{3wl}{8}x + \frac{wx^2}{2} - \frac{5wl}{4}(x - l) \qquad . \quad . \quad (28)$$

For any section, X in the first span,

$$EI\frac{d^2y}{dx^2} = \frac{-3wlx}{8} + \frac{wx^2}{2} \qquad . \quad . \quad . \quad (28a)$$

A maximum when $x = \frac{3}{8}l$

$$\text{At } x = \frac{3l}{8}, \quad M_x = -\frac{3wl}{8} \times \frac{3l}{8} + \frac{w}{2}\left(\frac{3l}{8}\right)^2$$

$$= -\frac{9wl^2}{128} = \text{maximum negative moment}$$

Integrating equation (28), for any section in the second span

$$EIy = \frac{-3wlx^3}{48} + \frac{wx^4}{24} - \frac{5wl(x - l)^3}{24}$$
$$+ Ax + B \qquad . \quad . \quad . \quad (28b)$$

$$x = 0, \qquad y = 0, \qquad B = 0$$

$$x = l, \qquad y = 0, \qquad A = \frac{wl^3}{48}$$

For any section in the first span,

$$EIy = \frac{-3wlx^3}{48} + \frac{wx^4}{24} + \frac{wl^3x}{48} \qquad . \quad . \quad (28c)$$

$$\text{Maximum value of } y \text{ when } \frac{dy}{dx} = 0$$

$$\text{Point of inflection when } \frac{d^2y}{dx^2} = 0$$

and is at the section where the moment is zero.

Differentiating (28c)

$$EI\frac{dy}{dx} = -\frac{9wl}{48}x^2 + \frac{wx^3}{6} + \frac{wl^3}{48} = 0$$

$$-9lx^2 + 8x^3 + l^3 = 0 \quad . \quad . \quad (28d)$$

Solving (28d) $x = l$ or $x = \cdot 42l$.

$$\text{when } x = l, \quad \frac{dy}{dx} = 0, \quad y = 0 : \text{a minimum}$$

$$\text{when } x = \cdot 42l, \quad \frac{dy}{dx} = 0, \quad y \text{ is a maximum}$$

$$EIy_{max} = -\frac{wl^4}{16} \cdot \cdot 075 + \frac{wl^4 \cdot 0316}{24} + \frac{wl^4 \cdot 42}{48} = \frac{wl^4}{186}$$

To find the points of inflection

$$EI\frac{d^2y}{dx^2} = -\frac{9wlx}{24} + \frac{wx^2}{2} = 0$$

$$-\frac{9lx}{24} = -\frac{12x^2}{24}$$

$$12x = 9l$$

$$\therefore x = \tfrac{3}{4}l$$

The point of inflection for each span is at a distance $\frac{3}{4}l$ from the free supports. The deflections for the second span may be obtained from equation (28b); they are the same as for the first span, working to the left from the free support C as origin.

Illustrative Problem 16.

A girder continuous over two spans each of length 20 ft. carries a uniformly-distributed load of 1 ton per foot run over the whole length of 40 ft. The girder is simply supported at the ends. The centre support is ·1 foot below the left-hand support and ·05 foot below the right-hand support. Take $EI = 40{,}000$ (feet)^2 ton units. Find the bending moment at the centre support and the maximum negative moments in the bays.

(1) If all the supports were at the same height, the moment at the centre support would be

$$M_B = +\frac{wl^2}{8} \text{ where } l = \text{length of one span}$$

$$= +\frac{400}{8} = +50 \text{ tons-ft.}$$

The moment at any section in the first bay (supports same height),

$$M_x = \left(-10 + \frac{50}{20}\right)x + \frac{x^2}{2}; \quad \left(10 - \frac{50}{20}\right)\text{tons} = \text{true reaction at the left support}$$

$$= -7{\cdot}5x + {\cdot}5x^2$$

$$\frac{dM_x}{dx} = -7{\cdot}5 + x = 0 \qquad \therefore\ x = 7{\cdot}5 \text{ ft.}$$

i.e. the moment is a maximum negative at a section distant 7·5 ft. from the origin.

The maximum negative moment

$$= -7{\cdot}5^2 + \tfrac{1}{2} \times 7{\cdot}5^2 = -28 \text{ tons-ft.}$$

(2) With the supports at the different levels,

$$2M_B(2l) = 2 \times 6\left(\frac{wl^2}{8} \times \frac{2}{3}l \times \frac{l}{2} \times \frac{1}{l}\right) - 6EI\left(\frac{y_1}{l} + \frac{y_2}{l}\right)$$

where y_1 = deflection centre support below the left-hand support ;

,, y_2 = deflection centre support below the right-hand support ;

$$80M_B = 4000 - 6 \times 40{,}000\left(\frac{{\cdot}1}{20} + \frac{{\cdot}05}{20}\right)$$

$$= 2200$$

$$M_B = +27{\cdot}5 \text{ tons-ft.}$$

The moment at any section in the first span is

$$M_x = \left(-10 + \frac{27{\cdot}5}{20}\right)x + \frac{x^2}{2} = -8{\cdot}62x + {\cdot}5x_2$$

For a maximum $\frac{dM_x}{dx} = 0 = -8{\cdot}62 + x = 0$

solving $x = 8{\cdot}62$ ft. from the origin.

The maximum negative moment is

$$-8{\cdot}62^2 + {\cdot}5 \times 8{\cdot}62^2$$

$$= -37{\cdot}2 \text{ tons-ft.}$$

In the second span working from the right support as origin, the results will be the same.

	M_B	Maximum negative moment for both spans and the position at which it occurs.
Supports all same level.	+ 50 tons-ft.	− 28 tons-ft. at 7·5 ft. from the left and right supports respectively.
At the different levels.	+ 27·5 tons-ft.	− 37·2 tons-ft. at 8·62 ft. from the supports respectively.

This example shows that if the props sink varying amounts, then the moments at the supports vary, and consequently the maximum negative moments in the span; if, say, the centre support sank a very great deal, the moment at the support would perhaps become a negative moment until, if the support was entirely removed, the girder would become a simply supported beam of 40 foot span, with a maximum negative moment at the centre,

$$= -\frac{40^2}{8} = -200 \text{ tons-ft.}$$

Illustrative Problem 17. (Fig. 70).

A continuous girder covers three spans: *AB*, 30 ft.; *BC*, 40 ft.; and *CD*, 20 ft. The uniformly-spread loads are 2, 1, and 3 tons per foot run on *AB*, *BC*, and *CD* respectively. If the girder is of the same cross-section throughout, find the moments at the supports *B* and *C*, and the pressures on the supports. Find the maximum negative moments for each span, the maximum deflections, and the points of inflection. Also the slopes of the beams at the supports.

$$2M_B(30+40) + M_C(40) = -\frac{6}{30}\left(-\frac{2\times 30^2}{8}\times\frac{30}{3}\times 2\times\frac{30}{2}\right)$$

$$-\frac{6}{40}\left(-\frac{40^2}{8}\times\frac{2}{3}\times 40\times\frac{40}{2}\right)$$

$$M_B\times 40 + 2M_C(40+20) = -\frac{6}{40}\left(-\frac{40^2}{8}\times\frac{2}{3}\times 40\times\frac{40}{2}\right)$$

$$-\frac{6}{20}\left(-\frac{3\times 20^2}{8}\times\frac{2}{3}\times 20\times\frac{20}{2}\right)$$

$$140M_B + 40M_C = 15\times 30^2 + 10\times 40^2$$

$$40M_B + 120M_C = 15\times 20^2 + 10\times 40^2$$

$$
\begin{aligned}
3{\cdot}5M_B + \quad M_C &= 737{\cdot}5 \\
M_B + \quad 3M_C &= 550{\cdot}0 \\
M_B &= 175 \text{ tons-ft.} \\
M_C &= 125 \quad \text{,,}
\end{aligned}
$$

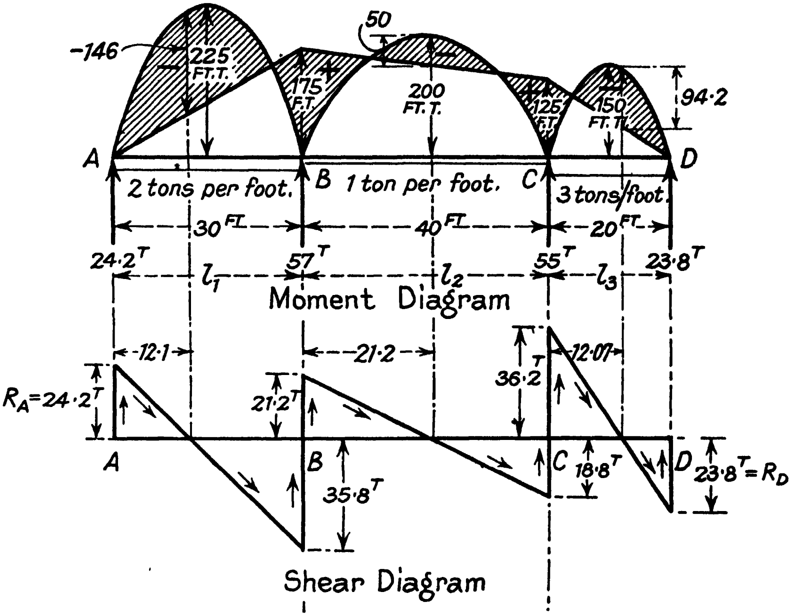


FIG. 70

$$R_A = \frac{2 \times 30}{2} - \frac{175}{30} = 24{\cdot}2 \text{ tons}$$

$$R_B = \frac{2 \times 30}{2} + \frac{175}{30} + \frac{40}{2} + \frac{175 - 125}{40}$$

$$= 57 \text{ tons } (35{\cdot}8 \text{ and } 21{\cdot}2)$$

$$R_C = \frac{40}{2} - \frac{175 - 125}{40} + \frac{20 \times 3}{2} + \frac{125}{20}$$

$$= 55 \text{ tons } (18{\cdot}8 \text{ and } 36{\cdot}2)$$

$$R_D = \frac{3 \times 20}{2} - \frac{125}{20} = 23{\cdot}80 \text{ tons}$$

The maximum moments occur wherever the shear force diagram changes sign.

Maximum negative moments occur in the bays at distances 12·1 ft. from A, 21·2 ft. from B, and 12·07 ft. from C.

Maximum positive moments at the supports,

$$M_B = 175 \text{ tons-ft.}, \; M_C = 125 \text{ tons-ft.}$$

Maximum negative moments,

$$\text{1st bay} = -\frac{24{\cdot}2 \times 12{\cdot}1}{2} = -146 \text{ tons-ft.}$$

$$\text{2nd bay} = 175\,(M_B) - \frac{21{\cdot}2 \times 21{\cdot}2}{2} = 175 - 225$$

$$= -50 \text{ tons-ft.}$$

$$\text{3rd bay (working from } D) = -\frac{23{\cdot}8 \times 7{\cdot}93}{2}$$

$$= -94{\cdot}2 \text{ tons-ft.}$$

The moment at any section is found by taking the area of the shear diagram from the origin to the section taken.

DEFLECTIONS

First Span. Using the correct reaction at the origin (A), (for no fixing moment at (A))

$$EI\frac{d^2y}{dx^2} = -24{\cdot}2x + \frac{2x^2}{2} \text{ for any section } X \text{ distant } x \text{ from } A$$

$$EIy = -24{\cdot}2\frac{x^3}{6} + \frac{2x^4}{24} + A_1x + B$$

when $x = 0$, $y = 0$, $B = 0$

„ $x = l = 30$ ft., $y = 0$

then $A_1 = +1377$, i.e. the slope at A is $\dfrac{1377}{EI}$

$$\therefore\; EIy = -\frac{24{\cdot}2x^3}{6} + \frac{x^4}{12} + 1377x$$

y_{max} for 1st span—

$$EI\frac{dy}{dx} = -12{\cdot}1x^2 + \frac{x^3}{3} + 1377 = 0$$

$$x = 13{\cdot}5 \text{ feet (by trial).}$$

$\therefore EIy_{max} = 11{,}527.$ EI in tons-feet units. y_{max} in feet.

At a point of inflection,

$$EI\frac{d^2y}{dx^2} = -24{\cdot}2x + x^2 = 0$$

$$\text{i.e. } x = 24{\cdot}2$$

$$\therefore \frac{dy}{dx} \text{ is a maximum } 24{\cdot}2 \text{ ft. from } A$$

Slope at A, $x = 0$, $EI\frac{dy}{dx} = 1377$

,, $x = 24{\cdot}2$, $EI\frac{dy}{dx} = -1000$

,, B, $x = 30$, $EI\frac{dy}{dx} = -513$

For the third span (all supports being at the same level), take D as origin, and taking x to the left positive and y downwards positive,

$$EI\frac{d^2y}{dx^2} = -23{\cdot}8x + \frac{3x^2}{2}$$

$$EIy = \frac{-23{\cdot}8x^3}{6} + \frac{3x^4}{24} + Ax + B;$$

when $x = 0$; $y = 0$ $\therefore B = 0$.

When $y = 0$; $x = 20$ and A becomes $= 585$

$$EIy = \frac{-23{\cdot}8x^3}{6} + \frac{3x^4}{24} + 585x$$

To find the maximum deflection in the third bay,

$$EI\frac{dy}{dx} = -11{\cdot}9x^2 + \frac{x^3}{2} + 585$$

$$= 0 \text{ for maximum.}$$

By trial $x = 9$ ft.

Max. deflection. $EIy_{max} = 3420$

The point of inflection is when $\frac{d^2y}{dx^2} = 0$

$$\frac{d^2y}{dx^2} = -23{\cdot}8x + 1{\cdot}5x^2 = 0$$

$$x = \frac{23{\cdot}8}{1{\cdot}5} = 15{\cdot}8 \text{ ft.}$$

The slope changes sign at the section 15·8 ft.

Note.—The slopes of the beam in this span will be of the opposite sign from those in the first span.

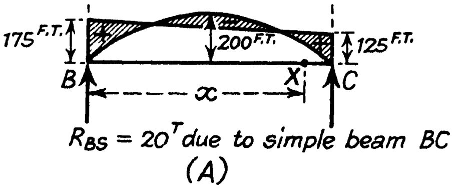


Fig. 70a

For the Second Span. It has been found better to work as follows. Take the origin at B; the reaction at B will be the one for the span simply supported. (Fig. 70a.)

Moment at any section X, x from B

$$= EI\frac{d^2y}{dx^2} = M_x = -20x + \frac{x^2}{2} + 175 - \frac{(175-125)}{40}x$$

$$= \frac{x^2}{2} - 21{\cdot}25x + 175 \quad . \quad . \quad . \quad (29)$$

When $M_x = 0$. $x = 11{\cdot}65$ or $31{\cdot}8$,

thus there is no moment at sections, these distances from B.

Integrating Equation (29),

$$EI\frac{dy}{dx} = \frac{x^3}{6} - 10{\cdot}6x^2 + 175x + A$$

when $x = 0$, $EIi_B = -513$ from the first span,

$$EI\frac{dy}{dx} = \frac{x^3}{6} - 10{\cdot}6x^2 + 175x - 513 \quad . \quad . \quad . \quad (30)$$

Giving x various values, it is found $\frac{dy}{dx} = 0$ when

$$x = 3{\cdot}8,\ 21{\cdot}6,\ \text{and}\ 38{\cdot}1\ \text{ft.}$$

Between $x = 0$ and $3{\cdot}8$, the slopes are negative
,, $3{\cdot}8$,, $21{\cdot}2$,, positive
,, $21{\cdot}2$,, $38{\cdot}1$,, negative
,, $38{\cdot}1$,, 40 ,, positive

$$x = 40. \quad EIi_{40} = EIi_c = +190$$

Integrating Equation (30),

$$EIy = \frac{x^4}{24} - \frac{10{\cdot}6x^3}{3} + \frac{175x^2}{2} - 513x + C$$

When $x = 0$, $y = 0$; therefore $C = 0$.
Then the deflection at any section is

$$EIy = +\frac{x^4}{24} - \frac{10{\cdot}6x^3}{3} + \frac{175x^2}{2} - 513x \quad . \quad . \quad (31)$$

$$\begin{aligned} \text{At } x &= 3{\cdot}8, EIy = -\ 876 \\ x &= 21{\cdot}6, EIy = +\ 14{,}630 \\ x &= 38{\cdot}1, EIy = -\ 1500 \end{aligned}$$

For this span, therefore, two portions of the beam deflect *above* the horizontal.

Illustrative Problem 18.

A beam is continuous over three spans of 100 ft., 100 ft., and 60 ft. The ends are both fixed horizontally. The beam is loaded along its whole length with a uniformly-distributed load of 1 ton per foot run. (Fig. 71.)

Find the fixing moments and the moments at the supports; also all the reactions.

For the first span and treating B as origin,

$$2M_A + M_B + \frac{6A_S(l - \bar{x})}{l^2} = 0^* \quad (A \text{ negative})$$

($\bar{x}$ = distance centroid simple moment diagram from A)

---

* $(0) + 2M_A(l + 0) + M_B(l) = -\frac{6A_S}{l}(l - \bar{x})$. {$\bar{x}$ from A.}

or, as AB is direction-fixed at A, imagine another beam B_1A to the left similar to AB and loaded similarly.

Then $M_{B_1} = M_B$ and we shall have

$$M_Bl + 4lM_A + M_Bl = -2 \times \frac{6A_S}{l}(l - \bar{x}).$$

(*See* Equation (9) for beams fixed at two ends.)

1st and 2nd spans :

$$M_A l + 2M_B(l + l_1) + M_C l_1 = -\frac{6A\bar{x}}{l} - \frac{6A_1\bar{x}_1}{l_1}$$

$$(\bar{x}_1 \text{ from } C)$$

2nd and 3rd spans :

$$M_B l_1 + 2M_C(l_1 + l_2) + M_D l_2 = -\frac{6A_1(l - \bar{x}_1)}{l_1} - \frac{6A_2\bar{x}_2}{l_2}$$

$$(\bar{x}_2 \text{ from } D, \quad \bar{x}_1 \text{ from } C)$$

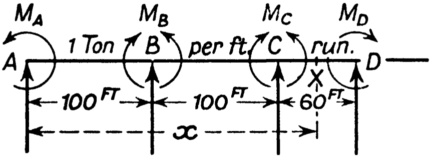


FIG 71

3rd span : Treating C as origin, and the beam as a beam fixed at two ends,

$$2M_D + M_C + \frac{6A_2(l_2 - \bar{x}_2)}{l_2^2} = 0. \quad (A_2 \text{ negative.})$$

Simplifying and solving the equations (1) to (4),

$M_A = +\ 807$ tons-ft.
$M_B = +\ 886$ „
$M_C = +\ 650$,
$M_D = +\ 125$ „

Reactions at the supports (working by the rules laid down) are—

$R_A = 48{\cdot}55$ tons ; $R_B = 104{\cdot}4$ tons ; $R_C = 85{\cdot}4$ tons;
$R_D = 21{\cdot}65$ tons.

Illustrative Problem 18*a*.

A beam is continuous over two spans of 21 ft. and 21 ft. It carries loads of 1 ton every 6 ft. from the left-hand support. Calculate the deflections of the beam at the load points : *EI* = constant in tons (ft.)² units. The end supports are free. (Fig. 71A.) Supports all at the same level.

The fixing moments at *A*, *C*, and *B* are 0, 0, and + 9·2 ft.-tons respectively.

The total reactions at *A*, *B*, and *C* are ·848, 4·304, and ·848 tons respectively.

Moment for a general section *X*, *x* ft. from *A*, and between the last load and *C* is

$$M_x = \frac{EId^2y}{dx^2} = -\cdot848x + 1(x-6) + 1(x-12) + 1(x-18)$$
$$-\,4\cdot304(x-21) + (x-24) + (x-30) + (x-36)$$

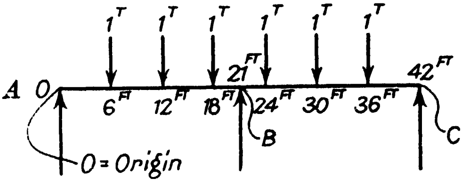


FIG. 71A

$$\therefore\ EIy = -\cdot848\frac{x^3}{6} - \frac{4\cdot304(x-21)^3}{6} + \frac{(x-6)^3}{6} \ldots + \frac{x-36^3}{6}$$
$$+ Ax + B$$

$$x = 0,\ y = 0,\ B = 0. \quad x = 21 \text{ ft.} \quad y = 0\ ;$$

$$\therefore\ 0 = -\cdot848 \times \frac{21^3}{6} + 21A. \quad A = 29\cdot7$$

Deflections at the load points,

$$EIy_6 = -\cdot848 \times \frac{6^3}{6} + 29\cdot7 \times 6 = +\,147\cdot6$$

$$EIy_{12} = -\cdot848 \times \frac{12^3}{6} + \frac{1 \times 6^3}{6} + 29\cdot7 \times 12 = +\,148\cdot2$$

$$EIy_{18} = -\frac{\cdot848 \times 18^3}{6} + \frac{12^3}{6} + \frac{6^3}{6} + 29\cdot7 \times 18 = +\,32\cdot8$$

$$EIy_{24} = -\frac{\cdot848 \times 24^3}{6} - \frac{4\cdot304 \times 3^2}{6} + \frac{18^3}{6} + \frac{12^3}{6} + \frac{6^3}{6}$$
$$+ 29\cdot7 \times 24 = +\,32\cdot3$$

y_{18} and y_{24} for the given loading should obviously be the same, the difference in the calculation being due to the limit of accuracy of calculating the constant *A* and the reactions. Working from *C* as origin, the deflections for *CB* will, however, correspond to the deflection of *AB* for the given loading.

54. **Propped Beams.** METHOD OF SOLUTION. (i) With a *non-elastic* prop to be of the same height as the supports in the case of a beam and the fixed point in the case of a cantilever, treat as a beam or cantilever without a prop, and calculate the deflection at the position of the prop. The prop then forces the beam or cantilever upwards until it is level with the supports : thus treat the beam or cantilever as simply loaded with the prop-force which causes the known deflection. Equate to the necessary deflection formulae, and the load in the prop will be obtained. (ii) For an *elastic* prop, the level of it will be below the support levels, depending on its elasticity.

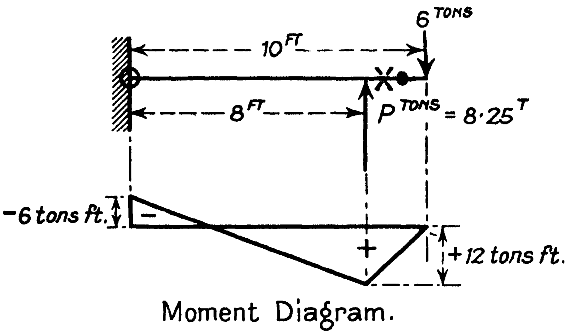


FIG. 72

If the prop is to be below the supports, treat the beam as having one concentrated load causing a deflection, which is less than the deflection of the beam without the prop by the distance below the supports.

Illustrative Problem 18*b*. Cantilever loaded as in Fig. 72. To find the reaction P.

$$EI\frac{d^2y}{dx^2} = 6(10 - x)$$

$$EI\frac{dy}{dx} = 60x - 3x^2 + A$$

$$x = 0, \qquad \frac{dy}{dx} = 0 \quad \therefore \quad A = 0$$

$$EIy = 30x^2 - x^3 + B$$

$$x = 0, \qquad y = 0, \quad \therefore \quad B = 0$$

$$EIy_8 = 30 \times 8^2 - 8^3 = 1408$$

(*a*) P acting alone causes a deflection upwards at section 8 ft.

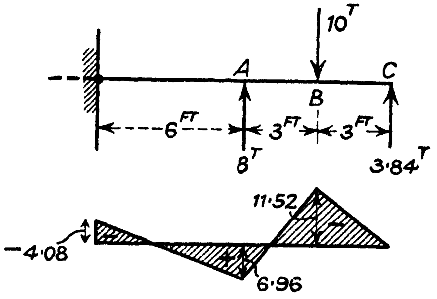
$$= \frac{1408}{EI}$$

By a previous formula, this will be $\frac{Wl^3}{3EI}$

$$= \frac{P \cdot 8^3}{3EI}$$

$$1408 = \frac{P \times 512}{3}$$

$$P = \frac{1408 \times 3}{512} \text{ tons} = 8{\cdot}25 \text{ tons}$$



Moment Diagram.
(*Foot tons.*)

FIG. 73

(*b*) If the prop is non-elastic, at a distance of $\frac{100}{EI}$ below the fixed point,

$$\frac{100}{EI} = \frac{1408}{EI} - \frac{P_1 \times 8^3}{3EI}$$

$$P_1 = \frac{1308 \times 3}{512} = 7{\cdot}68 \text{ tons}$$

Illustrative Problem 18*c*. A beam is fixed at one end and only free to move in a slide in a horizontal direction at the other. Forces of 8 tons and 10 tons act in opposite directions on it at distances of 6 ft. and 9 ft. from the fixed end. Find the reaction on the slide and draw the bending moment diagram. (Fig. 73.)

The deflection at the end of the beam with only the 10-ton load acting, using the general method for the deflection of a cantilever, and where $x = 12$ ft. and $n = 9$ ft.

$$EIy = \frac{10(9-12)^3}{6} + \frac{10 \times 9^2 \times 12}{2} - \frac{10 \times 9^3}{6}$$

$$= 10 \times 81 \times 6 - 10 \times 81 \times 1{\cdot}5$$

$$= 3645$$

neglecting the first term as $(9-12)$ is negative.

With only the 8-ton load acting,

$$EIy = \frac{8 \times 6^2 \times 12}{2} - \frac{8 \times 6^3}{6}$$

$$= 8 \times 36 \times 6 - 8 \times 36$$

$$= 1440$$

therefore the slide reaction will act in the same direction as the 8-ton load.

To find the reaction. Let it be P tons. By the method of superposition

$$EIy = 3645 = 1440 + \frac{Pl^3}{3}$$

$$3645 = 1440 + P \times \frac{144 \times 12}{3}$$

$$576\ P = 2205$$

$$P = 3{\cdot}84 \text{ tons}$$

55. A beam of length $2l$ ft. carries a uniformly distributed load of w tons per foot run. It is supported at the centre by a non-elastic prop level with the supports.

Find the loads carried by the prop and the non-elastic supports. (Fig. 74.)

Without the support, the deflection at the centre is

$$y_c = \frac{5w(2l)^4}{384\,EI}$$

Imagine the beam as a simple beam carrying a prop load acting upwards; the deflection upwards

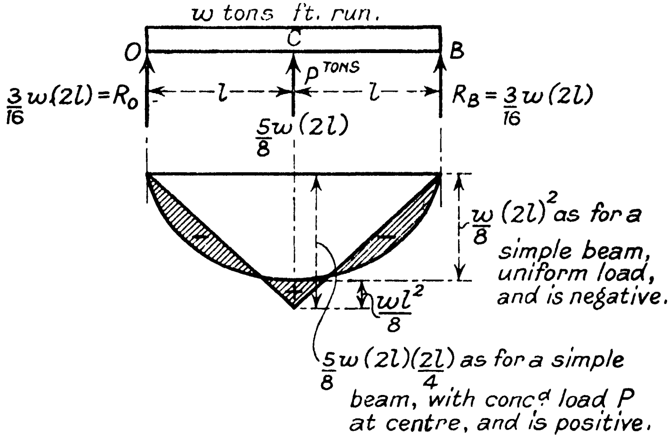
$$= \frac{5w(2l)^4}{384\,EI}$$

$$P = \text{load on prop}$$

$$\text{then } y_c \text{ due to prop} = \frac{P(2l)^3}{48\,EI}$$

$$\text{therefore } \frac{P(2l)^3}{48\,EI} = \frac{5w(2l)^4}{384\,EI}$$

$$\text{or } P = \tfrac{5}{8}w\,.\,(2l\,.\,) \qquad . \quad . \quad . \qquad (32)$$


w tons ft. run.
O
C
B
P TONS
3/16 w(2l) = R_O
R_B = 3/16 w(2l)
l
l
5/8 w(2l)
w/8 (2l)^2 as for a simple beam, uniform load, and is negative.
wl^2/8
5/8 w(2l)(2l/4) as for a simple beam, with concd load P at centre, and is positive.

Shaded Diagram is Resultant Moment Diagram.

FIG. 74

Reaction at the supports when the beam is propped

$$= \tfrac{3}{16}w\,.\,(2l\,.\,) \qquad . \quad . \quad . \qquad (33)$$

The moment diagram is negative, due to uniform load acting on a simple beam; then the moment diagram is positive due to prop load.

The effective moment at the centre

$$= +\frac{5w}{32}(2l)^2 - \frac{w}{8}(2l)^2 = +\frac{1}{32}w(2l)^2 = +\frac{wl^2}{8}$$

This propped beam is a case of a beam continuous over two equal spans which is discussed in this chapter.

Problem 18*c* can also be solved by the Theorem of Three Moments.

Illustrative Problem 18*d*.

Calculate the prop load P at A so that the deflection of A is zero, for the system given in Fig. 75. Find also the moment at B. EI = Constant.

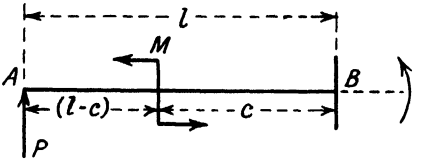


FIG. 75

AB is a beam direction fixed at B and simply supported at A. M is a couple applied at a point distant c from B.

$$EI\delta_A = Mc\left(l - \frac{c}{2}\right)$$

when considering the prop missing at A;

$$EI\delta_A = Pl^3/3$$

when the prop load P is at A and M missing.

$$\text{For} \qquad \delta_A = 0,\ Pl^3/3 = Mc\left(l - \frac{c}{2}\right)$$

$$\therefore \qquad P = \frac{3Mc}{l^3}\left(l - \frac{c}{2}\right)$$

$$\text{If} \quad c = l, \text{ then } P = \frac{3Ml^2}{2l^3} = \frac{3}{2}\frac{M}{l}$$

If $c = l$ the moment at B is M_B and it is equal to

$$M - \frac{3}{2}M = -\frac{M}{2}$$

Therefore if there is no displacement of the point A when the moment M is applied at A, then the moment at B is one-half the applied moment at A, but of the opposite sign.

If $c = l$, the slope of the tangent to the beam at A is

$$i_A = \frac{1}{EI}\left(Ml - \frac{1}{2}Pl^2\right) = \frac{1}{EI}\left(Ml - \frac{3}{4}Ml\right)$$
$$= \frac{Ml}{4EI}.$$

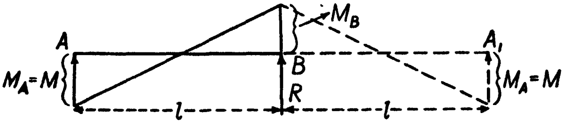


FIG. 76

Assuming the couple M is applied at the point A, the solution for the moment at B by the Theorem of Three Moments (*see* Fig. 76) is given by the following method—

By Theorem of Three Moments.

The span BA_1 is made a reflection of the span AB. B, A and A_1 are at the same level.

By the theorem,

$$M_A l + 2M_B(l + l) + M_A l = 0$$
$$\therefore \quad 2M_A = -4M_B$$
$$\therefore \quad M_B = -\frac{M_A}{2}$$

M_A is equal to M in the previous solution and therefore $M_B = -\frac{M}{2}$ when M is applied at A.*

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* Refer to pages 375 and 376 regarding another sign convention used for the slope-deflection and moment-distribution methods of analysis.

EXAMPLES

1. A continuous compound girder is carried across two spans, each of which is 20 ft., the three supports being at the same level. There is a uniformly-distributed load of 8 tons per foot run, including the weight of the girder. The moment of inertia of the girder section is 12,932 in inch units, and the depth is 26 in. Calculate the maximum intensity of flange stress, neglecting the effect of the rivet holes. Make a dimensioned sketch diagram showing the distribution of bending moment on the girder, and find the position of zero bending moment. (I.M.E.)

2. A continuous beam has three spans, the outside spans being each 15 ft. and the central span 20 ft. Draw the B.M. and shear diagrams, if the load on the central and one outside span is $1\frac{1}{2}$ tons per foot run and on the third span is $\frac{1}{2}$ ton per foot run. (I.Struc.Eng.)

3. A steel bar 2 in. wide and 4 in. deep is fixed rigidly at the ends; its effective length being 6 ft. It is subjected to a uniformly-distributed load of 4000 lb. and, in addition, there is a concentrated load of 2000 lb. at the centre, both loads acting in the plane of its depth. Calculate the bending moment at the ends and at the centre of the bar, also find the maximum intensity of stress.

4. Write down the theorem of three moments when the spans are of unequal length and are loaded with the same uniformly-distributed load. Apply it to find—

(*a*) The value of the maximum negative bending moment in a beam with fixed ends.

(*b*) The maximum negative bending moment in a beam with one end fixed and the other freely supported. (I.C.E.)

5. Use the theorem of three moments to prove that, in a beam uniformly loaded and supported at its two extremities, and continuous over an intermediate pier at its centre at the same level as the two supports, the load taken by this pier is five-eighths of the total load on the beam. (I.C.E.)

6. A beam of span 10 ft. is rigidly fixed horizontally at both ends. If a uniform load of 1 ton per foot run, including the weight of the beam, is applied, and the central deflection of the beam is found to be $\frac{1}{4}$ in., given $E = 12,500$ tons per square inch, calculate—

(*a*) The fixing moment.

(*b*) The moment of inertia of the beam. (U. of B.)

7. A continuous beam *ABCDE*, 50 ft. long, is supported on four props, at *A*, *B*, *C*, and *D*. The beam overhangs at *D*. *AB*, *BC*, *CD* are 15 ft., 20 ft., and 10 ft. respectively. Each of these spans carries a uniformly-distributed load. On the 15 ft. span, 2 tons; on the 20 ft. span, 1 ton; and on the 10 ft. span, and the overhung portion 3 tons per foot run. Determine the reaction and bending moment at each of the four props. Sketch roughly to scale complete bending moment and shear diagram. (U. of L.)

8. A cantilever 12 ft. long carries a uniformly-distributed load of 2 tons per foot run. A prop is inserted at a point 3 ft. from the free end so that the cantilever at this point is level with the built-in end. Find the load on the prop. (U. of B.)

9. A steel joist has one end *A* built horizontally into a vertical wall, the overhanging portion forming a cantilever 30 ft. long, and carrying a uniformly-distributed load of 500 lb. per foot. A tie rod is attached to the outer end *B* of the joist and is anchored to a point in the face of the wall 16 ft. vertically above the joist. If the tie rod is so adjusted that *B* is at the same level as *A* when the load is on the joist, find the tension in the tie rod, the resultant reaction and the bending moment at *A*. (U. of L.)

CHAPTER V

DISTRIBUTION OF SHEAR STRESS

56. Definitions. SHEAR STRESS exists between two parts of a body in contact when the two parts exert equal and opposite forces on each other in a direction tangential to their surface of contact.

Let a = area of contact in square inches;
P = total tangential or shear force;
q = intensity of shear stress in tons per sq. in.;

$$q = \frac{P}{a} \text{ tons/sq. in.} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

SHEAR STRAIN is alteration of shape due to shear stress. Considering the side CD fixed, Fig. 77, a square face $ABCD$ of a piece of material under simple shear will suffer a strain as indicated by the new shape $CA'B'D$.

AA' is extremely small, and it practically coincides with the arc of a circle of radius CA with C as centre.

The shear strain = θ radians = angle through which the edge CA has rotated on application of the shear stress.

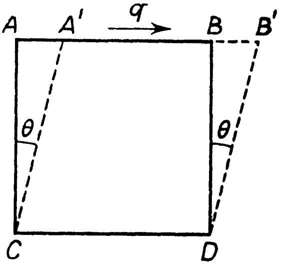


FIG. 77

$$\theta \text{ (very small)} = \frac{AA'}{AC} \text{ radians} \quad . \quad . \quad . \quad (2)$$

The *Modulus of Rigidity*, or shearing modulus, is the modulus expressing the relation between the intensity of shear stress and the amount of shear strain. It will be denoted by the letter G and has dimensions of force per unit area.

Let q = intensity of shear stress in tons per square inch.

Shear stress = shear strain × G

$$q = \theta G$$

$$G \text{ (tons per sq. in.)} = \frac{q}{\theta} = \frac{\text{shear stress}}{\text{shear strain}} \quad . \quad (3)$$

For steel, G is about $\frac{2}{5}E$, i.e. if $E = 30 \times 10^6$ lb. per sq. in. then $G = 12 \times 10^6$ lb. per sq. in.

57. **State of Simple Shear.** $ABCD$ is a rectangular block of unit thickness to the plane of the paper. (Fig. 78.)

A shear stress q is applied to the surface $AB \times 1$, then along CD there will be an equal and opposite intensity of shear stress q.

Total shear force on each face $= qAB = qCD$.

These forces acting alone would tend to rotate the block, the turning couple being $qAB \cdot BC$ units.

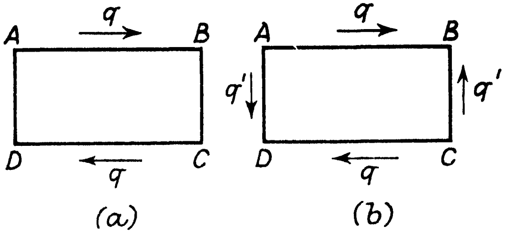


Fig. 78

For equilibrium, therefore, there must be some couple of equal magnitude, but acting in the opposite direction.

Hence tangential forces along AD and BC.

Let the tangential or shear stress on AD and BC be q'; the total shear force on each face $= q'AD = q'BC$.

The couple $= q'BC \times AB$ units,

then $q'BC \times AB = q \cdot AB \cdot BC$

$$q' = q \qquad \text{(4)}$$

that is, the intensities of shear stress across two planes at right angles are equal.

58. **Distribution of Shear Stress in Beams.** It has been shown that on each vertical cross-section of a beam there is a vertical shearing force acting, and it is now required to find how this force is distributed across the section. The vertical shear stress at any point is accompanied by a horizontal shear stress of equal intensity.

AB, CD are two sections of a beam δx apart. (Fig. 79.)

Moment at $AB = M$

" $CD = M + \delta M$

Consider a fibre of breadth z and thickness dy at a distance y from the neutral axis.

Suppose the cross-section is constant and therefore I is constant.

If f is the longitudinal stress in the fibre for section AB
,, f_1 ,, ,, ,, ,, ,, CD

$$\text{then } f = M\frac{y}{I} \text{ and } f_1 = (M + \delta M)\,.\,\frac{y}{I}$$

The total thrust on the fibre, at section AB

$$= fz \cdot dy$$
$$= \frac{M}{I}\, y \cdot z \cdot dy$$

The total thrust on the fibre, section CD

$$= f_1 z \cdot dy$$
$$= \frac{M + \delta M}{I}\,.\, y \cdot z \cdot dy$$

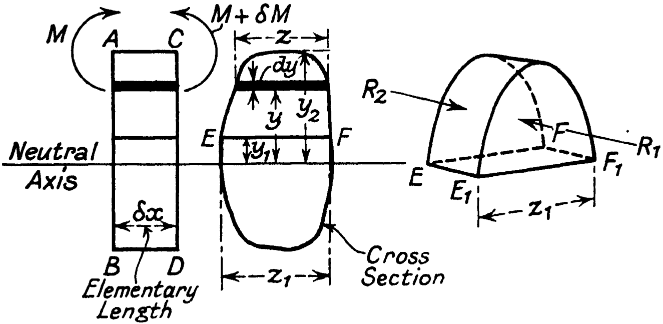


FIG. 79

EF is a line fixed on the section drawn parallel to the neutral axis at a distance y_1 from it; y_2 is the distance of the outer skin from the N.A.

R_2 = resultant thrust above EF for section AB
R_1 = ,, ,, ,, ,, CD ✓

$$R_2 = \sum_{y_1}^{y_2} \frac{M}{I}\,.\, yz \cdot dy \text{ and } R_1 = \sum_{y_1}^{y_2} \frac{M + \delta M}{I}\,.\, yz \cdot dy$$

R_2 is taken as acting from left to right, and R_1 as acting from right to left. Hence the small portion of the beam δx long, and contained by the two parallel sections AB and CD, and a horizontal plane EF situated at a distance y_1, width z_1, from the neutral surface is acted upon by two horizontal forces,

$$R_2 \longrightarrow \text{ and } R_1 \longleftarrow$$

The resultant force $R_1 - R_2$ tends to make this portion of the beam slide over the horizontal surface EF. This tendency to slide is resisted by the shearing action at the surface, and if q is the intensity of the shearing stress there,

$$q \cdot z_1 \delta x = R_1 - R_2$$

$$= \frac{\delta M}{I} \sum_{y_1}^{y_2} yz \cdot dy$$

$$q = \frac{\delta M}{\delta x} \cdot \frac{1}{I \cdot z_1} \sum_{y_1}^{y_2} yz \cdot dy$$

$$q = \frac{dM}{dx} \cdot \frac{1}{Iz_1} \cdot \int_{y_1}^{y_2} yz \cdot dy$$

when δx and δM made infinitely small,

but $\frac{dM}{dx} = S =$ total shearing force at the section

$$\therefore q = \frac{S}{Iz_1} \int_{y_1}^{y_2} yz \cdot dy \quad . \quad . \quad . \quad (5)$$

$$\text{or } q = \frac{S}{Iz_1} \int_{y_1}^{y_2} y \cdot dA \text{ i.e. } \frac{S}{Iz_1} . A\bar{y} = q \quad . \quad . \quad . \quad (6)$$

where A = area of the section between the limits y_2 and y_1 and $\bar{y}$ is the distance of the centroid of this area from the neutral axis of the section. $A\bar{y}$ therefore represents the moment of the area considered with respect to the neutral axis. This latter formula can with advantage be used for any beam section which can be taken as being made up of rectangles, such as I beams and built-up I beams.

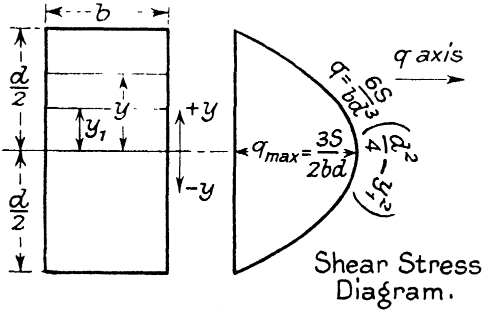


FIG. 80

59. EXAMPLES

(1) A beam of rectangular section, depth d, breadth b. Investigate the distribution of shear stress over a section at which the shearing force is S. (Fig. 80.)

Knowing $z = z_1 = b$,

and using the mathematical method given, the distribution of shear stress is as shown in Fig. 80. The maximum shear stress will occur at the neutral axis

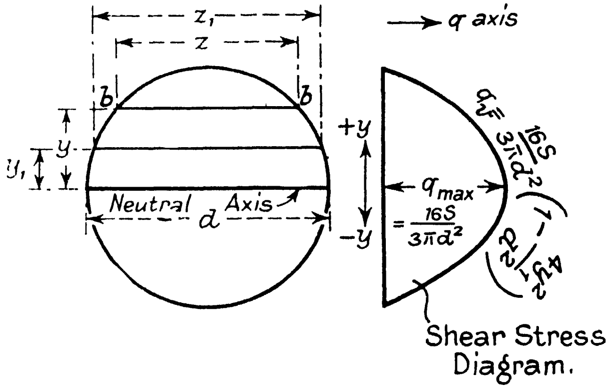
$$\text{where } q_{max} = \frac{6S}{4bd} = \frac{3S}{2bd} \quad . \quad . \quad . \quad (7)$$

$$q_{mean} = \frac{2}{3}\left(\frac{3S}{2bd}\right) = \frac{S}{bd}$$

$$= \frac{\text{shear force}}{\text{area cross-section}}$$

$$\frac{q_{max}}{q_{mean}} = \frac{3}{2} = m \text{ (see p. 126).}$$

(2) A beam of solid circular section is of diameter d. Investigate the distribution of shear stress over a section at which the shear force is S. (Fig. 81)



z_1
z
b
b
y
y_1
$+y$
$-y$
Neutral Axis
d
→ q axis
$q_{max} = \frac{16S}{3\pi d^2}$
$q_y = \frac{16S}{3\pi d^2}\left(1 - \frac{4y^2}{d^2}\right)$
Shear Stress Diagram.

FIG. 81

Note on the distribution of the shearing stresses in the case of a circular cross-section.

It has been shown* that at the boundary points such as b and b of the section of width z in Fig. 81 the shearing stresses are tangential to the boundary. At the mid-point of the chord bb (Fig. 81), symmetry requires that the shearing stress has the direction of the vertical shear force S. Then the shearing stresses at the boundaries and the mid-point pass through a common point. Assuming that the shearing stresses at the intermediate points also pass through this common point, then the shearing stresses can be determined in direction and magnitude, if we can find the magnitude of the vertical component. Further, we assume that the vertical components are the same at all points in the chord *BB*: using the method developed, this component can be found. After finding the point of intersection of the tangential shear stresses at the boundary ends of the chords, then the shear stresses can be determined completely.

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* ***Strength of Materials*****, by S. Timoshenko.**

Referring to Fig. 81, it is required to investigate the shearing stress along any section of breadth z_1 which is at a constant distance y_1 from the neutral axis.

Consider an elemental strip represented by *bb*, having a length z and a width dy.

$$z = 2\sqrt{\frac{d^2}{4} - y^2}$$

also $$z_1 = 2\sqrt{\frac{d^2}{4} - y_1^2}$$

The moment of the strip $z\,.\,dy$ about the neutral axis is $z\,.\,dy\,.\,y$ and the total moment of the area of the segment of the circle between the limits $\frac{d}{2}$ and y_1 is $A\bar{y}$

$$= \int_{y_1}^{\frac{d}{2}} 2\sqrt{\frac{d^2}{4} - y^2}\,.\,y\,.\,dy = \frac{2}{3}\left(\frac{d^2}{4} - y_1^2\right)^{3/2}$$

Consider equation (6), $q = \frac{S}{Iz_1}\,.\,A\bar{y}$.

Here q is a vertical shear stress and it will be equal to q_v, assumed constant as the vertical component of the shear stresses across the strip of the circle. Therefore q_v for the strip of breadth z_1 distant y_1 from the neutral axis will be equal to

$$\frac{S}{I}\cdot\frac{2}{3}\left(\frac{d^2}{4} - y_1^2\right)^{3/2} \times \frac{1}{2\sqrt{\frac{d^2}{4} - y_1^2}} = \frac{S}{3I}\cdot\left(\frac{d^2}{4} - y_1^2\right) \quad . \quad (7\text{A})$$

Now $$I = \frac{\pi d^4}{64}$$

$$\therefore \quad q_v = \frac{16S}{3\pi d^2}\left(1 - \frac{4y_1^2}{d^2}\right)$$

The total shear stress q_t at the end of the strip of length z_1 is

$$q_t = \frac{q_v \, . \, d}{2\sqrt{\frac{d^2}{4} - y_1^2}} = \frac{S \, . \, d}{6I}\sqrt{\left(\frac{d^2}{4} - y_1^2\right)}$$

When $y_1 = 0$, then q_t is a maximum.

Also at $y_1 = 0$, $q_{t\,max} = q_{v\,max} = q_{max}$ and there is no horizontal component of the shear stress.

Then $$q_{t\,max} = q_{v\,max} = q_{max} = \frac{Sd^2}{12I} = \frac{4S}{3A} = \frac{mS}{A}$$

where A is the area of the circle, see p. 126.

In the case of the circle, therefore, the maximum shearing stress is equal to four-thirds the average value obtained by dividing the shear force by the cross-sectional area.

$$q_{mean} = \frac{2}{3} \times \frac{16S}{3\pi d^2} = \frac{32S}{9\pi d^2} \text{ and is } < \frac{4S}{\pi d^2}$$

(3) **Investigate the distribution of the average vertical component of shear stress over the built-up section given in Fig. 82.**

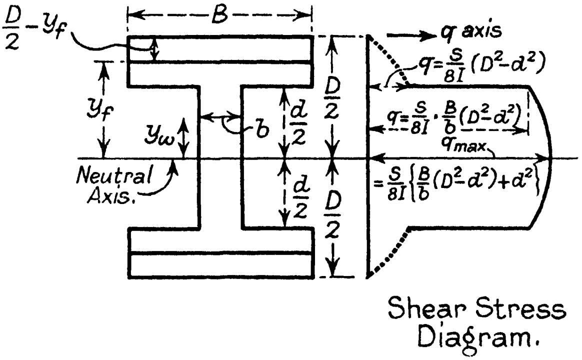
$\frac{D}{2} - y_f$
B
q axis
$q = \frac{S}{8I}(D^2 - d^2)$
y_f
y_w
b
$\frac{d}{2}$
$\frac{D}{2}$
$q = \frac{S}{8I} \cdot \frac{B}{b}(D^2 - d^2)$
q_{max}
$= \frac{S}{8I}\left\{\frac{B}{b}(D^2 - d^2) + d^2\right\}$
Neutral Axis.
$\frac{d}{2}$
$\frac{D}{2}$
Shear Stress Diagram.

FIG. 82

Note on the distribution of shear stress in the flange of an "I" section.

The actual distribution in the flanges of an R.S.J. section is somewhat complicated, and here the assumption that the vertical component of the shear stress on all horizontal sections is constant has been assumed and the curve as a consequence has been dotted in on Fig. 82. In the case of the web, the shear stresses are obviously all vertical, and can be determined by the method already developed.

For the Flange. q_v = vertical component of the shear stress at a distance y_f from the neutral axis. Using the form

$$q = \frac{SA\bar{y}}{Iz_1} \text{ and where } z_1 = B,$$

it can be shown that the average vertical component of the shear stress is

$$q_v = \frac{S}{2I}\left(\frac{D^2}{4} - y_f^2\right)$$

which holds good for values of y_f between $\frac{D}{2}$ and $\frac{d}{2}$

$$\text{when } y_f = \frac{d}{2}, \qquad q_v = \frac{S}{8I}\left(D^2 - d^2\right) \quad . \quad . \quad (10)$$

For the Web. q = shear stress at a distance y_w from the N.A. and it is vertical in direction.

$$q = \frac{SA\bar{y}}{Iz_1} \text{ and } z_1 = b,$$

$$\text{then } q = \frac{S}{8I} \times \frac{B}{b}\left(D^2 - d^2\right) + \frac{S}{2I}\left(\frac{d^2}{4} - y_w^2\right)$$

and only holds for values y_w between 0 and $\frac{d}{2}$

$$\text{When } y_w = \frac{d}{2}, \qquad q = \frac{S}{8I} \times \frac{B}{b}\left(D^2 - d^2\right) \quad . \quad . \quad (11)$$

$$\text{When } y_w = O, \qquad q = \frac{S}{8I}\left\{\frac{B}{b}\left(D^2 - d^2\right) + d^2\right\} \quad . \quad . \quad (12)$$

Notice. Passing from the flange to the web, the shear stress suddenly increases from

$$\frac{S}{8I}\left(D^2 - d^2\right) \text{ to } \frac{S}{8I} \cdot \frac{B}{b}\left(D^2 - d^2\right)$$

and a consideration of the stress diagram shows that the web takes almost all the shear.

60. **Elastic Energy in Shear Strain.** Elastic strain energy is stored by a material having shear strain within the elastic limit, just as in the case of direct stress and strain.

Consider a cube of side dx, of which one face is strained by an amount dy_s with respect to the opposite and parallel face by a tangential force

$$T = q \, . \, dx \, . \, dx ; \; dy_s/dx = q/G.$$

$$\text{Then shear resilience} = \frac{T}{2} dy_s = \frac{q}{2}(dx)^2 dy_s = \frac{q}{2}(dx)^2 \frac{q(dx)}{G}$$

$$= \frac{q^2}{2G}(dx)^3$$

$$= \frac{q^2}{2G} \text{ per unit volume.}$$

61. **Deflection of a Beam Due to Shearing.** In addition to ordinary deflections due to bending, there is a further deflection due to the vertical shear stress on transverse sections of a beam, except for those portions of a beam which bend to the arc of a circle.

The shear stress is not evenly distributed over the section, but varies from a maximum at the neutral surface to zero at the upper and lower edges of the section.

In many practical cases the shear deflection is negligible compared with the bending deflection. Also in some cases of design to allow for it a smaller value than the correct value of E for the material is assumed, and assuming the ordinary bending deflection formulae, gives a higher value of deflection for the beam than that using the correct value of E.

62. **Deflection Due to Shear.** (Fig. 83.) Let an elementary length (dx) of a beam deflect a small amount dy_s.

$$G = \frac{\text{shearing stress}}{\text{shearing strain}}$$

$$= \frac{q}{\frac{dy_s}{dx}}$$

$$\frac{q}{G} = \frac{dy_s}{dx}$$

If S = shear on the section due to external forces and A = area of cross-section,

$$q_{mean} = \frac{S}{A}$$

Assume q_{mean} constant over the whole section,

$$\text{then } \frac{dy_s}{dx} = \frac{S}{AG} \left(\begin{array}{l} \text{also applicable to } I \text{ beam} \\ \text{where } A = \text{area of the web} \end{array} \right) \quad . \quad (13)$$

THE EFFECT OF THE SHEARING FORCE ON THE DEFLECTION OF BEAMS

An additional deflection is produced by the shearing force in the form of a mutual sliding of adjacent cross-sections along each other. As a result of the non-uniform distribution of shearing stresses, the cross-sections, previously plane, become curved. At the neutral axis the shear strain is a maximum, and at the edges it is zero, and here the tangent to the warped plane is tangential to the beam flange. If the shear force is a constant along the beam, the warping at all cross-sections is the same, and therefore does not affect the longitudinal strain produced by the applied bending moment. Neglecting the deformation produced by the bending moment, under the shear force, assuming vertical forces acting on the beam, the elements of the cross-sections at the centroids remain vertical and slide along one another. The slope of the deflection curve, due to the shear force above, is equal at each cross-section to the shearing strain at the centroid of the section. Denoting by y_s the deflection due to shear, we obtain for any cross-section the following expression for the slope of the shear deflection curve—

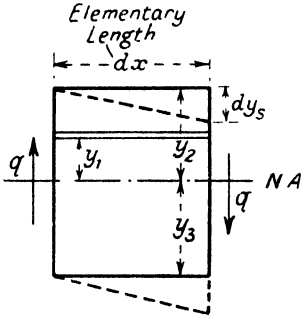

FIG. 83

$$\frac{dy_s}{dx} = \frac{q_{y=0}}{G} = \frac{m}{G} \times \frac{S}{A} = \frac{SC}{G}$$

in which S/A is the average shearing stress q, G is the modulus in shear and m is a numerical factor, by which the average shear stress must be multiplied in order to obtain the shear stress at the centre of the cross-section; also $C = \frac{m}{A}$

$m = 1 \cdot 5$ for rectangular sections and $4/3$ for circular cross-sections.

The assumptions made in the examples are that the beam can work freely, and in the case of the uniformly-loaded simply-supported beam, this condition is approximately satisfied: the condition of symmetry of deformation with respect to the middle section is satisfied. The warping will increase with shear from the middle where $q = 0$. In the case of a central point load, the condition of symmetry of deformation with respect to the centre will hold good. The centre section remains plane, but at sections immediately adjacent the shear force is $\frac{P}{2}$. Warping of the cross-sections of these must, however, take place: and there cannot be an abrupt change in warping from section to section. Therefore the warping at the central section cannot be that due to the shear forces on the basis given. Warping here must be partial, and the additional shear deflection less than that given by the elementary theory. The shear deflections given in the examples and based on the elementary theory will therefore be approximate displacements.

$$\frac{dy_s}{dx} = \frac{SC}{G} \text{ for varying } q, \text{ so that } y_s = \frac{C}{G}\int S \cdot dx \qquad (17)$$

$$\frac{dy_s}{dx} = \frac{S}{AG} \text{ for constant } q, \text{ so that } y_s = \frac{1}{AG}\int S \cdot dx \qquad (18)$$

For the rectangle $z_1 = z = b\,;\ y_2 = \frac{d}{2};\ y_3 = -\frac{d}{2}$

$$A = \text{whole area} = b \times d.$$

It has been shown that $C = \frac{1{\cdot}5}{bd} = \frac{1{\cdot}5}{A}$ . . (19)

For the circle, $C = \frac{4}{3A}$ . . . . . (20)

1·5 and 4/3 are the factors with which the average shearing stress must be multiplied, to give the shearing stress at the centroid of the rectangle and the circle respectively (see para. 59).

THE APPROXIMATE SHEAR DISPLACEMENTS FOR GIVEN BEAMS AND LOADINGS*

63. Simple Beam and Cantilever. y_s = shear deflection at a distance x below and from a convenient origin.

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* For a further discussion see *Strength of Materials*, Part I, by S. Timoshenko.

Now $\int_0^x S \cdot dx$ = area shear force diagram between the origin and x, and for a simply supported beam

= moment at distance x from the origin $= M_x$

for here there is no moment at the support (origin).

$$\text{Then } y_s = \frac{M_x}{AG} \text{ or } \frac{M_xC}{G} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

64. (*a*) **Simple Beam.** With a uniform load over the whole length of the beam,

$$y_s = \frac{1}{AG}\left(\frac{wlx}{2} - \frac{wx^2}{2}\right) = \frac{wx}{2AG}\left(l - x\right)$$

$$\text{or } = \frac{wxC}{2G}\left(l - x\right)$$

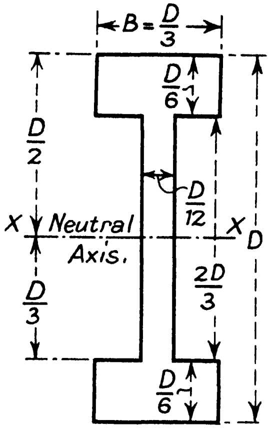

FIG. 84A

y_s is a maximum at the centre of the beam,

i.e. when $x = \frac{l}{2}$

$$\therefore y_{s\,max}$$

$$= \frac{wl^2}{8AG} \text{ for uniform } q \quad . \quad (22)$$

$$= \frac{wl^2C}{8G} \text{ for varying } q \quad . \quad (23)$$

(*b*) SIMPLE BEAM. With a single concentrated load W at the centre,

$$\text{at the centre : } y_{s\,max} = \frac{Wl}{4AG} \text{ for uniform } q \quad . \quad (24)$$

$$= \frac{WlC}{4G} \text{ for varying } q \quad . \quad (25)$$

65. **Cantilever.** (1) Uniform load the whole length of the beam.
(2) Concentrated load W at the free end of the beam.
The maximum deflection due to shear will occur at the free

end of the beam; i.e. the deflection below the fixed end, taken as the origin.

$\int_0^l S\,.\,dx =$ area of shear force diagram between the support and the free end.

For (1) $y_s = \dfrac{wl^2}{2AG}$ or $\dfrac{wl^2C}{2G}$ using Equation (21)

For (2) $y_s = \dfrac{Wl}{AG}$ or $\dfrac{WlC}{G}$ from Equation (21)

66. For a symmetrical I beam, find C,* or assume that the web takes all the shear and that it is evenly distributed over the web.

Illustrative Problem 19.

In a beam of *I* section the thickness of the web is half that of the flanges, which latter, i.e. thickness of the flanges, is half the breadth of the beam. If the breadth of the beam is one-third the depth, find the ratio of the maximum to mean shearing stress in the section. (Fig. 84A.)

The dimensions of the section will be as shown in Fig. 84A, and let them be in inches.

The neutral axis will be at a depth $\dfrac{D}{2}$ from the boundaries.

$$I_{xx} = \frac{D}{3} \times \frac{D^3}{12} - \left(\frac{D}{3} - \frac{D}{12}\right)\left(\frac{2D}{3}\right)^3 \times \frac{1}{12}$$

$$= \frac{7}{324} D^4 \text{ (in.)}^4$$

$$\text{Area} = 2 \times \frac{D}{3} \times \frac{D}{6} + \frac{2D}{3} \times \frac{D}{12} = \frac{D^2}{6} \text{ sq. in.}$$

Let S tons be the shearing force at the section, then the mean shearing stress

$$q_{mean} = \frac{S}{A} = \frac{S}{\frac{D^2}{6}} = \frac{6S}{D^2} \text{ tons/sq. in.}$$

---

* From Eqn. (12), page 124, and referring to Fig. 82.

$$\frac{mS}{A} = CS = \frac{S}{8I}\left\{\frac{B}{b}\left(D^2 - d^2\right) + d^2\right\}$$

$$C = \frac{1}{8I}\left\{\frac{B}{b}\left(D^2 - d^2\right) + d^2\right\}$$

The maximum shearing stress will be at the section at the neutral axis. Using the method in paragraph 58, Eqn. (6),

$$q_{max} = \frac{SA\bar{y}}{Iz_1}$$

where A is the area of the section above the neutral axis $\bar{y}$ the distance of the centre of gravity of A from the N.A.

$$z_1 = \frac{D}{12}, \quad A = \frac{D^2}{6 \times 2} = \frac{D^2}{12} \text{ sq. in.}$$

$$q_{max} = \frac{S324}{7D^4} \times \frac{12}{D}\left(\frac{D}{3} \times \frac{D}{6} \times \left(\frac{D}{3} + \frac{D}{12}\right) + \frac{D}{3} \times \frac{D}{12} \times \frac{D}{6}\right)$$

$$= \frac{15{\cdot}6S}{D^2}$$

$$\frac{q_{max}}{q_{mean}} = \frac{15{\cdot}6\,\frac{S}{D^2}}{6\,\frac{S}{D^2}} = \frac{2{\cdot}6}{1} = m^*$$

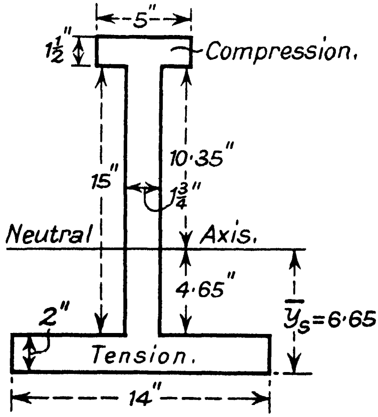

FIG. 84B

Illustrative Problem 20.

Obtain a formula giving the intensity of shear stress at any point in a vertical cross-section of a beam, and use it to find the ratio of the maximum to the mean shear stress at a section of a cast-iron beam in which the top flange is 5 in. by 1½ in., bottom flange 14 in. by 2 in., and web 15 in. by 1¾ in. (Fig. 84B.)

The maximum shear stress occurs for the section at the neutral axis.

Let $\bar{y}_s$ be the height of the centroid of the section above the tension flange (denoted BB). (*See* Fig. 84B.)

Area of cross-section $= (5 \times 1{\cdot}5 + 15 \times 1{\cdot}75 + 14 \times 2)$ sq. in.

$= 61{\cdot}75$ sq. in.

$$61{\cdot}75 \times \bar{y}_s = 7{\cdot}5 \times 17{\cdot}75 + 26{\cdot}25 \times 9{\cdot}5 + 28 \times 1$$

$$\bar{y}_s = 6{\cdot}65 \text{ in.}$$

I_{XX} about the neutral axis $= I_{BB} - A(\bar{y}_s)^2 = 2750$ in.^{4}

---

[*] *See* page 126.

z_1 = width of the cross-section at the neutral axis
= 1·75 in.

S = shearing force in tons

$$q_{max} = \frac{SA\bar{y}}{Iz_1} \qquad \begin{aligned} A &= \text{area above the neutral axis,} \\ \bar{y} &= \text{height centroid of } A \text{ above N.A.} \end{aligned}$$

$$q_{max} = \frac{S}{2750 \times 1{\cdot}75}\left(7{\cdot}5 \times 11{\cdot}1 + 1{\cdot}75 \times 10{\cdot}35 \times 5{\cdot}18\right)$$

$$= \frac{S \times 177{\cdot}3}{4820} = \frac{S}{27{\cdot}2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (A)$$

$$q_{mean} = \frac{S}{61{\cdot}75}$$

$$\text{Ratio } \frac{q_{max}}{q_{mean}} = \frac{61{\cdot}75}{27{\cdot}2} = \frac{2{\cdot}27}{1}$$

Working with the area below the N.A.,

$$q_{max} = \frac{S}{4820}\left(28 \times 5{\cdot}65 + 4{\cdot}65 \times 1{\cdot}75 \times 2{\cdot}34\right)$$

$$= \frac{177S}{4820} = \frac{S}{27{\cdot}2} \quad . \quad . \quad . \quad (B) \qquad \text{see } (A) \text{ above.}$$

REFERENCES

(1) *Structural Members and Connections*, Hool and Kinne. Examples in deflection due to shear.
(2) *Strength of Materials*, Part I, S. Timoshenko.

EXAMPLES

1. A rolled steel joist has the following cross-section: depth of section, 4 in.; width of flanges, $1\frac{3}{4}$ in.; thickness of flanges, $\frac{1}{4}$ in.; of web, $\frac{3}{16}$ in. This joist rests freely on two supports, 40 in. apart, and carries a load of $1\frac{1}{4}$ tons in the centre of the span. Calculate the maximum intensity of the shear stress in tons per square inch. Show by a diagram the actual distribution of the shear stress. Find the deflection of the beam at the centre due to shear. $G = 12 \times 10^6$ lb. per sq. in. (U. of L.)

2. A cast-iron beam of elliptical section is simply supported upon two supports 8 ft. apart, the major axis being horizontal. The greatest width of the beam is 12 in. and its greatest depth 8 in. The beam is uniformly loaded with a load of 2 tons per foot run. The weight of the beam should be taken into account. Weight of cubic foot of cast-iron, 460 lb. Find the maximum shear stress along the major axis. What is the deflection of the beam at the centre, (*a*) due to bending, (*b*) due to shear? (U. of B.)

3. Prove that the intensity of shear stress, q, at any point of the cross-section of a beam is $\frac{FAy,}{Ib}$ where F = shearing force at the section, I = moment of inertia of the cross-section, b = breadth of section at the point, A = area of cross-section on the farther side of the point to the neutral axis, and y = distance of C.G. of this area from the neutral axis. Show that for a rectangular cross-section the maximum shear stress is one and a half times the average shear stress. (I.C.E.)

4. The vertical cross-section of a horizontal tubular beam is 1 foot external and 6 in. internal diameter. Calculate the ratio of the maximum shear stress to mean shear stress at the section where the shear load is 6 tons.

5. A 9 in. by 3 in. wooden beam, span 10 ft., supports a uniformly-distributed load of 200 lb. foot run. Determine the deflection in inches at the centre of the beam due to shear. If the beam also supports a load of 0·4 ton at the centre, determine the deflection at the centre of the beam due to shear when the two loads are carried. Take $G = 9 \times 10^4$ lb./sq. in.

NOTE.—For cast-iron take $E = 6{,}000$ tons/sq. in.
$G = 2{,}500$ tons/sq. in.

6. Take the beam section of Problem 19, page 129. The beam is simply supported over a span L. It is loaded with a uniformly distributed load. Find the ratio of L/D at which the maximum deflection due to shear is one-tenth the maximum deflection due to bending. Take $G = 2/5\ E$.

7. Show that the deflection due to shear at a section X distant x from the support of a cantilever is equal to

$$y_s = \frac{wC}{G}\left[wlx - \frac{wx^2}{2}\right]$$

for a beam uniformly loaded over its whole length; and

$$y_s = \frac{WC}{2G}(x)$$

for a beam with a concentrated load W at the free end.

CHAPTER VI

COLUMNS

67. **Long and Short Struts.** A length of material, which may be of solid section or which may be built up, and which is subjected to thrust loads axially or non-axially, is a column or strut.

68. Short columns fail by the stress exceeding the yielding stress of the material in compression ; long columns fail by what is known as buckling, and between these two extremes, failure occurs by a combination of direct compressive stress and buckling.*

69. **Euler's Theory of Long Struts.** (Fig. 85A.) Long struts fail by buckling, or lateral bending, and the determination of the buckling load really becomes one of stability. The Euler formula tries to find what end force will cause a bending moment which will make the ratio of (increase of deflection) to (increase of load) equal to infinity. It neglects the effect of direct compression and only deals with the bending moments as a cause of failure. For usual shapes of columns it gives results which are far too high. Euler's solution is generally the one used in strength calculations of very long struts. The Euler strut is homogeneous, of uniform cross-section, very long in relation to its cross-sectional dimensions, and the load is supposed to be applied perfectly axially. It is also perfectly straight, that is, there is no initial deflection due to workmanship, etc., which would cause an initial moment to be applied to the strut. The end conditions greatly affect the strength to resist buckling.

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* Columns made of brittle materials, such as cast-iron, will fail suddenly in shear. Short solid specimens of a ductile material, such as mild steel, will squash or flatten out. (Refer to textbooks on strength of materials for diagrams and photographs of failures in compression.) On loading a long column of ductile material, for a small addition of load, the column will remain straight. On further applying load it will eventually begin to bend laterally, or buckle, until just above some critical load it will continue to keep on bending under this load until it reaches the failing strength of the material in compression, when it will fail suddenly by buckling.

The kind of failure depends on the type of column. In dealing with columns therefore, large deflections of the body as a whole have to be dealt with. (Photographs of failures of experimental aeroplane struts are given in a paper on the "Development of Metal Construction in Aircraft," by Major J. S. Nicholson, O.B.E., *Engineering*, March 12th, 19th, and 26th, 1920.)

LONG COLUMN OR STRUT, PIN-JOINTED AT BOTH ENDS

Let AB of length l be such a strut, pin-jointed at A and B, i.e. it has practically free motion about these points.

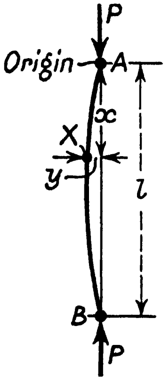

FIG. 85A

Consider the top joint A as the origin; take x downwards as positive, and let the strut under a load P deflect as shown in Fig. 85A. Let y be the amount of deflection of section X distant x from A. Let y be positive, as the strut bends concave upwards towards its original position.

Moment at $X = Py$ will be of the negative sense.

$$EI\frac{d^2y}{dx^2} = -Py$$

$$\frac{d^2y}{dx^2} = -\frac{P}{EI}y \quad . \quad . \quad . \quad (1)$$

Solving Equation (1),

$$y = A \sin\sqrt{\frac{P}{EI}}\,x + B\cos\sqrt{\frac{P}{EI}}\,x$$

where A and B are constants depending upon initial conditions.

$$x = 0, \quad y = 0, \quad \text{hence } B = 0$$

$$y = A\sin\sqrt{\frac{P}{EI}}\,x$$

$$x = l, \quad y = 0$$

$$0 = A\sin\sqrt{\frac{P}{EI}}\,l$$

A cannot be zero as if so no bending would take place, and the critical load has not been reached,

$$\text{therefore } \sin\sqrt{\frac{P}{EI}}\,.\,l = 0$$

$$\sqrt{\frac{P}{EI}}\,.\,l \text{ therefore} = 0,\ \pi,\ 2\pi, \text{ etc. (in radians.)}$$

Clearly the zero value is inadmissible, and confining ourselves to the smaller of the values π at which $\sin \sqrt{\frac{P}{EI}} \cdot l = 0$, then the crippling load is obtained from the equation

$$\sqrt{\frac{P}{EI}} \cdot l = \pi$$

$$\therefore \quad \text{Critical load} = P = \frac{\pi^2 EI}{l^2} = \text{Euler load} \quad . \quad (2)$$

70. **Long Strut or Column Fixed at Both Ends.** (Fig. 85B.) Owing to the end-fixing conditions, there are fixing moments at each end. When these are equal, the slopes dy/dx at each end are zero, as also at the middle of the strut. At the quarter points C and D, there are points of inflexion, where dy/dx is a maximum.

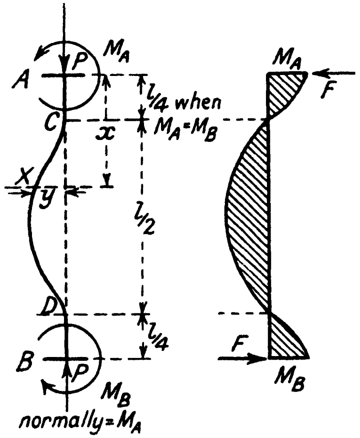

FIG. 85B

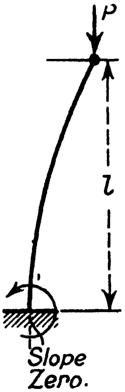

FIG. 85C

Let M_A and M_B be the end-fixing moments: normally $M_A = M_B$, and for which condition the points of inflexion are the quarter points. However, conditions alter if M_B is greater than M_A: the unbalanced moment $(M_B - M_A)$ tends to overturn the column and this calls into play the horizontal forces F, F at each end of the column, such that $Fl = M_B - M_A$.

Consider normal conditions when $M_A = M_B = u$.

Then the bending moment at X, distant x from A, is

$M_x = Py + u$. (Py is taken as of the negative sense.)

Then $$EI\frac{d^2y}{dx^2} = u - Py; \ \frac{d^2y}{dx^2} = -\frac{P}{EI}\left(y - \frac{u}{P}\right)$$

Let $$P/EI = n^2 \text{ and } u/P = a$$

$$\frac{d^2y}{dx^2} = -n^2(y-a) = -n^2z$$

$$\frac{d^2y}{dx^2} = \frac{d^2z}{dx^2} = -n^2z$$

As before, $$z = A \sin nx + B \cos nx$$

$$\therefore \ y - a = A \sin nx + B \cos nx$$

Differentiating $$\frac{dy}{dx} = nA \cos nx - nB \sin nx$$

when $x = 0$, $dy/dx = 0$: as previously $\cos nx = 1 \quad \therefore \ A = 0$

$$\therefore \ y - a = B \cos nx$$

Again if $x = 0, y = 0$: $\therefore \ B = -a$

$$\therefore \ y - a = -a \cos nx$$

$$\therefore \ y = a(1 - \cos nx)$$

If $x = l, y = 0$

$$\therefore \ 0 = a(1 - \cos nl) \qquad \therefore \ \cos nl = 1$$

Hence nl can have values 0, 2π, 4π, etc.

By previous reasoning take the value $2\pi = nl$

$$\therefore \ n^2 = \frac{4\pi^2}{l^2} \text{ and}$$

$$P = \frac{4\pi^2EI}{l^2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This value for P gives a value four times as large as that obtained in the case of the pin-jointed column considered in paragraph 69. The end-fixed column thus has four times the strength of the other.

It can be shown that the points of inflexion occur at the quarter points: the distance between these points is $l/2$ and this length is called the effective length of the strut; it is an equivalent strut, pin-jointed at the ends. Using the formula for this strut

$$P = \frac{\pi^2 EI}{l^2}$$

And replacing l^2 by $l^2/4$

$$\text{then } P = 4\pi^2 EI/l^2$$

obtained by mathematical analysis.

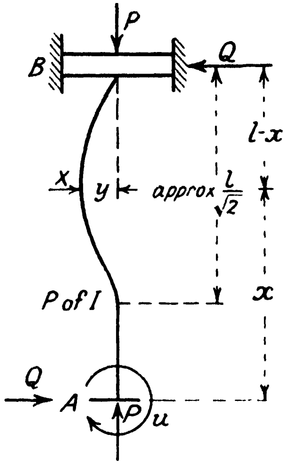

Fig. 85D

71. **Long Column or Strut, Pin-jointed at One End and Fixed at the Other.** (Fig. 85D.) The pin-jointed end is considered as being able to slide freely in a frictionless guide, the normal reaction at the guide being Q; this normal force is balanced at the fixed end.

Take the origin at the fixed end A.

$$\text{B.M. at } X = M_X = -Py + Q(l-x) = \frac{EId^2y}{dx^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{Q(l-x)}{EI} - \frac{P}{EI}\cdot y$$

$$= -\frac{P}{EI}\left\{y - \frac{Q}{P}(l-x)\right\}$$

Let $\qquad n^2 = P/EI$, and $z = \left\{y - \frac{Q}{P}(l-x)\right\}$

Then $\qquad d^2y/dx^2 = -n^2z$ and $\frac{dz}{dx} = \frac{dy}{dx} + \frac{Q}{P}$

and $d^2z/dx^2 = d^2y/dx^2 = -n^2z$

By the general solution $z = A \sin nx + B \cos nx$

$$\text{so that, } y - \frac{Q}{P}(l-x) = A \sin nx + B \cos nx$$

when $\qquad x = 0,\ y = 0,\ \therefore\ B = -\frac{Ql}{P}.$

Differentiating $\frac{dy}{dx} + \frac{Q}{P} = nA \cos nx - nB \sin nx$

When $x = 0,\ dy/dx = 0$

So that, $Q/P = nA$, thus $A = Q/nP$

$$\therefore\ y = -\frac{Q}{P}(l - x) = \frac{Q}{nP} \sin nx - \frac{Ql}{P} \cos nx$$

$$= \frac{Q}{P}\left(\frac{1}{n} \sin nx - l \cos nx\right)$$

When $x = l\ y = 0\ \therefore\ 0 = Q/P(\sin nl/n - l \cos nl)$

Simplifying $\tan nl = \cancel{\theta}.\ nl$

The solution of this equation—most easily carried out by a graphical method—for the least value of nl, is tan nl = 4·493 radians or 257·5°.

Substituting $n^2l^2 = (4{\cdot}493)^2 = 2{\cdot}047\pi^2$

$$\therefore\ \frac{P}{EI} = \frac{2{\cdot}047\pi^2}{l^2}$$

$$\therefore\ P = 2{\cdot}047\pi^2EI/l^2 \quad . \quad . \quad . \quad (4)$$

The result for the value of P shows that a strut pin-jointed at one end, and fixed at the other, is just more than twice as strong as a pin-jointed strut.

The effective length of the strut is approx. $l/\sqrt{2}$.

72. **Long Column Fixed at One End and Freedom of Movement in Any Direction at the Other.** The strut bends as shown in Fig. 85C, and its shape is similar to that of half a strut pin-jointed at both ends. Its effective length is $2l$.

$$\therefore\ \text{Crippling load } P = \frac{\pi^2EI}{(2l)^2} = \frac{\pi^2EI}{4l^2} \quad . \quad . \quad (5)$$

73. It has been shown that areas have maximum and minimum moments of inertia ; therefore, if a strut is free to bend in any direction, it will bend about that axis for which the moment of inertia is a minimum. This, obviously, makes P a minimum. The strut may be constrained to bend in a given direction ; in this case, take the required I, which can be found by using, say, the momental ellipse in Chapter II

But secondary failure may take place due to the minimum value of I, unless precautions are taken to prevent this by side supports or other means.

$$\text{Euler formula.}\quad P = \frac{C\pi^2EI}{l^2} \quad . \quad . \quad . \quad (6)$$

$$= \frac{C\pi^2EA}{\left(\frac{l}{k}\right)^2} \quad . \quad . \quad . \quad (7)$$

where I = moment of inertia for the axis about which the strut is made to bend,

C = constant depending on the fixity of the ends
= 1 for pin ends,
= 4 for fixed ends,
= $\frac{1}{4}$ for 1 fixed and 1 free end ;

A = cross-sectional area,
k = radius of gyration.

For a constant cross-sectional area of strut,

$$P\left(\frac{l}{k}\right)^2 = \text{constant} \quad . \quad . \quad (8)$$

Plotting P against $\frac{l}{k}$, a rectangular hyperbola is obtained, which is the Euler curve. (*See* Fig. 86.)

74. **Rankine Formula for Struts.** For very short struts, the failing load would be Af_c, where f_c is the crushing or yielding stress of the material, and A the cross-sectional area. For a long strut, the Euler buckling load

$$= \frac{C\pi^2EI}{l^2} \qquad (\textit{See}\ \text{Eqn. } 6)$$

C a constant depending on the fixity of the ends.

For struts which are sufficiently long to have some tendency to buckle and yet are not long enough for the direct compressive stress to be negligible, it is clear that the ultimate failing stress must lie between the two limits

$$P = Af_c \text{ and } P = \frac{C\pi^2EI}{l^2}$$

Many formulae have been suggested, but the most used one is that known as the Rankine formula. This is

$$P = \frac{Af_c}{1 + Af_c \times \frac{l^2}{C\pi^2 EI}} = \frac{Af_c}{1 + \frac{a}{C}\left(\frac{l}{k}\right)^2} \quad (7) \quad \text{where } a = \frac{f_c}{\pi^2 E} \text{ (theo.)}$$

It is often written $P = \dfrac{Af_c}{1 + a\left(\dfrac{l_1}{k}\right)^2}$ . . . . (9)

where l_1 is the effective length of the strut $= \dfrac{l}{\sqrt{C}}$

If l is very large compared with k,

$$P = \frac{Af_c}{a\left(\frac{l_1}{k}\right)^2} = \frac{Af_c}{\frac{Af_c l_1^2}{\pi^2 EI}} = \frac{\pi^2 EI}{l_1^2}$$

If l is very short, then $\dfrac{f_c}{\pi^2 E}\left(\dfrac{l_1}{k}\right)^2$

will tend to become small and negligible compared with 1 when $P = Af_c$

The value of a for some materials has been ascertained by experiment; but the theoretical value, though not so good as the experimental, is very useful where a practical value has not been obtained.

The Rankine curve for a mild steel tube, where $a = \dfrac{1}{7500}$ is shown in Fig. 86.

RANKINE'S CONSTANTS.

Material.	f_c tons per square inch.	a.
Mild steel . .	21	$\frac{1}{7500}$
Wrought-iron .	16	$\frac{1}{9000}$
Cast-iron . .	36	$\frac{1}{1600}$

As $\dfrac{l}{k}$ increases in value, the Rankine and Euler curves tend to meet. Above the meeting point value of $\dfrac{l}{k}$. that is, values of $\dfrac{l}{k}$ greater than this, use Euler values for the failing loads.

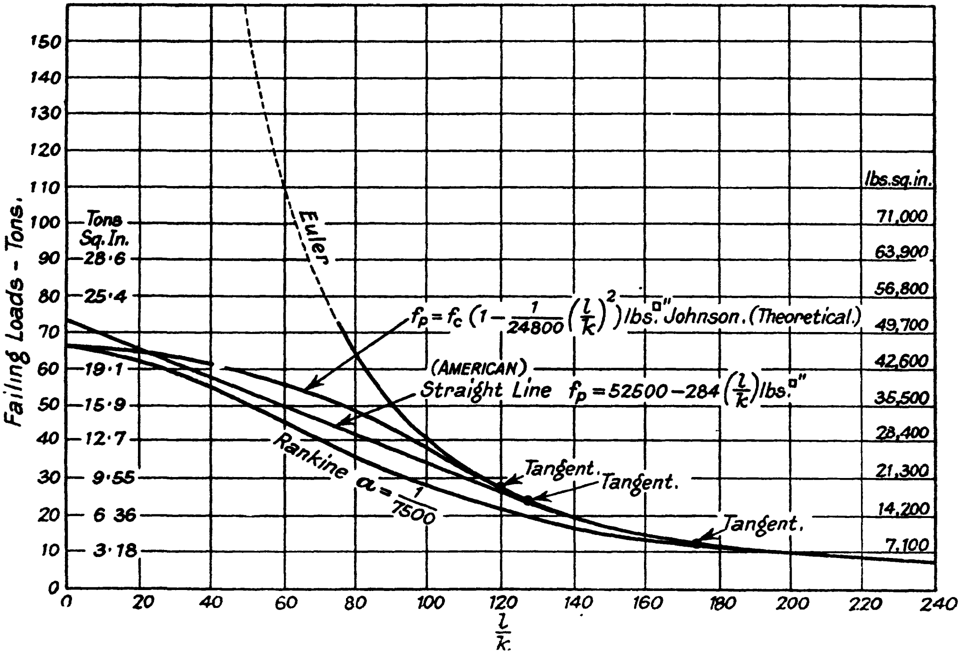
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
Failing Loads - Tons.
Tons Sq. In.
28·6
25·4
19·1
15·9
12·7
9·55
6 36
3·18
lbs.sq.in.
71,000
63,900
56,800
49,700
42,600
35,500
28,400
21,300
14,200
7,100
0
20
40
60
80
100
120
140
160
180
200
220
240
$\frac{l}{k}$
Euler
$f_p = f_c\left(1 - \frac{1}{24800}\left(\frac{l}{k}\right)^2\right)$ lbs.□" Johnson. (Theoretical.)
(AMERICAN) Straight Line $f_p = 52500 - 284\left(\frac{l}{k}\right)$ lbs.□"
Rankine $\alpha = \frac{1}{7500}$
Tangent.
Tangent.
Tangent.

FIG. 86

For struts of a channel section, it has been found by experiment that for

Mild steel, values of $\frac{l}{k} > 150$, Euler curve holds good.
High tensile steel, values of $\frac{l}{k} > 120$, ,, ,,
Duralumin, values of $\frac{l}{k} > 170$, ,, ,,

75. To satisfy the experimental curves of $\left(P, \frac{l}{k}\right)$ for the intermediate lengths of strut, various other empirical formulae have been developed, chief of which is

J. B. Johnson's Parabolic Equation. (*See* Fig. 86.)

$$\frac{P}{A} = f_c - g\left(\frac{l}{k}\right)^2 \quad . \quad . \quad . \quad . \quad (10)$$

f_c = Yield stress of the material in lb./sq. inch ; g is a constant of such a value to make the parabolic curve tangential to the Euler curve at some fairly high value of $\frac{l}{k}$; l is the exact test length of the strut.

MILD STEEL (*American constants*)*

End conditions	f_c lb./sq. in.	g.	Limit of $\frac{l}{k}$.
Pin ends .	42,000	0·97	150
Flat ends .	42,000	0·62	190

NOTE.—When $\frac{l}{k} = O$; f_c = 42,000 lb. per square inch is less than the yielding stress of mild steel, which is about 52,500 lb. per square inch.

NOTE.—For a strut absolutely freely hinged, and to make the parabola meet the Euler curve tangentially, g must be theoretically equal to $\frac{f_c^2}{4\pi^2 E}$

NOTE ON J. B. JOHNSON'S PARABOLIC EQUATION.

The maximum load which can be carried by a strut is P_M. Then, by Johnson's equation

$$P_M = Af_c\left(1 - \frac{f_c}{4C\pi^2 E} \cdot \frac{l^2}{k^2}\right) \quad . \quad . \quad (A)$$

---

* The constants in paragraphs 75, 76 and 77 are from *Structural Members and Connections*, by Hool and Kinne

where C is the fraction for the end conditions.

Or, if P is the safe working load

$$A = \frac{P}{f_c} + \frac{f_c x l^2}{4C\pi^2 E} \text{ where } x = \frac{A}{k^2}$$

x is a constant depending upon the shape of the cross-section.

The formula for A is one which can be used for design purposes.

The corresponding values of P_M and A on the Euler basis are

$$P_M = C\pi^2 EA \cdot \frac{k^2}{l^2} \text{ and } \frac{P_M}{Af_c} = \frac{C\pi^2 E}{f_c} \cdot \frac{k^2}{l^2} \quad . \quad (B)$$

and $$A = \frac{P}{f_c}\left(\frac{xf_c^2 l^2}{4C\pi^2 E}\right)^{\frac{1}{2}}$$

If $\frac{f_c}{4C\pi^2 E} \cdot \frac{l^2}{k^2}$ for a column design is equal to, or is less than, 1/2, the Johnson formula is true: if this discriminant is greater than 0·5, the Euler formula is true.

The Johnson formula is an empirical formula to change from $P_M = Af_c$ true for short columns, to tangency with the Euler curve, at the point where the strength has fallen to one-half that of a short column of the same material and section. Used in its proper range, it fits the maximum loads (engineering strength) with surprising accuracy. It is a good design formula for long columns, whose probable strength P_M is greater than one-half of Af_c.

From equations (A) and (B) if

$$\frac{P_M}{Af_c} = \frac{1}{2} \text{ then } \frac{C\pi^2 E}{f_c} \cdot \frac{k^2}{l^2} = \frac{1}{2} \text{ or } 2 = \frac{f_c}{C\pi^2 E} \cdot \frac{l^2}{k^2}$$

and $$\sqrt{\frac{f_c}{C\pi^2 E}} \cdot \frac{l}{k} = 1{\cdot}414 = \sqrt{2}$$

Therefore for values of $\sqrt{\frac{f_c}{C\pi^2 E}} \cdot \frac{l}{k} <$ or $= \sqrt{2}$, use the Johnson formula for calculating P_M/Af_c and for values of $\sqrt{\frac{f_c}{C\pi^2 E}} \cdot \frac{l}{k} > \sqrt{2}$, calculate P_M/Af_c by the Euler formula. A curve of P_M/Af_c against $\sqrt{\frac{f_c}{C\pi^2 E}} \cdot \frac{l}{k}$ can be plotted which is suitable for use for the design of the practical column. (*See* Fig. 86A.) The change for the Johnson curve to the Euler curve will occur at the value of $\sqrt{\frac{f_c}{C\pi^2 E}} \cdot \frac{l}{k} = \sqrt{2}$.

76. **American Straight Line Formula.** (*See* Fig. 86.) A formula easy to work with, and giving approximate values of $\frac{P}{A}$, useful for preliminary calculations.

$$\frac{P}{A} = f_c - g\left(\frac{l}{k}\right) \quad . \quad . \quad . \quad . \quad (10a)$$

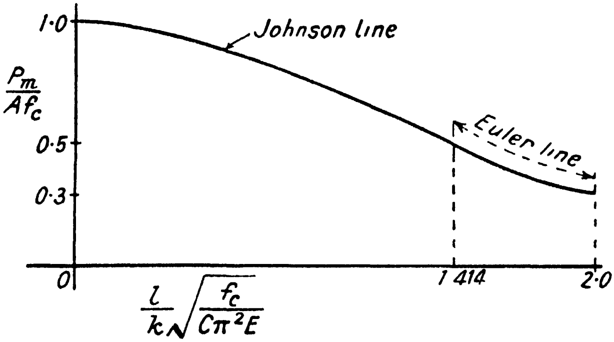

FIG. 86A

The point of tangency of the straight line with the Euler curve is the limiting value for which this formula can be used :

f_c and g are empirical constants, so chosen so as to make the formula fit the results of column tests ;

l is the exact test length of the strut.

STRUCTURAL STEEL (*American Values*)

End conditions.	f_c lb./sq. in.	g.	Limit of $\frac{l}{k}$.
Flat ends	52,500	179	195
Hinged ends . . .	52,500	220	159
Round ends . . .	52,500	284	123

CAST-IRON

$$\text{Failing stress} = \frac{P}{A} = 34{,}000 - 88\left(\frac{l}{k}\right)$$

77. Other Straight Line Formulae.

(A) *American Railway Association for Working Loads*—

$$\frac{P}{A}\left(\text{lb./sq. in.}\right) = 16{,}000 - 70\left(\frac{l}{k}\right);\ \frac{P}{A} \ngtr 14{,}000\,\text{lb./sq. in.} \quad (11)$$

$$\therefore\ \text{From } \frac{l}{k} = 0 \text{ to } 30.\ \frac{P}{A} = 14{,}000\,\text{lb./sq. in.}$$

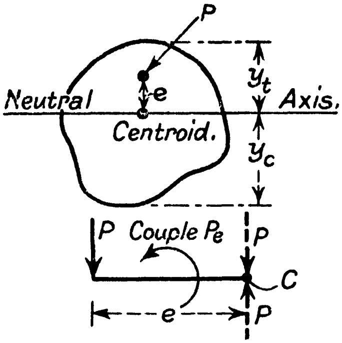

FIG. 87A

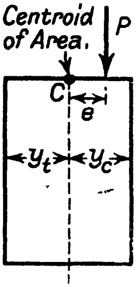

FIG. 87B

(B) *American Bridge Co.'s Formula for Structural Steel Working Loads*—

$$\frac{P}{A}\left(\text{lb./sq. in.}\right) = 19{,}000 - 100\left(\frac{l}{k}\right):\ \frac{P}{A} \ngtr 13{,}000\,\text{lb./sq. in.} \quad (12)$$

$$\text{i.e. from } \frac{l}{k} = 0 \text{ to } 60,\ \frac{P}{A} = 13{,}000\,\text{lb./sq. in.}$$

78. Struts—Eccentrically Loaded. In Fig. 87A, the load P is acting at an eccentricity e from the centroid of the cross-sectional area of a column. This force P is equivalent to a couple of magnitude Pe plus an extra force acting downwards at the centroid.

The couple will produce bending, and consequently compressive and tensile stresses in the material; whilst the additional P (thrust) at C will cause a direct compressive stress of equal intensity over the whole of the cross-section.

79. For a Short Column. Cross-sectional area A (Fig. 87B).

Maximum compressive stress for P applied at an eccentricity e on one of the two principal axes of the cross-section.

$$= \frac{P}{A} + \frac{Pey_c}{I} = \frac{P}{A} + f_{cB} \quad . \quad . \quad (13)$$

Maximum stress on the other edge distant y_t from C

$$= \frac{P}{A} - \frac{Pey_t}{I} = \frac{P}{A} - f_{tB} \quad . \quad . \quad . \quad (14)$$

If $\frac{Pey_t}{I} > \frac{P}{A}$ a resultant tensile stress will be developed, as shown in Fig. 88.

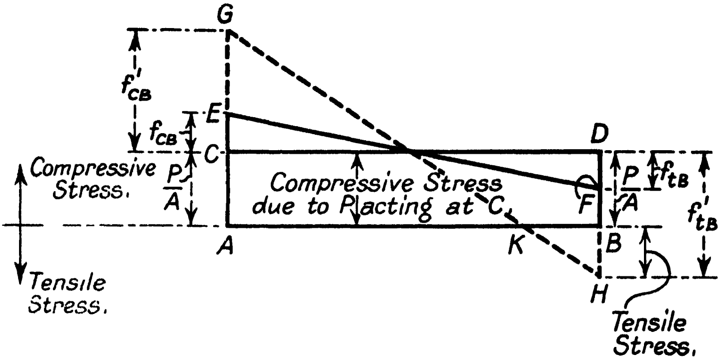

FIG. 88

Referring to Fig. 88,

$ABCD$ = normal stress distribution

$AEFB$ = stress distribution when $\frac{Pey_t}{I} < \frac{P}{A}$

$AGKHBK$ = ,, ,, $\frac{Pey_t}{I} > \frac{P}{A}$

80. **Long Struts—Euler Form.** PIN JOINTS (Fig. 89). Let P act on one of the two principal axes at a distance e from the centroid of the cross-sectional area. Let A be the origin. It is required to find the deflection y of some section X distant x from the origin. Take y as positive.

$$\text{Moment at } X = -P(e + y)$$

$$EI\frac{d^2y}{dx^2} = -Pe - Py \quad . \quad . \quad . \quad (15)$$

$$\text{or } \frac{d^2y}{dx^2} + \frac{Py}{EI} + \frac{Pe}{EI} = 0$$

The solution of this differential equation is

$$y + e = A \sin \sqrt{\frac{P}{EI}} \cdot x + B \cos \sqrt{\frac{P}{EI}} \cdot x \qquad (16)$$

$$x = 0, \quad y = 0, \quad B = e$$

$$\frac{dy}{dx} = A \sqrt{\frac{P}{EI}} \cos \left(\sqrt{\frac{P}{EI}}\, x\right) - e \sqrt{\frac{P}{EI}} \sin \left(\sqrt{\frac{P}{EI}} \cdot x\right)$$

$$\frac{dy}{dx} = 0, \text{ when } x = \frac{l}{2}$$

$$\therefore A = e \tan \sqrt{\frac{P}{EI}} \cdot \frac{l}{2}$$

$$y + e = e \tan \left(\sqrt{\frac{P}{EI}} \cdot \frac{l}{2}\right) \sin \left(\sqrt{\frac{P}{EI}} \cdot x\right) + e \cos \left(\sqrt{\frac{P}{EI}} \cdot x\right) \quad . \quad . \quad (17)$$

Particular values of $\sqrt{\frac{P}{EI}}$ that make $\tan\left(\sqrt{\frac{P}{EI}} \cdot \frac{l}{2}\right)$ infinite will also make y infinite. The only interpretation of this mathematical infinity is that the column is unstable for values of $\sqrt{\frac{P}{EI}}$ satisfying the equation

$$\tan \sqrt{\frac{P}{EI}} \cdot \frac{l}{2} = \pm \infty$$

i.e. values of $\sqrt{\frac{P}{EI}}$. given by

$$\sqrt{\frac{P}{EI}} \cdot \frac{l}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \text{ etc.}$$

The load at which instability really appears is the first one given by

$$\frac{P}{EI} = \frac{\pi^2}{l^2}$$

i.e. the critical value of P is $= \frac{\pi^2 EI}{l^2}$ which is the same as for the axially loaded Euler column. In this case, the strut bends, however small the load, and the deflection is correctly given by the equation for y before instability is reached, but at the particular load given by $P = \frac{\pi^2 EI}{l^2}$ the rod is really unstable. If the length is great, the stress in the rod may become so great, before the load is large enough to produce instability, that the strut just fails as a beam would fail under transverse loads.

If this is possible two separate calculations are necessary, one to find the safe load assuming instability impossible, the other to find the buckling or critical load assuming the stresses do not become unsafe before buckling begins. The smaller of these loads is the safe load for the strut or column.

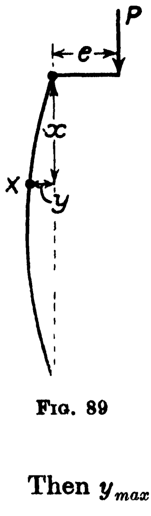

FIG. 89

From equation (17) the deflection y is obtained for any value of x.

$$y \text{ is a maximum when } x = \frac{l}{2}$$

$$y_{max} + e = \frac{e \sin^2\left(\sqrt{\frac{P}{EI}} \cdot \frac{l}{2}\right) + e \cos^2\left(\sqrt{\frac{P}{EI}} \cdot \frac{l}{2}\right)}{\cos\left(\sqrt{\frac{P}{EI}} \cdot \frac{l}{2}\right)}$$

$$\text{Then } y_{max} + e = e \sec\left(\sqrt{\frac{P}{EI}} \cdot \frac{l}{2}\right) \quad . \quad . \quad . \quad (18)$$

$$\text{or } \quad y_{max} = + e\left[(1 - \cos\left(\sqrt{\frac{P}{EI}} \cdot \frac{l}{2}\right)\right] \Big/ \cos\sqrt{\frac{P}{EI}} \cdot \frac{l}{2} \quad . \quad (18a)$$

If $\sqrt{\frac{P}{EI}} \cdot \frac{l}{2}$ is small, it is accurate to assume that

$$\cos\sqrt{\frac{P}{EI}} \cdot \frac{l}{2} = 1 - \frac{Pl^2}{8EI}$$

in the numerator and $= 1$ in the denominator of equation (18a).

Then $y_{max} = \frac{Pel^2}{8EI}$. or y_{max} varies directly as P.

This coincides with the expression for the deflection of a beam bent by couples Pe at the ends.

For other cases, equation (18) will give the displacement under a load P.

The moment at the centre is

$$P(y_{max} + e) = Pe \sec\left(\sqrt{\frac{P}{EI}} \cdot \frac{l}{2}\right) = M_c = Pe\left(\frac{Pl^2}{8EI} + 1\right)$$

The maximum compressive stress in the material at the centre will be

$$f_{max} = \frac{P}{A} + M_c \frac{y_c}{I} \quad . \quad . \quad . \quad (19)$$

(This latter, assuming the elastic moment form holds good for all values of P, and y_c = distance from the N.A. to the outer compression fibre—see Fig. 87B.)

The minimum stress will be $\frac{P}{A} - M_c \frac{y_t}{I}$

which may be compressive or tensile. From equation (19)

$$f_{max} = Pe\frac{y_c}{I} \sec\left(\sqrt{\frac{P}{EI}} \cdot \frac{l}{2}\right) + \frac{P}{A} = Pe\frac{y_c}{I}\left(\frac{Pl^2}{8EI} + 1\right) + \frac{P}{A} \quad (20)$$

Knowing the ultimate compressive stress of the material, the probable failing load P can be calculated for a given strut.

81. Failure of the strut will occur *usually* in the compression flange when the maximum compressive stress reaches the ultimate stress of the material.

With built-up struts, failure should occur by the compression flange buckling; if not properly designed, it may not do so, but fail in a secondary manner due to rivets failing or webs buckling, etc.

82. Professor Pippard and J. L. Pritchard have shown in their book *Aeroplane Structures** how the Rankine formula for axially-loaded struts may be adjusted to apply to an eccentrically-loaded strut.

For a short strut eccentrically loaded, the maximum compressive stress developed is

$$f_c = \frac{P}{A} + \frac{Pey_c}{I}$$

From which $\frac{P}{A} = \frac{f_c}{\left(1 + \frac{ey_c}{k^2}\right)}$

---

* *Aeroplane Structures*, by A. J. Pippard and J. L. Pritchard. (Longmans, Green & Co.)

In Rankine's formula, replace in the numerator

$$f_c \text{ by } \frac{f_c}{\left(1 + \frac{ey_c}{k^2}\right)}, \text{ the } \textit{eccentricity factor}$$

then $P = \dfrac{Af_c}{\left\{\left(1 + a\left(\frac{l_1}{k}\right)^2\right\}\left\{1 + \left(\frac{ey_c}{k^2}\right)\right\}\right.}$ = Rankine's failing load for eccentrically-loaded struts . . (21)

83. **Combined End and Transverse Loads.** In certain structures, members acting as struts may also have transverse loads applied to them, e.g. members of the frames of an aeroplane wing. In these cases a compressive stress will be developed in the strut due to the end load P, the moment Py (where y is the deflection of the strut), and the moment caused by the transverse load.

84. **Case I.** A concentrated transverse load of W acting at the centre of the strut having pin-ends. Assume the strut

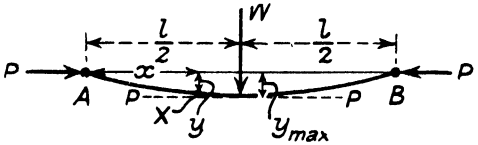

FIG. 90

elastic to failure and that it will fail by buckling of the compression flange, find the approximate axial end load P to cause failure.

The strut is constrained to bend in the direction of one of the principal axes.

Let A be the origin. (Fig. 90.)

Upward reactions at A and B due to $W = \dfrac{W}{2}$

x to the right positive : y downwards positive,

then $EI\dfrac{d^2y}{dx^2} = M_x = -\dfrac{W}{2}x - Py$

$$\text{or } \frac{d^2y}{dx^2} + \frac{P}{EI}\cdot y + \frac{W}{2EI}\cdot x = 0 \quad . \quad . \quad (22)$$

The solution of this equation is—

$$y = A \sin\left(\sqrt{\frac{P}{EI}} \cdot x\right) + B \cos\left(\sqrt{\frac{P}{EI}} \cdot x\right) - \frac{Wx}{2P} \quad . \quad (23)$$

$$x = 0, \quad y = 0, \quad B = 0$$

when $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \sqrt{\frac{P}{EI}} A \cdot \cos\left(\sqrt{\frac{P}{EI}} \cdot x\right) - \frac{W}{2P}$$

$$0 = \sqrt{\frac{P}{EI}} \cdot A \cos\left(\sqrt{\frac{P}{EI}} \cdot \frac{l}{2}\right) - \frac{W}{2P}$$

$$A = \frac{W}{2P} \cdot \sqrt{\frac{EI}{P}} \cdot \frac{1}{\cos\left(\frac{l}{2} \cdot \sqrt{\frac{P}{EI}}\right)}$$

$$\therefore y = \frac{W}{2P} \cdot \sqrt{\frac{EI}{P}} \cdot \frac{\sin\left(x \cdot \sqrt{\frac{P}{EI}}\right)}{\cos\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)} - \frac{Wx}{2P} \quad . \quad . \quad (24)$$

An equation giving the deflection for any value of x, knowing P and W,

y is a maximum when $x = \frac{l}{2}$

$$y_{max} = \frac{W}{2P}\sqrt{\frac{EI}{P}} \cdot \tan\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) - \frac{Wl}{4P} \quad . \quad . \quad (25)$$

The bending moment will be a maximum at the centre, and will be of the negative sign for known values of P and W.

$$-M_{max} = \frac{W}{2}\sqrt{\frac{EI}{P}} \cdot \tan\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) - \frac{Wl}{4} + \frac{Wl}{4}$$

$$= \frac{W}{2}\sqrt{\frac{EI}{P}} \tan . \left(\frac{l}{2} \cdot \sqrt{\frac{P}{EI}}\right) \quad . \quad (26)$$

Assuming elastic conditions hold up to failure, the critical load for P, will be that value when the maximum compression stress becomes the failing stress of the material $= f_c$.

$y_c =$ distance of the compression flange from the neutral axis.

There will be an equivalent load P acting at the centroid to complete the couple Py_{max}

$$f_c = \frac{W}{2}\sqrt{\frac{EI}{P}}\tan\left(\frac{l}{2}\cdot\sqrt{\frac{P}{EI}}\right)\cdot\frac{y_c}{I}+\frac{P}{A} \qquad . \quad . \quad (27)$$

Knowing f_c, W, I, A, and y_c, the Equation (27) can be solved to give the failing value of P.

85. **Case II.** Strut pin-jointed, end load applied axially, with a uniformly-distributed transverse load (over the whole length of the strut). (Fig. 91.) The strut is constrained to bend in the direction of one of the principal axes.

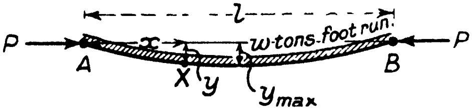

FIG. 91

$$M_x = EI\frac{d^2y}{dx^2} = -Py - \frac{wl}{2}x + \frac{wx^2}{2} \qquad . \quad . \quad . \quad (28)$$

The solution of this equation gives

$$y = A\sin\left(\sqrt{\frac{P}{EI}}\cdot x\right) + B\cos\left(\sqrt{\frac{P}{EI}}\cdot x\right) - \frac{wlx}{2P} + \frac{wx^2}{2P} - w\frac{EI}{P^2} \qquad . \quad . \quad . \quad (29)$$

$$\text{when } x = 0, \quad y = 0, \quad B = +w\frac{EI}{P^2}$$

$$\text{when } x = \frac{l}{2}, \quad \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 0 = A\sqrt{\frac{P}{EI}}\cdot\cos\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) - w\frac{EI}{P^2}\sqrt{\frac{P}{EI}}\sin\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) - \frac{wl}{2P} + \frac{2wl}{4P}$$

$$A = +\frac{wEI}{P^2}\cdot\tan\left(\frac{l}{2}\cdot\sqrt{\frac{P}{EI}}\right)$$

when $x = \frac{l}{2}$, $y =$ maximum

$$y_{max} = + \frac{wEI}{P^2} \frac{\sin^2\left(\frac{l}{2} \cdot \sqrt{\frac{P}{EI}}\right)}{\cos\left(\frac{l}{2} \cdot \sqrt{\frac{P}{EI}}\right)} + w\frac{EI}{P^2} \cos\left(\frac{l}{2} \cdot \sqrt{\frac{P}{EI}}\right)$$

$$- \frac{wl^2}{4P} + \frac{wl^2}{8P} - \frac{wEI}{P^2}$$

$$= - \frac{wEI}{P^2}\left[1 - \sec\left(\frac{l}{2} \cdot \sqrt{\frac{P}{EI}}\right)\right] - \frac{wl^2}{8P} \quad . \quad .. \qquad (30)$$

The moment at the centre will be a maximum

$$- M_{max} = Py_{max} + \frac{wl^2}{8} = \frac{wEI}{P}\left[\sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) - 1\right]$$

Assuming elastic conditions to hold to failure,

$f_c =$ failing stress in compression of the material,

$$f_c = \frac{wEI}{P}\left[\sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) - 1\right]\frac{y_c}{I} + \frac{P}{A} \quad . \quad . \qquad (31)$$

85*a*. If $P_e =$ Euler failing load for the struts with transverse loads,

$$P_e = \frac{\pi^2 EI}{l^2}$$

$$\sqrt{P_e} = \frac{\pi}{l}\sqrt{EI} \text{ or } \frac{l}{\sqrt{EI}} = \frac{\pi}{\sqrt{P_e}}$$

86. **Case I.** Concentrated transverse load at the centre of the strut,

$$f_c = \frac{W}{2} \frac{l}{\pi}\sqrt{\frac{P_e}{P}} \tan\left(\frac{\pi}{2}\sqrt{\frac{P}{P_e}}\right) \cdot \frac{y_c}{I} + \frac{P}{A} \quad . \quad . \qquad (32)$$

It can be shown* that if $P < P_e$

$$\text{then } f_c = \frac{P}{A} + \frac{Wl}{4Z} + \frac{Wl^3}{48EI} \cdot \frac{P}{Z} \cdot \frac{P_e}{P_e - P} \quad . \quad . \qquad (32a)$$

where $Z = I/y_c$

---

* *Materials and Structures*, Vol. I, by E. H. Salmon.

The longitudinal load P increases the deflection due to the lateral load, in the ratio $P_e : (P_e - P)$.

87. **Case II.** Uniformly-distributed transverse loads,

$$f_c = w\frac{l^2}{\pi^2}\frac{P_e}{P}\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_e}}\right) - 1\right]\frac{y_c}{I} + \frac{P}{A} \quad . \qquad (33)$$

It can be shown* that if $P < P_e$

$$\text{then } f_c = \frac{P}{A} + \frac{wl^2}{8Z} + \frac{5}{384}\cdot\frac{wl^4}{EI}\cdot\frac{P}{Z}\cdot\frac{P_e}{P_e - P} \quad . \qquad (33a)$$

where $Z = I/y_c$

The longitudinal load P increases the deflection due to the lateral load in the ratio $P_e : (P_e - P)$.

Knowing w or $W \cdot f_c \cdot P_e$ and A, P can be ascertained from equations (32*a*) and (33*a*).

Illustrative Problem 21.

1. Calculate the thickness of a mild steel tube 6 in. in internal diameter, 12 ft. long, to support an axial load of 15 tons. The tube is fixed at its ends.

Taking a factor of safety of 4,

Let $D_1 =$ external diameter in inches

$$t = \text{thickness in inches} = \frac{D_1 - 6}{2}$$

$$I = \frac{\pi}{4}\left(D_1^2 - 6^2\right)\left(\frac{D_1^2 + 6^2}{16}\right) = \frac{\pi}{64}\left(D_1^4 - 6^4\right) = \text{area} \times k^2$$

$l = 144$ in.

$$\text{the effective length} = \frac{l}{2} = 72 \text{ in.}$$

Take E for mild steel = 13,000 tons per sq. in.

Using Euler's formula, the buckling load is

$$4 \times 15 = \frac{\pi^2 \times 13{,}000 \times \frac{\pi}{64}\left(D_1^4 - 6^4\right)}{72 \times 72}$$

Take $\pi^2 = 10$,

$$D_1^4 = 1345$$

$$D_1 = 6{\cdot}06 \text{ in.}$$

$$\text{then } t = \frac{0{\cdot}06}{2} = 0{\cdot}03 \text{ in.}$$

---

* *Materials and Structures*, Vol. I, by E. H. Salmon.

Rankine's Formula. Take f_c for mild steel as 22 tons per square inch.

$$a = \frac{1}{7000}$$

$$60 = \frac{22 \times \frac{\pi}{4}\left(D_1^2 - 36\right)}{1 + \frac{1}{7000}\left(\frac{72 \times 72}{D_1^2 + 36}\right) \times 16}$$

$$\text{Solving } D_1 = \sqrt{40} = 6{\cdot}32 \text{ in.}$$

The thickness of the tube is, therefore, $\frac{0{\cdot}32}{2} = 0{\cdot}16$ in.

$\frac{l}{k} = 15{\cdot}2$. Rankine

$\frac{l}{k} = 15{\cdot}8$. Euler

Illustrative Problem 22.

An *I* beam, 12 in. × 6 in., 54 lb. per foot length, is used as a horizontal strut hinged at both ends, and is 20 ft. long centre to centre. The beam is arranged with its web horizontal. The least moment of inertia is 28·3 in.^{4}. There is an axial load of 24 tons, and a vertical load of 1 ton at the centre of the length. Calculate the maximum bending moment approximately, and find the maximum and minimum normal stresses. (U. of L., 1923.)

Cross-sectional area of the beam is 15·88 sq. in.

(1) Neglecting the weight of the strut, the maximum moment is (*see* Equation 26)

$$= \frac{W}{2}\sqrt{\frac{EI}{P}} \,.\, \tan \frac{l}{2}\sqrt{\frac{P}{EI}}: \text{ Note } \frac{l}{2}\sqrt{\frac{P}{EI}} \text{ is in radians.}$$

Take $E = 13{,}000$ tons/square inch. l to be in inches.

$$M_{max} \text{ tons-in.} = \frac{1}{2}\sqrt{\frac{13{,}000 \times 28{\cdot}3}{24}} \,.\, \tan \frac{240}{2}\sqrt{\frac{24}{13{,}000 \times 28{\cdot}3}}$$
$$= 86{\cdot}6 \text{ tons-in.}$$

$$\text{Max. compressive stress} = 86{\cdot}6 \times \frac{3}{28{\cdot}3} + \frac{24}{15{\cdot}88}$$

$$= 9{\cdot}24 + 1{\cdot}51 = 10{\cdot}75 \text{ tons/sq. in.}^*$$

$$\text{Max. tensile stress} = 9{\cdot}24 - 1{\cdot}51 = 7{\cdot}73 \quad ,,$$

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* By Eqn. (32*a*) $f_e = 11{\cdot}08$ tons/sq. in.

Allowing for the weight of the beam, it can be shown that

$$-M_{max} = \frac{wEI}{P}\left[\left(\sec\frac{l}{2}\sqrt{\frac{P}{EI}}\right) - 1\right] + \frac{W}{2}\sqrt{\frac{EI}{P}}\,.\,\tan\frac{l}{2}\sqrt{\frac{P}{EI}}$$

that is, the sum of the maximum moments due to a transverse concentrated and a uniform transverse load acting individually in conjunction with the end load;

$w = 54$ lb. per ft. run $= 4{\cdot}5$ lb. per in. run $= \frac{1}{500}$ ton per in. run

$$-M_{max} = \frac{1}{500} \times \frac{13{,}000 \times 28{\cdot}3}{24}\left(-1 + \sec\,(55{\cdot}5^\circ)\right)$$

$$+\ 86{\cdot}6 \text{ tons-in.}$$

$$= 109{\cdot}9 \text{ tons-in.}$$

$$\text{Max. compressive stress} = 109{\cdot}9 \times \frac{3}{28{\cdot}3} + 1{\cdot}51$$

$$= 11{\cdot}6 + 1{\cdot}51 = 13{\cdot}11 \text{ tons/sq. in.}$$

$$\text{Max. tensile stress} = 11{\cdot}6 - 1{\cdot}51 = 10{\cdot}09 \text{ tons/sq. in.}$$

Illustrative Problem 23.

A strut consisting of a steel tube 4 in. outside diameter and $\frac{3}{16}$ in. thick is loaded along an axis parallel to the centre line and $\frac{1}{8}$ in. from it. The tube is 120 in. long. The yield stress of the steel is 28 tons per square inch. Find the crushing load of the strut. (The method of solution of the equation you devise must be clearly indicated.) (U. of L., 1923.)

Take the strut as hinged at both ends.

Using the Euler form for eccentric loading (Eqn. 20),

$$f_{max} = Pe\frac{y_c}{I}\sec\left(\sqrt{\frac{P}{EI}}\cdot\frac{l}{2}\right) + \frac{P}{A}$$

$f_{max} = 28$ tons/sq. in. $l = 120$ in. $= l_1$

$E = 13{,}000$ tons/sq. in. $y_c = 2$ in. $e = \frac{1}{8}$ in.

$$I = 4{\cdot}12 \text{ in.}^4$$

$$A = 2{\cdot}27 \text{ sq. in.}$$

$$k^2 = 1{\cdot}82 \text{ in.}^2$$

$$\frac{l}{k} = \frac{120}{1{\cdot}35} = 89$$

$$28 = P\frac{1}{8} \times \frac{2}{4{\cdot}12}\sec\left(\sqrt{\frac{P}{13{,}000 \times 4{\cdot}12}} \times 60\right) + \frac{P}{2{\cdot}27}$$

$$= \frac{P}{16{\cdot}48}\sec\frac{\sqrt{P}}{3{\cdot}86} + \frac{P}{2{\cdot}27}$$

By trial, $P =$ nearly $30{\cdot}55$ tons.

By Rankine's formula for eccentric loads,

$$P = \frac{f_c A}{\left[1 + a\left(\frac{l_1}{k}\right)^2\right]\left[1 + e\frac{y_c}{k^2}\right]}$$

$$\text{Taking } a = \frac{1}{7000}$$

$$P = \frac{28 \times 2{\cdot}27}{\left[1 + \frac{1}{7000} \times \frac{120^2}{1{\cdot}82}\right]\left[1 + \frac{1}{8} \times \frac{2}{1{\cdot}82}\right]} = 26 \text{ tons}$$

Eccentrically-loaded strut: failing load
is 30·55 tons (Euler method)
and is 26·0 tons (Rankine formula).

Note—Euler's Critical Load $= P = \frac{\pi^2 EI}{l^2} = 36{\cdot}6$ tons

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EXAMPLES

1. Find the buckling load of a channel section strut 12 in. × 7 in. × 1 in., length 16 ft., both ends rigidly fixed, by Rankine's formula. Check your result by Euler's formula and, if there is any discrepancy between the two, state which result you consider to be the more reliable, and why? Draw the Euler and Rankine curves.

2. A strut, whose length is 100 times its diameter, is compressed by a load applied axially. Describe how the strut gives way, and show how the nature of the end constraints affects the strength. Discuss the application to this case of Euler's formula,

$$\text{Limiting load} = \frac{\pi^2 EI}{n^2 l^2}$$

Derive the Rankine formula and explain its use.

3. A column of circular section, the internal and external diameters of which are d and D respectively, has to support a non-axial load of W tons acting at a distance x from the centre of the column. Find the value of x in order that there may be no tension in the material. (I.C.E.)

4. A strut is formed by two channels 7 in. × 3½ in. braced together back to back and 3½ in. apart. Their flanges are 0·5 in. and the web 0·4 in. in thickness. If the strut is 20 ft. long and hinged at the ends, find the safe load, given that the safe working load on a column with fixed ends

$$= \frac{6A}{1 + \left(\frac{1}{20{,}000}\right)\frac{L^2}{k^2}}$$

when A is the cross-sectional area of the strut ;

L is its length ; and

k the radius of gyration about an axis at right angles to the webs (I.C.E.)

5. The vertical pillar of a crane is of I section, 18 in. deep, area 22 sq. in., maximum moment of inertia 1150 in inch units. Find the maximum intensities of compressive and tensile stress in the pillar, when a load of 6 tons—acting in a plane containing the lengthwise centre line of the web—is being carried at a radius of 12 ft. from the centre of the section. (I.M.E.)

6. A bar of steel under a tension test is gripped so that the line of pull is ⅛ in. from the axis of the bar. It is 2 in. diameter. What is the maximum stress produced in the specimen when the load registered by the testing machine is 10 tons ? Deduce any formula you use. (U. of B.)

7. A short timber prop 6 in. diameter is loaded with a load of 10 tons along a line parallel to the axis and at ½ in. from it. Find the maximum and minimum stress in the section of the prop. (U. of B.)

8. You are required to ascertain how the strength of mild steel hollow tubes (say, 4 in. external and 3½ in. internal diameters) used as struts depends upon the length of the tube. How would you carry out tests ? Describe any special fittings you would use. What kind of curve would you plot ? Compare the failure of a tube 1 foot long with one 8 ft. long. (U. of B.)

9. A square hollow cast-iron column 12 ft. long, sides 6 in., and walls 1½ in. thick, is fixed at the lower end and pin-connected at its upper end. Calculate the load the column will safely carry, allowing a reasonable factor of safety.

$$f_c = 36 \text{ tons per sq. in.} \quad \text{Constant} = \frac{1}{1600}$$

(U. of B.)

10. A cast-iron column, rigidly fixed in the ground, with its upper end free, is 20 ft. long ; the cross-section is a hollow cylinder 12 in. outside diameter, 8 in. inside diameter. Calculate the safe load this column will carry.

(f_c = 36 tons per square inch. The constant for a column pin connected at both ends is $\frac{1}{1800}$) (U. of B.)

11. A round bar 2 in. diameter is subject to a pull of 5 tons along a line parallel to the axis and ¼ in. from it. Determine the maximum and minimum stresses on the sections of the bar.

12. A solid cast-iron column, 4 in. in diameter and 2 ft. long, is fixed at both ends. Calculate the load the column will safely carry by Rankine's and Euler's formula.

In Rankine's formula, take $a = \dfrac{1}{1800}$ and safe compressive stress for short lengths of the material = 7 tons per square inch.

In Euler's formula, take E for cast-iron = 10,500 tons per square inch.

(U. of B.)

13. Compare the strength of columns 12 ft. long containing the same volume of metal: (*a*) the column being rolled steel joist of I section 10 in. × 8 in. × $\frac{3}{4}$ in.; (*b*) cast-iron hollow cylindrical column, the metal being $\frac{3}{4}$ in. thick.

Use Rankine's formula—

$$P = \frac{f_c A}{1 + a\left(\frac{l}{k}\right)^2}$$

f_c for steel 21 for cast-iron 36

a for steel $\dfrac{1}{7500}$ for cast-iron $\dfrac{1}{1600}$

(U. of L.)

14. Discuss briefly the limits of application of Euler's and Rankine's formula for struts.

A mild steel tube is used as a vertical strut. It is 2 in. internal diameter and the metal is $\frac{3}{16}$ in. thick. It is 20 ft. long and firmly set in a foundation of concrete. The upper end is quite free. Making reasonable assumptions as to constants, calculate the load at which it will fail.

If the strut was reduced to 3 ft. in length, what load could it carry ?

(U. of B.)

15. Find the external diameter and thickness of metal for a hollow steel strut 10 ft. in length and capable of carrying an axial thrust of 20 tons. The ratio of diameters to be 10 to 8, and the ends of the struts are firmly fixed. Use the Rankine formula : f_c = 21 tons per square inch ; $a = 1/7500$ (for rounded ends) ; factor of safety = 5. (I.M.E.)

16. A rolled steel joist, 8 in. deep and 6 in. wide, is used as a strut. Its moment of inertia is 110·5 in inch units, and the cross-sectional area is 10·3 in. A compressive load of 40 tons acts along a line lying in the centre of the web and parallel to the longitudinal axis of the joist, but at a distance of 2 in. from it. Determine the maximum intensity of stress induced.

(I.M.E.)

17. Find the safe working load for a column of the section indicated in the sketch, 20 ft. long, and having its ends fixed, using the formula

$$\text{Working load} = \frac{6A}{1 + \dfrac{1}{20{,}000}\dfrac{L^2}{k^2}} \text{ tons,}$$

where A is the cross-sectional area in square inches,
L is the length of the column,
and k is the least radius of gyration about a diameter.

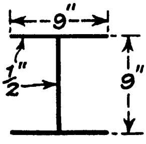

FIG. 92

(I.C.E.)

18. A steel built-up stanchion is made of two plates 16 in × $\frac{7}{8}$ in., two plates 12 in. × $\frac{7}{8}$ in., and four angles $3\frac{1}{2}$ in. × $3\frac{1}{2}$ in. × $\frac{1}{2}$ in., the section forming an internal rectangular space of 12 in. × 7 in. The stanchion is 30 ft. high and loaded centrally. Considering the stanchion is fixed at both ends, determine the safe load. (U. of L., 1923.)

19. Find the necessary diameter of a mild steel strut 8 ft. 4 in. long, freely hinged at each end, if it has to carry a thrust of 10 tons with a possible deviation from the axis of one-eighth the diameter, the greatest compressive stress not to exceed 5 tons per square inch. (E = 13,000 tons/square inch.)

20. A mild steel strut, pin-jointed, 5 ft. long, $I = 10{\cdot}0$ (in.)^4, carries an axial thrust of 5 tons. What transverse load can it carry at the centre to develop a maximum stress of 22 tons per square inch, assuming the strut remains elastic? Cross-sectional area of the strut, 0·5 sq. in. ($E = 13{,}000$ tons per square inch.)

21. If the strut in Question 16 is pin-jointed, 100 in. long, and the eccentricity is as before, find the buckling load of the strut (*a*) by Euler's method, (*b*) by Rankine's method, assuming the failing stress of the steel is 22 tons per square inch in compression.

22. A strut, consisting of a steel tube 4 in. outside diameter and $\frac{3}{16}$ in. thick, is loaded along an axis parallel to the centre line and $\frac{1}{8}$ in. from it. The tube is 120 in. long. The yield stress of the steel is 28 tons per square inch. Find the crushing load of the strut. (L.U.)

Students are requested to solve the problems, where possible, by the straight line and parabolic formulae, using any constants necessary from the tables.

CHAPTER VII

FRAMED STRUCTURES WITH DEAD LOADS

88. **Frames or Trusses.** A frame is a structure which consists of ties and struts pin-jointed or riveted together; its individual members are in tension or compression. *Ties* take pulls, *struts* take pushes.

Notes on the Frames given in Fig. 93. (*a*) is the Lunville or *N*-girder, or Pratt Truss. The verticals are struts and the diagonals are ties.

(*c*) is the Howe Truss : the verticals are in tension and the diagonals are in compression.

(*b*) and (*d*) are respectively the single and double Warren girders, both ties and struts being inclined, usually at angles between 45° to 60°.

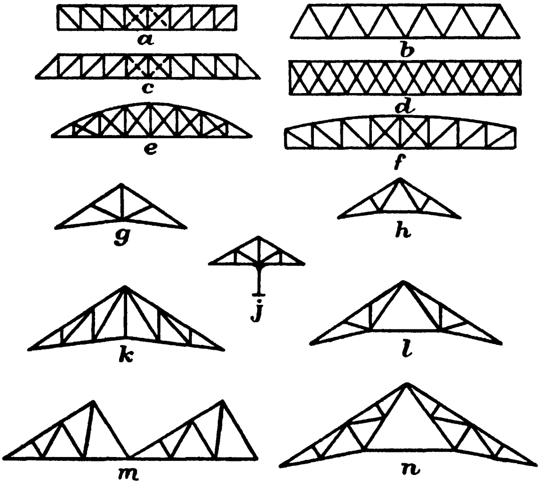

FIG. 93

(*e*) and (*f*) are hog-backed lattice girders or lattice bow girders.

(*g*) to (*n*) are various types of roof-trusses, (*g*) and (*h*) being used for small spans up to 30 ft. or so ; (*k*) and (*l*) for large spans of 45 to 50 ft. ; (*n*) is a French truss, which is used for a span of 60 ft.; (*m*) is a common factory roof, known as a northern light or saw-tooth roof. The short side is glazed to admit a northern light without direct sunshine.

89. All frames in this chapter and the following ones are taken as pin-jointed.

Frames may be divided into three classes, viz., perfect, incomplete or imperfect, and redundant.

A perfect frame has just sufficient members to keep it in equilibrium under all systems of loading. Frames in one plane are called plane frames. The simplest plane and perfect frame is the triangle: the basis of all perfect frames. The frame in Fig. 94 is a perfect frame.

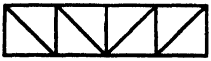

FIG. 94

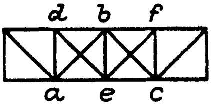

FIG. 95

If the diagonals of the frame in Fig. 94 were removed, then the frame would be incomplete or imperfect. Under certain conditions of loading it will be in equilibrium.

The frame in Fig. 95 has too many members (by *ab* and *bc*), and it is a redundant frame. If the diagonals *ab* and *bc* were removed, the frame would be perfect, as shown in Fig. 94.

If *ab* and *bc* are only capable of taking tension and are consequently quite light, the structure would be just braced; for if *de* and *ef* were taking tension, then *ab* and *bc* would slacken off. Counter bracing as shown is inserted if the sign of the shear in the bay is likely to change due to loads moving over the frame. Overbraced frameworks are perhaps more rigid than perfect frames, but the loads in the members cannot be calculated by simple processes. Often they are divided up into perfect frames with certain common members; the loads in each are calculated separately and then added algebraically. This is not an accurate method, but it gives a close approximation in many cases.* The number of members in a perfect frame is $2n - 3$, where n is the number of joints.† Any number greater than $2n - 3$ will represent the number of "internal" redundant members.

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* *See* Chapter IX.

† *See also* Chapter IV for notes on supports and "external" redundants.

90. **Methods of Determining the Stresses in Framed Perfect (Plane) Structures.** The forces, including the reactions, acting on the structure being known or calculated, the loads in the members themselves may be found by one, or a combination of three methods—

(1) The stress or force diagram ;
(2) The method of sections ;
(3) Resolution.

The supports for the frames will be such that the reactions can be calculated by statical methods. There will be no redundant restraints at the supports. (*See* Chapter IV.)

91. **Stress or Force Diagrams.** Probably the most satisfactory method of determining the forces in the individual members is by means of a stress diagram.

*The following method is given by Capito in his *Applied Mechanics*—

Fig. 96 (*a*) represents a triangular link frame with three forces P_1, P_2, and P_3 acting respectively at the three joints in the plane of the figure. The lettering of the frame and the forces may be as shown in the figure ; it will be seen that every bar and force has a letter to its right and left which gives a name to it. The bars are named thus : *OA*, *OB*, *OC*. This is called Bow's notation.

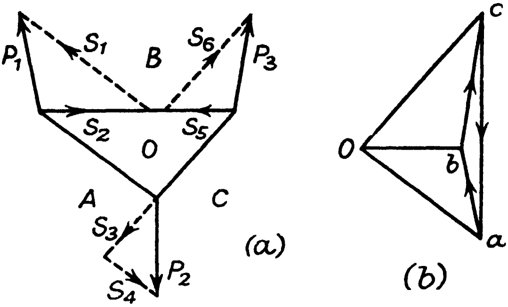

FIG. 96

As every force has also a letter on each side, they may be named, when taking a clockwise direction round the frame, *AB*, *BC*, *CA* instead of P_1, P_2, P_3. Finally the joints are named *OAB*, *OBC*, and *OCA*.

Resolve each of the forces P_1, P_2, P_3 in the direction of its adjacent bars. At the joint *OAB*, P_1 is resolved into S_1 parallel to *OA*, and S_2 along *OB* ; similarly for the other joints. Obviously S_1 and S_4 act in the same direction but not in the same sense, and similarly for the other forces.

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* Capito, *Applied Mechanics*, Part II. (Griffin.)

The given forces will obviously be in equilibrium when

$$S_1 + S_4 = O \qquad S_2 + S_5 = O \qquad S_3 + S_6 = O$$

Choose any pole O in Fig. 96(b), and draw the three straight lines Oa, Ob, and Oc respectively parallel to the bars OA, OB, and OC.

Set off to some scale

$$\begin{aligned} Oa &= S_4 = -S_1 \\ Ob &= S_2 = -S_5 \\ Oc &= S_6 = -S_3, \end{aligned}$$

then $ab = P_1$, $bc = P_3$, and $ca = P_2$.

As these three latter form a triangle, they are, therefore, in equilibrium.

NOTE.—At each joint there are two internal forces (in the members) and one external load; as the frame is in equilibrium, then these forces are in equilibrium, and so form the sides of a triangle. If there are more than two members at a joint, then all the internal forces and the external force are in equilibrium, and the forces to scale must form a closed polygon.

The diagram in Fig. 96(b) is called the force-stress diagram.

When n coplanar straight lines emanate from a point O and an n-sided polygon be drawn whose sides are parallel to, or perpendicular to, the corresponding lines through O. then the polygon is called the reciprocal for the point O.

In Fig. 96(b) the triangles Oab, Oca, and Obc are reciprocals for the joints OAB, OCA, and OBC respectively; hence the force-stress diagram is the reciprocal force diagram for the frame shown. Fig. 96(b) represents the three triangles placed in one figure. In practice, the external loadings on a frame are given; for known conditions, and by means of moments or otherwise, the reactions may be found.

A simple frame is shown in Fig. 97 loaded with a single load W acting vertically at the joint OBC. The frame is supported on rollers at the joints OAB and OCA.

Let $OB = OC$,

then obviously the reactions R_1 and R_2 acting vertically upwards are each equal to $\frac{W}{2}$.

The lettering of the diagram, as indicated, starts at the centre space O, and the remaining spaces formed are then lettered in a clockwise direction, that is, lettering all the spaces between the external forces first and then between the members themselves (as also shown for another frame in Fig. 98).

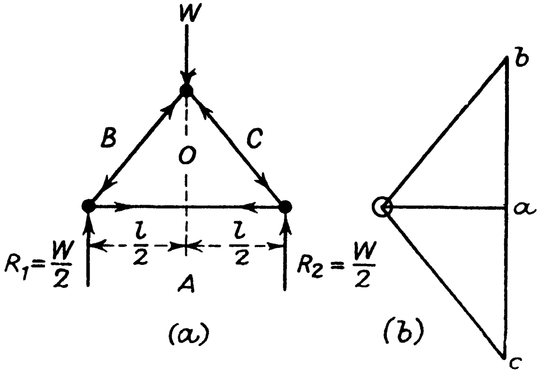


FIG. 97

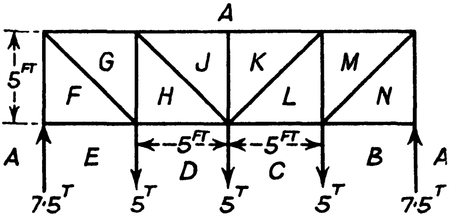


FIG. 98

Continuing, for the simple frame in Fig. 97, draw a load diagram (Fig. 97 (b)), which in this case will be a vertical straight line bc. The force-stress diagram for the frame consists of a number of triangles.

$bc = W$ to scale and acting from b to c; that is, downwards.
$ca = R_2$ upwards;
$ab = R_1$ „

The triangle abc in this case is a single line, and $bc = ca + ab$.

Considering joint OAB. From b on the load diagram draw bo parallel to BO, and from a, ao parallel to AO; then

Oab is the triangle of forces for the joint *OAB*; *ob* and *Oa* to scale represent the internal forces in *OB* and *OA* acting at the joint.

Considering the joint *OBC*, *W* is the external force, and from the previous triangle one internal force *Ob* is known; i.e. on the force-stress diagram $bc = W$; $Ob =$ internal stress in *OB*. Joining *O* to *c* completes the stress diagram, which is a triangle *Obc* for the joint *OBC*. *Oc* to scale is the internal force in the member *OC*.

92. **The Kind of Stress in a Member.** (Fig. 99.) (i) Considering joint *OBC*. Start with *W* and work round the joint in a clockwise direction. On the force-stress diagram, starting at *b*. $bc = W$ which is in a downward direction.

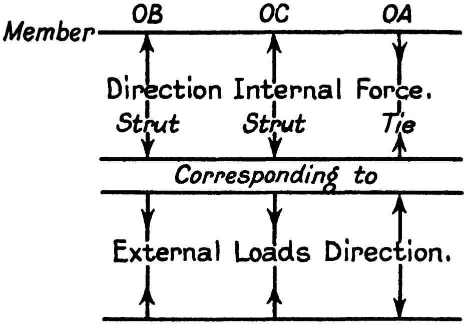


FIG. 99

Consider the member *OC*; in the stress diagram, proceed from *c* to *o* ↖. This indicates the internal force in the member *OC* acting towards the joint *OBC*; similarly for *OB*, completing the direction round the stress diagram *O* to *b* is in ↗ direction, again indicating an internal force acting towards the joint *OBC*. The directions of the internal forces at the joint are indicated in Fig. 97.

(ii) Consider the joint *OAB*; on the stress diagram *a* to *b* is ↑ upwards; *b* to *O* is ↙, indicating a force acting towards the joint *OAB*; *O* to *a* is ⟶, indicating an internal force acting away from the joint.

(iii) Similar reasoning applies for bars at the joint *OAC*.

Internal forces acting towards a joint indicate a strut; away from a joint, a tie. (Fig. 99.)

93. With this method, if there are three unknowns to solve for at a joint, it fails. It is necessary to find one of the unknowns by another method, thus leaving two which can be found by continuing the force-stress diagram.

94. **The Method of Sections.** If the internal forces acting at the joints of the link frame *DA*, *DB*, and *DC* are balanced, equilibrium will not be disturbed by cutting the bars *DB* and *DC* (Fig. 100), provided that the necessary external forces be added at the points where the bars are cut. The two latter forces and force *BC* must balance; but only *BC* is known. As, however, the resultant of any two of the three forces acts in the line of the third force, we may choose any point *O* in the line of action of one of the unknown forces, say *DB*, as fulcrum, and measure the lengths of the perpendiculars p_1 and p_2, let fall from *O* on the forces *BC* and *CD* respectively. We have thus,

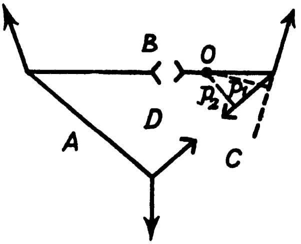


Fig. 100

$$p_1 \times \text{external force } BC = p_2 \times \text{internal force in } DC.$$

$$\text{Hence, force in } DC = \frac{p_1}{p_2} \times \text{force } BC.$$

By this method, known as the method of sections, all the internal forces in the bars may be determined.

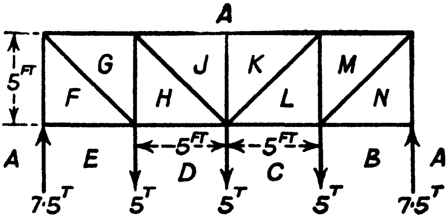


Fig. 100A

Illustrative Problem 24.

A loaded frame is given in Fig. 100 A. It is required to find the forces and kind of stress in the members *DH* and *AJ*.

Always take a dividing line, so that when taking moments about a point *O*, the unknown force required will be the only unknown force having a moment about this point. The reactions are equal and of 7·5 tons each.

To Find the Force in DH. The dividing line will cut the members AJ, JH, DH as indicated in Fig. 101(a). This portion of the beam is in equilibrium under the loading shown.

The force in DH is required. Take moments about the joint $AGHJ$, i.e. point Y. Then from the diagram,

$$7{\cdot}5 \times 5 \ (\circlearrowright) = DH \times 5 \ (\circlearrowleft)$$

Force in $DH = dh = 7{\cdot}5$ tons, and it acts away from its joint $EFGHD$, thus indicating a tensile force in the member.

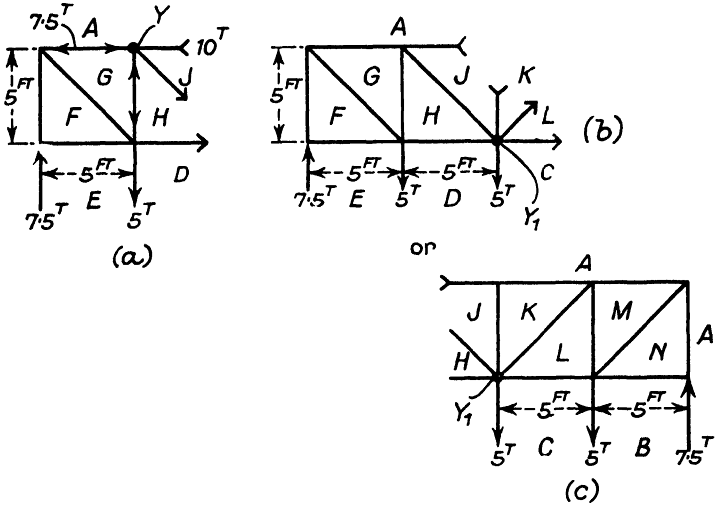


Fig. 101

Similarly for the Member AJ,

Moments about the joint Y_1, *see* Fig. 101(b).

$$\text{Then } AJ \times 5 \ (\circlearrowleft) = 7{\cdot}5 \times 10 \ (\circlearrowright) + 5 \times 5 \ (\circlearrowleft)$$

$$= 50 \ (\circlearrowright) \text{ tons-ft.}$$

$$\text{Hence the internal force in } AJ = \frac{50}{5} = 10 \text{ tons } (\circlearrowleft)$$

and it acts towards the joint $AGHJ$ and, therefore, the member AJ is a strut.

The frame may be cut also as indicated in Fig. 101(c).

In this case take moments again about Y_1 for the left-hand portion.

95. The Force in the Member *JH* is found from the method of **Resolution of Forces.** This method consists in resolving the resultant force at any point in a structure or across a particular section, along the members of the frame meeting at that point or cut by the section. If there are more than two members meeting at a point, then all the remainder must be known in order to obtain the loads in the remaining two by this method. Fig. 101 (*a*).

At the joint *AJHG*, four members meet. It can be shown by the previous method that the internal compressive force in the member *AG* is 7·5 tons.

The joint is in equilibrium, therefore, the algebraic sum of all the horizontal components of the forces in the members must be zero, and similarly for the vertical components.

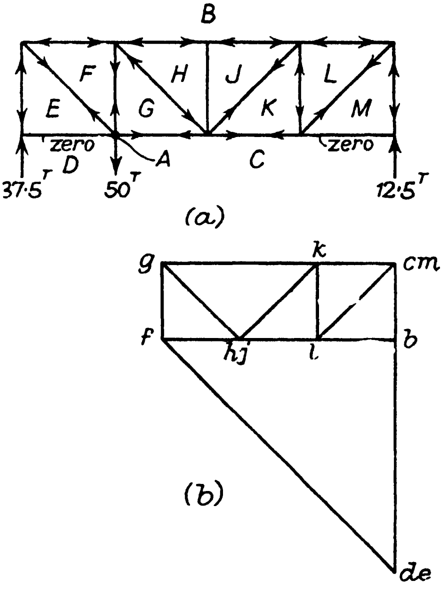


Fig. 102

The members *AG* and *AJ* are horizontal, *GH* vertical, and *JH* (from the dimensions of the frame) is at an angle of 45° to the horizontal.

Let tensile forces be positive and compressive forces negative.

Resolving horizontally,

$$7{\cdot}5\ (AG) - 10\ (AJ) = -\ (\text{horizontal component})\ JH$$

$$\text{Horizontal component in } JH \ \therefore\ +\ 2{\cdot}5$$

and *JH* must be a tie.

$$JH \cos 45^\circ = 2{\cdot}5$$

therefore, tensile force in $JH = 2{\cdot}5\sqrt{2}$ tons.

The vertical component of the force in $JH = +\ 2{\cdot}5$ tons; therefore, the vertical force in the member *GH* is $-\ 2{\cdot}5$ tons: as these are the only two members acting at the joint *Y*;

therefore, *GH* is a strut having an internal compressive force of 2·5 tons.

[Note.—*GH* takes the shear force (10 − 7·5) tons = 2·5 tons in the second bay from the left-hand support.]

96. The method of resolution is not so elegant as the two previous as a whole. Where members meet at right angles to one another, however, it is often the quicker method. A combination of the methods may be the most useful.

Illustrative Problem 25. (Fig. 102.)

The lattice girder shown in Fig. 102 is loaded at the joint *A* with a load of 50 tons. Find the forces in the members, stating the kind of force.

The force-stress diagram is shown in Fig. 102(*b*): the arrows on the members in Fig. 102 indicate whether the member is a strut or a tie.

TABLE OF FORCES

Member.	Strut (*S*) or Tie (*T*).	Force in Member. Tons.	Member.	Strut (*S*) or Tie (*T*).	Force in Member Tons.
BE	*S*	− 37·5	*DE*	–	0
BF	*S*	− 37·5	*EF*	*T*	$+ \sqrt{2} \times 37{\cdot}5$
BH	*S*	− 25	*FG*	*T*	+ 12·5
BJ	*S*	− 25	*GH*	*S*	$- \sqrt{2} \times 12{\cdot}5$
BL	*S*	− 12·5	*HJ*	–	0
BM	*S*	− 12·5	*JK*	*T*	$+ \sqrt{2} \times 12{\cdot}5$
CM	–	0	*KL*	*S*	− 12·5
CK	*T*	+ 12·5	*LM*	*T*	$+ \sqrt{2} \times 12{\cdot}5$
CG	*T*	+ 37·5			

Determination of Some of the Forces by the Method of Sections. Take a dividing line through the three members in the second bay from the left and moments for the left-hand portion of the girder about the joint *FBHG*.

$$37{\cdot}5 \times 15\,(\circlearrowright) = GC \times 15\,(\circlearrowleft)$$

Force in *GC* = 37·5 tons and acting away from its joint, so that *GC* is a tie.

Taking moments of the forces to the left of and about the joint *CGHJK* to give the force in *BH*.

$$37{\cdot}5 \times 30\,(\circlearrowright) + 50 \times 15\,(\circlearrowleft) + BH \times 15\,(\circlearrowleft) = 0$$

$$\text{or} \quad 37{\cdot}5 \times 30 - 50 \times 15 - BH \times 15 = 0$$

Force in BH = 25 tons and acting towards its joint, therefore, negative so that BH is a strut.

The Force in EF by the Method of Resolution of Forces.

The force in BF can be shown equal to 37·5 tons.

Horizontal component of force in EF + force in BF = 0

$$EF \cos 45 + (-37{\cdot}5) = 0$$

therefore, force in EF

$$= \frac{+37{\cdot}5}{\cos 45^\circ} = \sqrt{2} \times 37{\cdot}5 \text{ tons}$$

and EF is a tie.

Illustrative Problem 26.

A Warren girder, 30 ft. span, has three equal bays in the lower boom. All the diagonals are inclined at 60° to the horizontal. There are loads of 15 tons at each of the two joints in the lower boom. Find the forces and the kind of force in the members of the girder. (Fig. 103.)

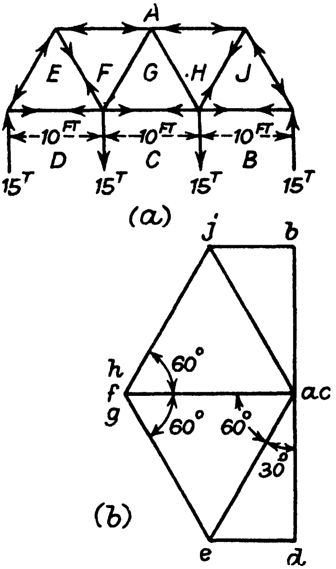


FIG. 103

All the members are 10 ft. long.

The force-stress diagram is given in Fig. 103(b).

TABLE OF FORCES

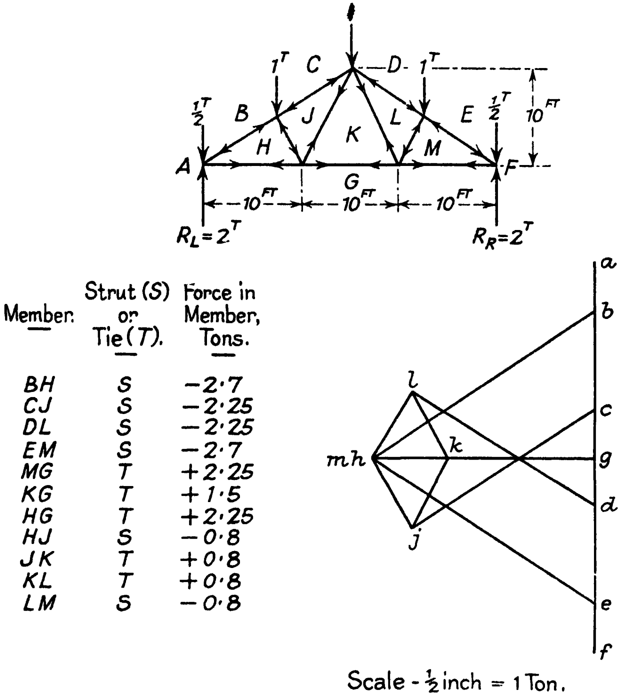
Member.	Tie (T) or Strut (S).	Force in Member.	Member	Tie (T) or Strut (S).	Force in Member.
AF	*S*	$-\frac{30}{\sqrt{3}}$	*EF*	*T*	$+\frac{30}{\sqrt{3}}$
AH	*S*	$-\frac{30}{\sqrt{3}}$	*FG*	—	0
BJ	*T*	$+\frac{15}{\sqrt{3}}$	*GH*	—	0
CG	*T*	$+\frac{30}{\sqrt{3}}$	*HJ*	*T*	$+\frac{30}{\sqrt{3}}$
DE	*T*	$+\frac{15}{\sqrt{3}}$	*JA*	*S*	$-\frac{30}{\sqrt{3}}$
AE	*S*	$-\frac{30}{\sqrt{3}}$			

The student is asked to check the forces by the use of the other methods.

Illustrative Problem 27 (*a*). (Fig. 104.)

Find the stresses in the members of the truss in the sketch, due to the loads indicated, and distinguish which are tensile and which compressive.

(I.C.E., 1923.)



Member	Strut (*S*) or Tie (*T*).	Force in Member, Tons.
BH	S	−2·7
CJ	S	−2·25
DL	S	−2·25
EM	S	−2·7
MG	T	+2·25
KG	T	+1·5
HG	T	+2·25
HJ	S	−0·8
JK	T	+0·8
KL	T	+0·8
LM	S	−0·8

Scale - ½ inch = 1 Ton.

FIG. 104

For the given loading, $R_L = R_R = 2$ tons.

Member	Strut (*S*) or Tie (*T*)	Force in Member. Tons.	Member.	Strut (*S*) or Tie (*T*)	Force in Member Tons.
BH	*S*	− 2·7	*HG*	*T*	+ 2·25
CJ	*S*	− 2·25	*HJ*	*S*	− 0·8
DL	*S*	− 2·25	*JK*	*T*	+ 0·8
EM	*S*	− 2·7	*KL*	*T*	+ 0·8
MG	*T*	+ 2·25	*LM*	*S*	− 0·8
KG	*T*	+ 1·5			

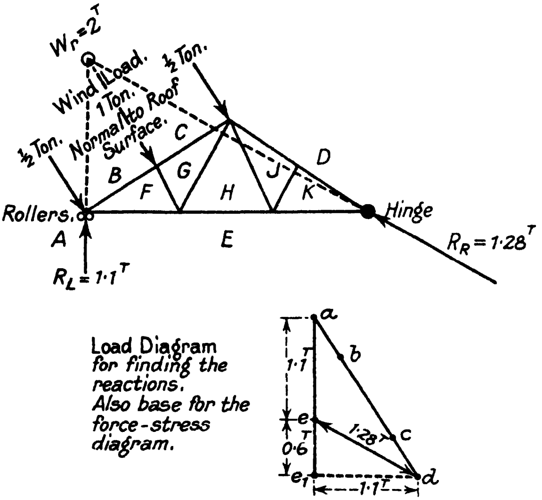
Illustrative Problem 27 (*b*). (Fig. 105.)

The truss in Problem 27*a* is fixed at one end and on rollers on the other. Wind loads, as indicated in Fig. 105, act on the roof. Find the reactions and also the forces in the members due to this loading.

The resultant of the wind loads W_R = 2 tons will act normally to the surface at the joint *BCGF*. The roof truss is in equilibrium under W_R and the two reactions ; hence the three forces must form the three sides of a triangle. R_L at the rollers will act vertically upwards. Draw the triangle of external forces, including the reactions as shown in Fig. 105.

ad = 2 tons : ae = 1·1 ton ; then $de = R_R$ in magnitude and direction.
$ea = R_L$ = 1·1 ton ; $de = R_R$ = 1·28 tons.

Draw the stress diagram as before, starting at the left-hand reaction.



Load Diagram for finding the reactions. Also base for the force-stress diagram.

FIG. 105

Illustrative Problem 27 (*c*). (Fig. 106.)

The roof truss in the previous problems 27*a* and 27*b* is loaded with the same dead loads and wind loads acting together. Find the forces in the members.

Method 1. The two loads act independently and so obtain a force-stress diagram for each ; add algebraically the forces in the members due to the two kinds of loads to give the resultant forces in them.

Method 2. On the left side of the truss in Fig. 106*a*, find the resultant loads and their lines of action for the dead and wind loads at each joint separately. As the external loads and the reactions are in equilibrium, it does not matter the type of truss. Divide it up into single triangles as illustrated. Working from the ridge joint, draw a force-stress diagram (for the newly-arranged truss), which will eventually be closed by the reactions which can be taken off to scale. Draw the force-stress diagram for the original truss, the load polygon for which will be *abcdefg*.

The diagrams are given in Fig. 106.

97. ROOF TRUSS. WITH THREE UNKNOWNS AT A JOINT (Fig. 107). Calculate the reactions in the usual way, starting from the left-hand reaction. Drawing the force-stress diagram, the forces in the members CK, KA are found. Proceeding to either of the joints $CDMLK$ or $KLPA$, it is found that there are three unknowns at each joint.

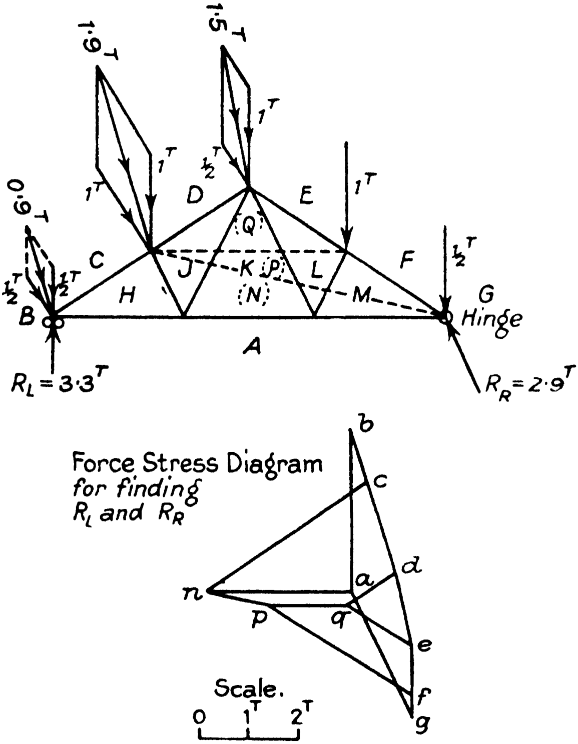


FIG. 106

Divide the truss into two parts by a dividing line through the ridge joint and the member PA; then the left-hand half of the truss is in equilibrium under the external loading, and the internal force in $PA = T = pa$ acting as an external force. Then from moments about the ridge joint,

$$\Sigma\,(\text{moments of the external forces and } R_L) + (pa)h = 0$$

In this case, T will act away from the joint, and so PA is a tie; pa is thus found, leaving two unknowns at the joint

AKLP. Force in *KL* can be found, leaving two unknowns at the joint *CDMLK*. Proceed with the completion of the stress diagram by the usual methods.

97*a*. NOTES ON WIND PRESSURES.

P = Intensity of wind pressure on a plane normal to the direction of the wind.

P_n = Normal Intensity of wind pressure on a plane inclined at an angle θ to the horizontal.

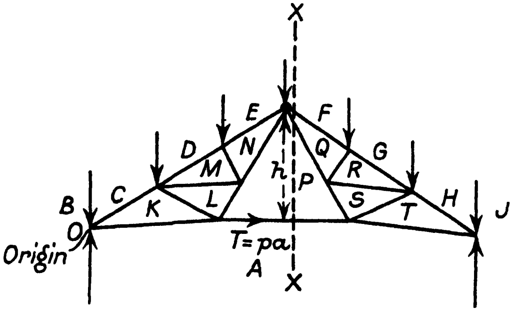


FIG. 107

FORMULAE FOR P_n

(1) $P_n = P \sin\theta$. (See Goodman's *Applied Mechanics*.)

(2) $P_n = P \frac{2\sin\theta}{1+\sin^2\theta}$. (Duchemin's formula: see Arrol's Handbook by A. Hunter.)

(3) $P_n = P \sin\theta^{(1\cdot 84\cos\theta - 1)}$ (Hutton's formula: see *Applied Mechanics*, Duncan.)

The tangential component of the pressure P on the surface inclined at the angle θ to the horizontal is assumed to exert no pressure on the plane.

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(1) *Mechanics* (Part II), Capito. (Messrs. Griffin & Son.) Methods and examples for finding loads in frames.
(2) *Applied Mechanics*, Duncan. (McMillan.) Further examples.
(3) *Elements of Graphic Statics*, Sprague. Examples.
(4) *Roofs and Bridges*, Merriman and Jacoby. (Wiley.)
Graphic Statics (Part II). Methods and examples.
(5) *Theory of Structures*, Jamieson. (Griffin.) Further examples.
(6) Arrol's *Bridge and Structural Engineer's Handbook*, A. Hunter. Notes on wind pressures, and scantlings of roof principals.
(7) *Design of Modern Steel Structures*, L. E. Grinter. (Macmillan.)
(8) *Modern Framed Structures*, Part I, Simple Structures, Johnson, Bryan and Turneaure. (Wiley.)

EXAMPLES

1. If a number of forces, some of which are inclined to the vertical, act on a hinged structure, and the direction of the reaction at one point of support is known, show how the magnitude of both reactions and the direction of the one at the other point of support can be found.

2. Find the stresses in the members of the truss in Fig. 108 due to the loads indicated, and show which are tensions and which compressions. (I.C.E.)

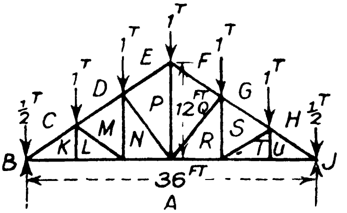


FIG. 108

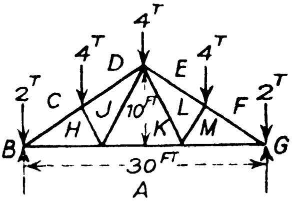


FIG. 110

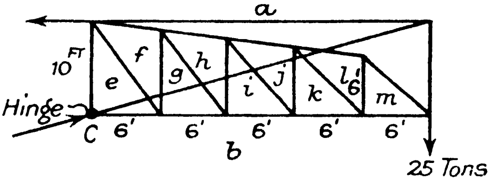


FIG. 109

3. If the left extremity in the sketch in Fig. 108 is hinged and the right can slide horizontally, find the stresses in the members due to a wind of an intensity equal to 40 lb. per square foot on a vertical surface blowing from left to right. Trusses 10 ft. apart. (I.C.E.)

4. Draw a reciprocal diagram giving the forces of the members in the framed cantilever shown. (Fig. 109.) (I.C.E.)

5. Find the stresses in the members of the truss in the sketch due to the loads indicated, and distinguish which are tensile and which compressive. (Fig. 110.) (I.C.E.)

6. If the right extremity of a truss of the dimensions in the above sketch (Fig 110) is hinged, and the left one is capable of sliding horizontally, find the stresses in the members due to a horizontal wind of intensity equal to 35 lb. per square foot on a vertical surface, blowing from left to right, assuming that the total wind pressure on the inclined surface is equal to its normal component. Trusses 10 ft. centres.

7. If a roof truss, as in the sketch (Fig. 111), is loaded as indicated, draw the stress diagram to scale, showing which stresses are tensions and which compressions.

8. If trusses similar to that in the sketch (Fig. 111), and loaded as shown, are placed 10 ft. apart, and the support at *A* is hinged and that at *B* can slide horizontally, and a wind is blowing horizontally from the right-hand side with a force of 30 lb. per square foot on a vertical surface, draw the stress diagram. (I.C.E.)

9. In the braced cantilever shown (Fig. 112), the three members of the lower chord are each 10 ft. in length, and the upper chord is straight and

meets the vertical face of the wall 12 ft. above the lower chord. The diagonal bracings are attached to the upper chord at points vertically above the centres of the three members of the lower chord. Neglecting the weight of the structure, draw the complete stress diagram for it, and determine the reactions

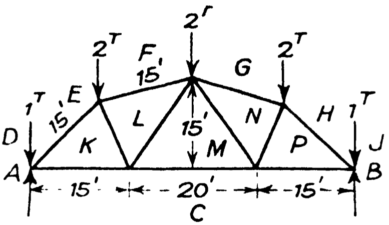


FIG. 111

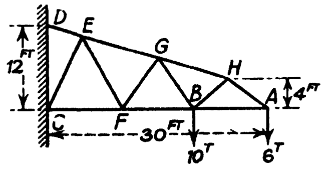


FIG. 112

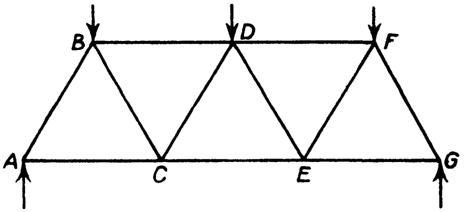


FIG. 113

at *D* and *C*, distinguishing between tensions and compressions when loads of 6 tons and 10 tons are suspended from the joints *A* and *B* respectively. Tabulate the stresses in the members. (U. of L.)

10. A Warren girder has two bays in the upper and three in the lower boom. It carries a load of 2 tons at each panel point in the upper boom. Draw, to scale, the stress diagram for the girder. (Fig. 113.) (I.S.E.)

11. Determine the reactions and the forces acting in the members of the vertical frame shown in Fig. 114.

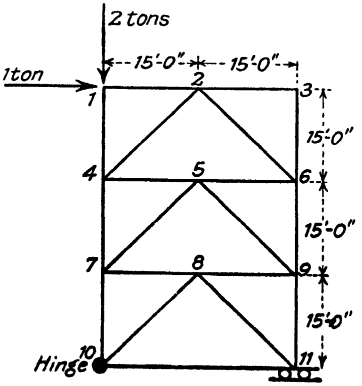


FIG. 114

CHAPTER VIII

DEFLECTION OF PERFECT FRAMES UNDER DEAD LOADS

98. It has been shown that, due to loads acting on a frame, tensile or compressive forces are induced in the various members of the frame, causing lengthening or shortening of the members. Assuming that the strains are within the elastic limits of the material, then the resilience* of each of the prismatic members can be shown to be equal to

$$\frac{1}{2}\frac{F^2L}{AE}$$

where F is the total force, L the length of the member, A the cross-sectional area, E the modulus of elasticity.

Let x in. be the extension or compression of a member L in. long acting under a load of F tons. A is the cross-sectional area,

$$\frac{x}{L} = \frac{\frac{F}{A}}{E} \text{ or } x = \frac{F}{A}\frac{L}{E}$$

The average internal load acting through the distance x is $\frac{F}{2}$ therefore the total internal work per member

$$= \tfrac{1}{2}Fx = \frac{1}{2}\frac{F^2}{A}\frac{L}{E}$$

99. **Castigliano's Theorem.** For any frame, assuming no work done by the reactions,

$$\begin{aligned}\text{Total internal work} &= \Sigma \text{ internal work of all the members}\\ &= \text{total external work.}\end{aligned}$$

Consider any structure (Fig. 115) loaded with a number of concentrated loads,

$$W,\ W_1,\ W_2,\ W_3,\ \ldots\ W_n$$

Total external work done on the structure

$$= U = \tfrac{1}{2}Wy + \tfrac{1}{2}W_1y_1 + \tfrac{1}{2}W_2y_2 + \tfrac{1}{2}W_3y_3 + \quad . \quad (1)$$

---

* Resilience = internal strain energy (*vide* Art. 37).

where y, y_1, y_2, y_3, etc., are the deflections of the structure at the load points and in the same direction as that in which the loads act.

Now let W be increased by δW, then the deflections under all the loads will be increased by δy, δy_1, δy_2, etc. And a small increase in external work $= \delta U =$ internal work

$$= W\delta y + \tfrac{1}{2}\delta W \cdot \delta y + W_1 \delta y_1 + W_2 \delta y_2 + \quad . \quad (2)$$

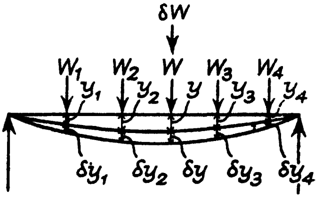


FIG. 115

Now let the loads $W + \delta W$, W_1, W_2, etc., be applied initially; then the work done

$$\begin{aligned} &= U_1 = \tfrac{1}{2}(W + \delta W)(y + \delta y) \\ &\quad + \tfrac{1}{2}W_1(y_1 + \delta y_1) + \tfrac{1}{2}W_2(y_2 + \delta y_2) + \ \cdots \\ &= \tfrac{1}{2}Wy + \tfrac{1}{2}\delta W . y + \tfrac{1}{2}W \cdot \delta y \\ &\quad + \tfrac{1}{2}\delta W \cdot \delta y + \tfrac{1}{2}W_1 y_1 + \tfrac{1}{2}W_1 \delta y_1 + \ \cdots . \qquad (3) \end{aligned}$$

Subtract (1) from (3), then

$$\begin{aligned} U_1 - U = \delta U &= \tfrac{1}{2} \cdot \delta W \cdot y + \tfrac{1}{2}W \cdot \delta y + \tfrac{1}{2}W_1 \delta y_1 \\ &\quad + \tfrac{1}{2}W_2 \delta y_2 + \ \cdots \quad . \quad . \qquad (4) \end{aligned}$$

Divide Equation (2) by 2 and take the limit, then

$$\frac{\delta U}{2} = \tfrac{1}{2}W \cdot \delta y + \tfrac{1}{2}W_1 \delta y_1 + \tfrac{1}{2}W_2 \delta y_2 + \ \cdots . \qquad (5)$$

Subtract (5) from (4),

$$\frac{\delta U}{2} = \frac{1}{2}\,\delta W \cdot y$$

$$\text{i.e. } \frac{\delta U}{\delta W} = y = \frac{\partial U}{\partial W} \text{ (in the limit)} \quad . \quad . \qquad (6)$$

This is known as *Castigliano's First Theorem*, which postulates that in any beam or truss, subjected to any set of loads, the deflection of an arbitrary point X is equal to the first partial derivative of the internal work of deformation with respect to a load W at the point acting in the direction of the desired deflection.

100. EXAMPLE. Consider the Warren frame or girder in Fig. 116.

(*a*) Let W_1 be the only external load acting at the joint *DEFGC*.

The force-stress diagram can be drawn for the frame with only this load acting.

The force in any member, say, $DE = F_{DE} = k_1 W_1$ where k_1 is a numerical coefficient.

Similarly the force in member $FA = F_{FA} = k_1' W_1$ where k_1' is a numerical coefficient.

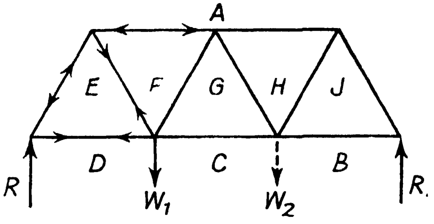


FIG. 116

Use similar notation for the forces in the other members.

(*b*) Let another load W_2 act as indicated at the joint *CGHJB*; a force-stress diagram for this load acting alone can be drawn.

Let the force in *DE* due to W_2 acting alone be $F_{DE}' = k_2 W_2$.

Force in *FA* due to $W_2 = F'_{FA} = k_2' W_2$ - where k_2' is a coefficient; and similarly for the other members; k_1, k_2, etc., and k_1', k_2', etc., are numerical coefficients.

If a girder consists of a number of bays and there are loads acting at several joints, then forces are developed in each member of the frame due to each load acting separately.

(*c*) Consider the joint where W_1 is acting; imagine another load of unit magnitude at the same joint. A force-stress diagram can be drawn for the load of 1 ton acting alone, and the shape of this diagram is the same as for W_1 acting alone.

Then force in *DE* due to 1 ton acting alone will be therefore,

$$F_{1DE} = k_1 \times 1, \text{ because 1 replaces } W_1.$$

Therefore k_1 is the force in the member *DE* for unit load at the point of application; k_1' = force in member *FA* due to 1 ton acting alone instead of W_1; and so on for all the other members.

(*d*) The total work done in any member $= \frac{1}{2}\frac{F^2l}{AE}$, where $F =$ total force in the member due to external loads

$$= \tfrac{1}{2}(k_1W_1 + k_2W_2 + k_3W_3 + \ldots k_nW_n)^2 \frac{l}{AE}$$

(*e*) Total work done on the whole structure

$$= \sum \frac{1}{2}\frac{F^2l}{AE}$$

therefore $U = \sum \frac{1}{2}\frac{F^2l}{AE}$; for one load $U = \frac{1}{2}Wy = \sum \frac{F^2l}{2AE}$

$\therefore\ y = \sum \frac{F^2l}{AEW}$ so that the deflection may be found for *one* load from this equation,

$$(f)\quad U = \sum \frac{1}{2}\frac{F^2l}{AE} = \tfrac{1}{2}(k_1W_1 + k_2W_2 + \ldots k_nW_n)^2 \frac{l}{AE}$$
$$+ \tfrac{1}{2}(k_1'W_1 + k_2'W_2 + k_3'W_3 + \ldots k_n'W_n)^2 \frac{l_1}{A_1E}$$
$$+ \ldots \text{etc.} \qquad . \quad . \qquad (7)$$

(*g*) To find the deflection y of the point of application X of any load W_1 which is one of a number of loads acting on the structure differentiate U partially with respect to the load W_1.

$$\frac{\partial U}{\partial W_1} = \sum \frac{2F}{2} \cdot \frac{\partial F}{\partial W_1} \cdot \frac{l}{AE}$$
$$= k_1(k_1W_1 + k_2W_2 + \ldots k_nW_n)\frac{l}{AE}$$
$$+ k_1'(k_1'W_1 + k_2'W_2 + \ldots k_n'W_n)\frac{l_1}{AE}$$
$$+ \ldots \text{etc.} \quad . \quad . \quad . \qquad (8)$$

Now $(k_1W_1 + k_2W_2 + k_3W_3 + K_nW_n) =$ line on force-stress diagram for all loads for one member; $(k_1'W_1 + k_2'W_2 + k_3'W_3)$ $=$ line on force-stress diagram for all loads for another member; and similarly for all the other members.

(k_1, k_1', etc., are forces in the members due to 1 ton at the point of application.)

from eqn. (6)

but $\frac{\partial U}{\partial W_1} = y$ the deflection in the direction of W_1.

$$\therefore\ y = \sum \frac{\begin{pmatrix}\text{Total force in a member}\\ \text{due to external loads}\end{pmatrix}\begin{pmatrix}\text{force in a member}\\ \text{due to unit load at } X\end{pmatrix}}{AE} \times l \quad (9)$$

$$= \sum F \cdot \frac{\partial F}{\partial W_1} \cdot \frac{l}{AE}$$

$$= \sum F \cdot u \cdot \frac{l}{AE} \qquad \text{where } u = \frac{\partial F}{\partial W_1} \quad . \quad . \quad . \quad (9a)$$

The displacement at a point where no load is acting

$$W_1 = \text{zero.} \quad k_1 W_1 = 0\,;$$

therefore k_1 is not equal to zero, but equal to the force in a member due to unit load acting at the point where $W_1 = 0$.

The procedure to get the deflection at any point on a loaded framed structure is (1) find the load in each bar due to the loading on the structure; (2) take a unit load acting in the given direction and treat it as the only force on the structure and find the force due to it in each bar; then (3) use equation (9).

In solving problems for displacements, it is best to draw up a Table as outlined below.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Name or No. of Member	Force in Member due to all loads	Force in Member due to unit load at place where deflection required	Product (2) × (3)	l	A	$\frac{l}{A}$	(4) × (7)
	F	$\frac{\partial F}{\partial W_1}$	$F\frac{\partial F}{\partial W_1}$				$F\frac{\partial F}{\partial W_1}\frac{l}{A}$

Sum Total of (8) . . X

Deflection required $= \frac{\text{Sum Total of (8)}}{E}$

In bridge frames or girders the loads on the bridge are generally transmitted to the main girders by cross girders at the joints, and the dead loads act vertically downwards; it is only therefore necessary in these cases to find the vertical displacements.

101. If at a joint the load does not act vertically downwards, the loads in the members can be found for the load acting in its true direction; unit load will act in the same direction, and the displacement in the direction of the load may be found. This displacement can be resolved vertically and horizontally.

The horizontal and vertical displacements can also be found by imagining unit horizontal and vertical loads at the point of application.

The displacement in the direction of the load may be found by compounding the two component displacements.

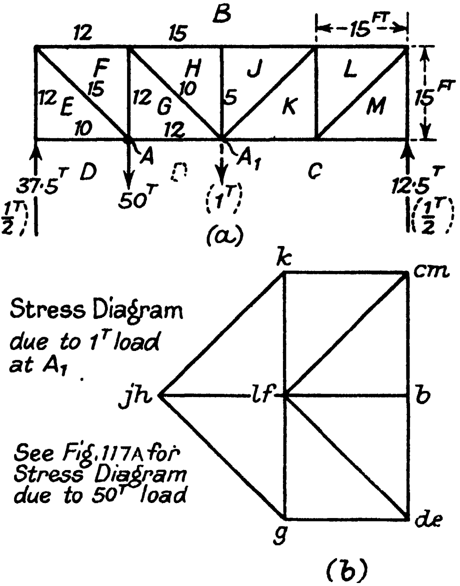


FIG. 117

To obtain the displacement of a single point in a truss, the equation

$$y = \Sigma F \cdot \frac{\delta F}{\delta W} \cdot \frac{l}{AE} = \Sigma F \cdot u \cdot \frac{l}{AE}$$

will usually give the readiest solution. To obtain the simultaneous displacements of a number of points in a truss, the Williot diagram is the simplest and quickest method. This is a graphical method of constructing the deflection diagram. Space does not permit of the discussion of this diagram and reference should be made to it in other works. It is largely used for truss deflection problems in the field of statical indeterminancy.

Illustrative Problem 28.

Part I. The lattice girder shown in Fig. 117 is loaded at the joint *A* with a load of 50 tons. Find the amount of deflection of the girder at the point A. The figures against the members in the left-hand half of the girder indicate the cross-sectional area of the member in square inches (and are the same for the right-hand half). $E = 13{,}000$ tons per square inch. (I.St.E., 1923.)

The force-stress diagram is as in Fig. 117A (*b*).

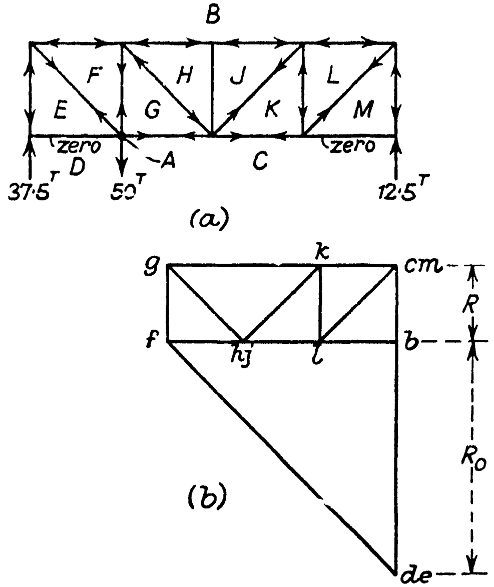


FIG. 117A

For the deflection of A vertically downwards, as only load is at A, y is the displacement;

$$\frac{1}{2}Wy = \frac{1}{2} \times y \times 50$$

$$= \text{external work}$$

$$= \frac{1}{2}\sum \frac{F^2 l}{AE}$$

F = force in a member due to 50 tons only acting.

(*See* Table I, page 186, for forces, etc., in the members.)

$$\sum \frac{F^2 l}{A} = 158{,}900$$

tons-in. units

$$\frac{1}{2} \times 50 \times y = \frac{1}{2} \times \frac{158{,}900}{13{,}000}$$

$$y\ (\text{inches}) = \frac{158{,}900}{13{,}000} \times \frac{1}{50} = \cdot 245 \text{ in.}$$

Part II. For the given conditions of loading of the girder in Fig. 117, find the deflection of the joint A_1. In this case, imagine a vertical load of 1 ton at A_1. Find the forces in the members due to this unit load. Next draw up a table as given in Table II.

Let tensile forces be positive and compressive forces negative.

T is for tensile and C for compressive forces in the following table.

TABLE II FOR PROBLEM 28, PART II, PAGE 184

Member. (1).	F Force in Tons due to 50-ton Load (kind) at A. (2).	Force in Tons due to 1-ton Load at A_1 (and kind). $u = \frac{\partial F}{\partial W_1}$ (3).	$F\frac{\partial F}{\partial W_1}$ (2) × (3) (4).	l in. (5).	A sq. in. (6).	$\frac{l}{A}$ (7).	$F\frac{\partial F}{\partial W_1} \times \frac{l}{A}$ (4) × (7). (8).
BE	− 37·5 C	− ·5 C	+ 18·75	180	12	15	+ 281
BF	− 37·5 C	− ·5 C	+ 18·75	180	12	15	+ 281
BH	− 25 C	− 1 C	+ 25	180	15	12	+ 300
BJ	− 25 C	− 1 C	+ 25	180	15	12	+ 300
BL	− 12·5 C	− ·5 C	+ 6·25	180	12	15	+ 94
BM	− 12·5 C	− ·5 C	+ 6·25	180	12	15	+ 94
CM	+ 0	0	0	180	10	18	0
CK	+ 12·5 T	+ ·5 T	+ 6·25	180	12	15	+ 94
CG(*DG*)	+ 37·5 T	+ ·5 T	+ 18·75	180	12	15	+ 281
DE	+ 0	0	0	180	10	18	0
EF	+ $\sqrt{2} \times 37·5$ T	+ $\frac{\sqrt{2}}{2}$ T	+ 37·5	$\sqrt{2} \times 180$	15	$12\sqrt{2}$	+ 636
FG	+ 12·5 T	− ·5 C	− 6·25	180	12	15	− 94
GH	− $\sqrt{2} \times 12·5$ C	+ $\frac{\sqrt{2}}{2}$ T	− 12·5	$\sqrt{2} \times 180$	10	$18\sqrt{2}$	− 318
HJ	+ 0	0	0	180	5	36	0
JK	+ $\sqrt{2} \times 12·5$ T	+ $\frac{\sqrt{2}}{2}$ T	+ 12·5	$\sqrt{2} \times 180$	10	$18\sqrt{2}$	+ 318
KL	− 12·5 C	− ·5 C	+ 6·25	180	12	15	+ 94
LM	+ $\sqrt{2} \times 12·5$ T	+ $\frac{\sqrt{2}}{2}$ T	+ 12·5	$\sqrt{2} \times 180$	15	$12\sqrt{2}$	+ 212
				Total of (8) +			2573

TABLE FOR PROBLEM 28, PART I, PAGE 183

Member.	Kind.	Force in Member. Tons = F.	l in.	A sq. in.	F^2.	$\frac{l}{A}$	$\frac{F^2l}{A}$
BE	Strut (S)	− 37·5	180	12	1406	15	21,100
BF	S	− 37·5	180	12	1406	15	21,100
BH	S	− 25·0	180	15	625	12	7500
BJ	S	− 25·0	180	15	625	12	7500
BL	S	− 12·5	180	12	156	15	2340
BM	S	− 12·5	180	12	156	15	2340
CM	Tie (T)	0	180	10	0	18	0
CK	T	+ 12·5	180	12	156	15	2340
CG	T	+ 37·5	180	12	1406	15	21,100
DE	T	0	180	10	0	18	0
EF	T	$+ \sqrt{2} \times 37·5$	$\sqrt{2} \times 180$	15	2812	$12\sqrt{2}$	47,700
FG	T	+ 12·5	180	12	156	15	2340
GH	S	$- \sqrt{2} \times 12·5$	$\sqrt{2} \times 180$	10	312	$18\sqrt{2}$	7950
HJ	—	0	180	5	0	36	0
JK	T	$+ \sqrt{2} \times 12·5$	$\sqrt{2} \times 180$	10	312	$18\sqrt{2}$	7950
KL	S	− 12·5	180	12	156	15	2340
LM	T	$+ \sqrt{2} \times 12·5$	$\sqrt{2} \times 180$	15	312	$12\sqrt{2}$	5300
							158,900

NOTE.—Tensile forces plus, compressive forces minus, and F^2 is always positive.

(Problem 28, Part II, *continued.*)

The unit force-stress diagram for 1 ton only acting at A_1 is given in Fig. 117 (*b*).

Sum of column (8) = + 2573 tons-in. units ;

therefore, deflection of the joint $A_1 = \frac{2573}{13,000}$ in. = ·198 in.

Illustrative Problem 29.

A Warren girder 30 ft. span has three equal bays in the lower boom. All the diagonals are inclined at 60° to the horizontal. There are loads of 15 tons at each of the two joints in the lower boom. The stress in the tension members is 5 tons per square inch, and in the compression members 3 tons per square inch. Find the deflection at the points of application of the loads. E = 13,000 tons/sq. in. All members are 10 ft. long. (L.U., 1923.)

For the calculation of the loads in the members due to the external loads of 15 tons acting at the two lower joints, *see* Fig. 103 (Chap. VII), and Table, page 187.

The girder is symmetrically loaded ; therefore the deflection at the two lower joints will be the same, so that

TABLE FOR ILLUSTRATIVE PROBLEM 29

Member.	Tie or Strut.	F tons. Force in Member.	l in.	A sq. ins.	F^2.	$\frac{l}{A}$	$\frac{F^2l}{A}$
AF	S	$-\frac{30}{\sqrt{3}}$	120	$\frac{10}{\sqrt{3}}$	300	$12\sqrt{3}$	$3600\sqrt{3}$
AH	S	$-\frac{30}{\sqrt{3}}$	120	$\frac{10}{\sqrt{3}}$	300	$12\sqrt{3}$	$3600\sqrt{3}$
BJ	T	$+\frac{15}{\sqrt{3}}$	120	$\frac{3}{\sqrt{3}}$	75	$40\sqrt{3}$	$3000\sqrt{3}$
CG	T	$+\frac{30}{\sqrt{3}}$	120	$\frac{6}{\sqrt{3}}$	300	$20\sqrt{3}$	$6000\sqrt{3}$
DE	T	$+\frac{15}{\sqrt{3}}$	120	$\frac{3}{\sqrt{3}}$	75	$40\sqrt{3}$	$3000\sqrt{3}$
AE	S	$-\frac{30}{\sqrt{3}}$	120	$\frac{10}{\sqrt{3}}$	300	$12\sqrt{3}$	$3600\sqrt{3}$
EF	T	$+\frac{30}{\sqrt{3}}$	120	$\frac{6}{\sqrt{3}}$	300	$20\sqrt{3}$	$6000\sqrt{3}$
FG	—	0	120	—	—	—	—
GH	—	0	120	—	—	—	—
HJ	T	$+\frac{30}{\sqrt{3}}$	120	$\frac{6}{\sqrt{3}}$	300	$20\sqrt{3}$	$6000\sqrt{3}$
JA	S	$-\frac{30}{\sqrt{3}}$	120	$\frac{10}{\sqrt{3}}$	300	$12\sqrt{3}$	$3600\sqrt{3}$
					Sum $\frac{F^2l}{A}$ =		$38{,}400\sqrt{3}$ tons-in. units

$$\frac{1}{2}Wy + \frac{1}{2}Wy = \text{total external work} = \frac{1}{2}\sum\frac{F^2l}{AE}$$
$$= \text{total internal work}$$

that is, $\frac{1}{2} \times 15 \times y + \frac{1}{2} \times 15 \times y = \frac{1}{2} \times \frac{38{,}400\sqrt{3}}{13{,}000}$

$$y \text{ in.} = \frac{38{,}400\sqrt{3}}{13{,}000 \times 30} = \cdot\ 71 \text{ in.}$$

The deflection of each of the lower joints is equal to ·171 in.

Illustrative Problem 30.

The girder in the previous problem is loaded with 15 tons at the left joint in the lower boom and with 10 tons at the right joint. Find the deflection of the joint loaded with the 10 tons, the cross-sectional areas being the same

The force-stress diagrams required are given in Fig. 118, and the calculations in Table III.

TABLE III

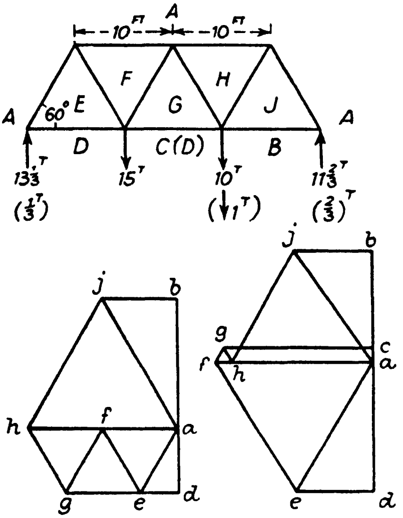
Member. (1).	Force in Tons due to 15-and 10-ton Loads. F (2).	Force in Tons due to 1-ton Load at the 10-ton Point. $\frac{\partial F}{\partial W_1} \cdot U$ (3).	$F\frac{\partial F}{\partial W_1}$ (2) × (3). (4).	l in. (5).	A sq. in. (6).	$\frac{l}{A}$ (7).	$F\frac{\partial F}{\partial W_1} \cdot \frac{l}{A}$ (4) × (7). (8).
AF . . .	− 15·4	− ·4	+ 6·16	120	$\frac{10}{\sqrt{3}}$	$12\sqrt{3}$	+ 128
AH . . .	− 13·5	− ·8	+ 10·8	120	$\frac{10}{\sqrt{3}}$	$12\sqrt{3}$	+ 224
BJ . .	+ 6·8	+ ·4	+ 2·72	120	$\frac{3}{\sqrt{3}}$	$40\sqrt{3}$	+ 188
CG (*GD*) .	+ 15·0	+ ·6	+ 9·00	120	$\frac{6}{\sqrt{3}}$	$20\sqrt{3}$	+ 311
DE . .	+ 7·7	+ ·2	+ 1·74	120	$\frac{3}{\sqrt{3}}$	$40\sqrt{3}$	+ 120
AE . . .	− 15·4	− ·4	+ 6·16	120	$\frac{10}{\sqrt{3}}$	$12\sqrt{3}$	+ 128
EF . . .	+ 15·4	+ ·4	+ 6·16	120	$\frac{6}{\sqrt{3}}$	$20\sqrt{3}$	+ 213
FG . . .	+ 1·5	− ·4	− ·6	120	$\frac{3}{\sqrt{3}}$	$40\sqrt{3}$	− 42
GH . . .	− 1·5	+ ·4	− ·6	120	$\frac{3}{\sqrt{3}}$	$40\sqrt{3}$	− 42
HJ . . .	+ 1·5	+ ·78	+ 1·17	120	$\frac{6}{\sqrt{3}}$	$20\sqrt{3}$	+ 41
JA . . .	− 13·5	− ·78	+ 10·5	120	$\frac{10}{\sqrt{3}}$	$12\sqrt{3}$	+ 218
					Sum of (8) =		1487 tons-in. units

NOTE.—Tensile forces plus, compressive forces minus.

The deflection of the girder at the joint with the 10-ton load

$$= \frac{1487}{13{,}000} = \cdot 114 \text{ in.}$$

101*a*. In Chapter III, paragraph 37, it was shown that the work done on a beam by bending $= U = \int \frac{M^2 \, . \, dx}{2EI}$ (between



Force-Stress Diagram *for load of 1 Ton only acting at 10T joint.* Scale – 1 Inch = 1 Ton.

Force-Stress Diagram *with loads of 15 Tons and 10 Tons acting.* Scale – ½ Inch = 10 Tons.

See Table for kinds of stresses in the members.

FIG. 118

the required limits) where M = Moment due to the external loads acting at a section X distant x from the origin.

Consider a beam loaded in any manner: it is required to find the deflection under any load F. By Castigliano's Theorem,

$$\frac{\partial U}{\partial F} = y = \frac{1}{EI} \int M \left(\frac{\partial M}{\partial F} \right) . \, dx. \quad (EI \text{ being constant.})$$

Now M will be of the form (for a beam, irregularly loaded, and working from the left support as origin) as given below.

$$R_o = \frac{W(l-a)}{l} + \frac{W_1(l-b)}{l} \dots$$

$$\therefore M = -\frac{W(l-a)x}{l} - \frac{W_1}{l}(l-b)x - \dots - \frac{F}{l}(l-n)x - \dots$$

$$+ W(x-a) + W_1(x-b) + \dots F(x-n) - \dots$$

where W, W_1, W_2, ... F, etc., are at distances $a, b, c, \dots n$, etc., from the origin, and where $n < x$.

$$\text{Thus } \frac{\partial M}{\partial F} = -\frac{1 \times (l-n)x}{l} + 1 \times (x-n).$$

= Moment at the section X. due to an imaginary unit load acting at the point of application of F.

$= m$.

Therefore,

$$\frac{\partial U}{\partial F} = y = \frac{1}{EI}\int \begin{array}{l}\text{(Moment at the section due to all the real} \\ \text{external loads)} \times \text{(Moment at the section} \\ \text{due to an imaginary unit load at the} \\ \text{point of application)} \, . \, dx\end{array}$$

$$= \frac{1}{EI}\int M \, . \, m \, . \, dx,$$

the integration being taken between the required limits.

If the section X considered lies nearer to the origin than the point of application of F, then $(x-n)$ is neglected, as n would be equal to, or greater than, x.

If it is required to find the deflection of the beam at a point where there is no load, place the imaginary unit load at this point. M will be the moment at any section due simply to the real external loads and m the moment at the same section due to the imaginary unit load at the point for which the deflection is required.

EXAMPLE.

A beam AC simply supported and of length l carries a load of W tons at B a distance nl from A. Calculate the deflection (1) under the load at B, and (2) at a section distant z from A and between A and B. EI = constant. (Fig. 118A.)

(1) Moment at any section X between A and B due to W (A as origin) $= - W(1-n)x$.

Moment at any section X_1 between B and C (C as origin) $= - Wn\,x_1$.

$\therefore$ For deflection at B,

$$U = \frac{1}{2}Wy = \frac{1}{2EI}\int_0^{nl} M^2 \, . \, dx + \int_0^{l(1-n} M_1^2 dx_1$$

so that $\frac{1}{2}\,Wy = \frac{1}{2EI}\left[\int_0^{nl} W^2(1 - n)^2x^2 \, . \, dx + \int_0^{l(1-n)} W^2n^2x_1^2 \, . \, dx_1\right]$

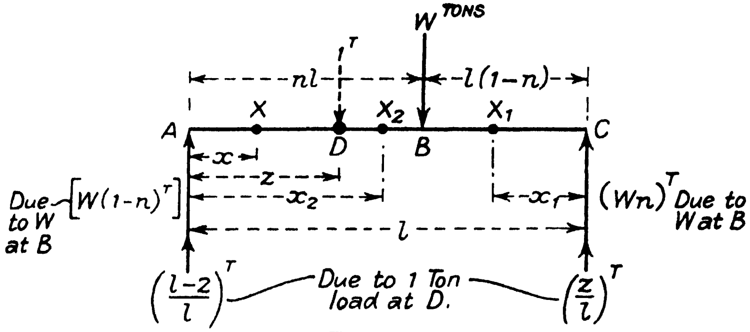


FIG. 118A

An equation from which y can be found

$$y = \frac{Wl^3n^2}{3EI}\,(1 - 2n + n^2)$$

$$\text{If } n = \tfrac{1}{2},\ y = \frac{Wl^3}{48EI}$$

(2) Deflection at D, distant z from A.

Reaction at A, due to 1 ton at D, $= \dfrac{1(l - z)}{l}$ ton.

Reaction at C due to 1 ton at $D = \dfrac{z}{l}$ ton.

Moment at any Section X between A and D (A as origin) due to unit load at $D = - \dfrac{1 \times (l - z)x}{l}$

Moment at any section X_2 between D and B (A as origin) due to unit load at $D = - \dfrac{1(l - z)}{l} \, . \, x_2 + 1(x_2 - z)$.

Moment at any section X_1 between B and C (C as origin) due to unit load at $D = - \dfrac{1 \times z \times x_1}{l}$

$$\therefore \text{ Deflection at } D \text{ (by Castigliano's Theorem)} = \frac{1}{EI}\int (Mmdx)$$

$$= y_{\mathrm{D}} = \frac{1}{EI}\left[\int_0^z W(1-n)x \,.\, \frac{(l-z)}{l} x \,.\, dx + \int_z^{nl} W(1-n)x_2 \,.\right.$$

$$\left.\left\{\frac{(l-z)}{l} \,.\, x_2 - (x_2 - z)\right\} dx_2 + \int_0^{l(1-n)} Wnx_1 \,.\, \frac{zx_1}{l} \,.\, dx_1\right]$$

Solving the integrals, y_{D} can be found in terms of W, z, l, and n.

REFERENCES

(1) *Further Problems in the Theory and Design of Structures,* Andrews.
(2) *Theory of Structures,* Morley (other methods and examples).
(3) *Elastic Stresses in Structures.* Translation of Castigliano's work by Andrews. (Scott, Greenwood & Sons.)
(4) *Mechanics of Internal Work,* Church. (Wiley.)
(5) Arrol's *Bridge and Structural Engineers' Handbook,* A. Hunter Deflection of framed structures. Graphical and analytical examples given.
(6) *Statically Indeterminate Stresses* (Chapter I). Parcel and Maney. (Examples of Williot diagrams are given.)
(7) *Kinetic Theory of Engineering Structures,* Molitor. (McGraw-Hill.)
(8) *Fundamentals of Indeterminate Structures,* Plummer. (Pitman Publishing Corporation.)

EXAMPLES

1. Explain Castigliano's Theorem with reference to the deflection of a structure due to a system of forces acting on it, and give a proof of same. (I.C.E.)

2. Prove that the deflection in the direction of any one of a system of forces applied to a structure at its point of application is equal to the differential coefficient of the total work done on the structure with respect to the particular force.

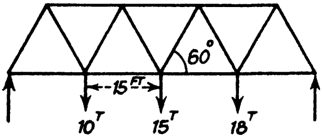


FIG. 119

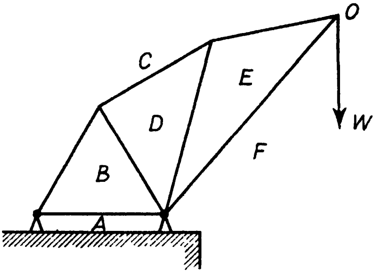


FIG. 120

A Warren girder (Fig. 119) has four equal bays in the lower boom, and all the triangles are equilateral. There are loads applied at the panel points of the lower boom, as shown. The sectional area of all the members is the same. Determine the deflection at the centre of the girder. $E = 13{,}000$ tons per sq. in. (U. of B.)

3. In the frame shown (Fig. 120), find the displacement of the point O perpendicular to the bar AB. The lengths of the bars are as follows—

AB, BC, BD, EC, 13 ft. each ; DE, 18 ft. ; EF, 27 ft.
Stress in compression members = 4 tons per square inch.
Stress in tension members = 5 tons per square inch.

E = 12,000 tons per square inch.
Angle between DB and $CD = 90°$.

4. Show how to find the deflection at a joint in a pin-jointed structure, due to the action of a number of forces on it, expressed in terms of the stresses in its members due to the acting forces, and those due to a force applied at the given joint and in the direction the deflection is required. (I.C.E.)

5. Referring to Fig. 112 (Chap. VII), page 177, find the deflection of the braced cantilever in the direction (*a*) of the 6-ton load, (*b*) of the 10-ton load. Assume l/A for tension members = 80, and for compression members = 24. E = 30,000,000 lb. per sq. in.

6. Taking the Warren girder in Fig. 113 (Chap. VII), page 177, find the vertical deflection of the centre joint of the top boom, and also of the joints in the lower boom. Length of bay = 10 ft. E = 30,000,000 lb. per sq. in.

7. A Warren girder 30 ft. span has three equal bays in the lower boom. All the diagonals are inclined at 60° to the horizontal. There are loads of 15 tons at each of the two joints in the lower boom. The stress in the tension members is 5 tons/square inch ; in the compression members, 3 tons/square inch. Find the deflection at the points of application of the loads. E = 13,000 tons/square inch.

8. A beam simply supported is 20 ft. long and carries a load of 4 tons at a point 12 ft. from the left support. Calculate the deflection (using the methods in Chapter VIII) of the beam under the load and at a point 6 ft. from the left support.

E = 12,000 tons per sq. in.
I = 100 in. units.

9. If the beam in the previous example (9) is rigidly fixed at both ends, and the loading is the same, calculate the deflections of the beam under the load and at a point 6 ft. from the left support.

(Check the results of questions 9 and 10 by methods given in Chapters III and IV.)

10. A triangular truss rests on two supports A and B at the same level. A is a hinged immovable support, and B is a hinge on frictionless rollers. The span AB is 20 ft. The members of the truss are AC, CB, BD, DA and CD. $AC = CB$ = 120 in. $AD = DB$ = 134 in. CD = 60 in. The cross-sectional areas are: of AC and CB, 3 sq. in.; of AD and DB, 2 sq. in.; and of CD, 4 sq. in. A vertical load of 8,000 lb. and a horizontal load of 5,000 lb. in direction A to B act at the joint C. Calculate the horizontal deflections of B and C. Take E = 28,000,000 lb. per sq. in.

CHAPTER IX

THE PRINCIPLE OF LEAST WORK—THE DETERMINATION OF THE STRESSES IN THE MEMBERS OF REDUNDANT FRAMES AND OF EXTERNAL REDUNDANT RESTRAINTS

102. THE stresses in redundant frames cannot be determined by the ordinary methods of graphic or analytical statics. The usual procedure has been to work by the method of superposition, by which the redundant frame is considered divided up into a number of superposed firm or perfect frames, and the load divided between them, the stresses in common members being added together. The results obtained by this method are fairly accurate.

An analytical method is that dependent upon the PRINCIPLE OF LEAST WORK. By this method the cross-sectional areas of the members must be initially known. The agreement between the two methods depends, therefore, upon these areas.*

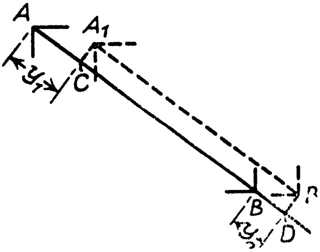


FIG. 121

103. **Consider Any Structure with One Redundant Member.** Let AB be the position of the redundant bar before loading and A_1B_1 after loading. (Fig. 121.)

From A_1 and B_1 drop perpendiculars on to AB at C and D.

Let $AC = y_1$, $BD = y_2$; and these are the components of the displacements in direction AB.

Let T tons be the load in AB after loading : l = length AB.

AB after loading will be strained by an amount $y_2 - y_1$, which will be equal to $\frac{Tl}{AE}$ where A = cross-section of bar and E = Young's modulus.

$$y_2 - y_1 = \frac{Tl}{AE} \quad . \quad . \quad . \quad . \quad (1)$$

Now let the bar AB be removed and replaced by loads T_1 and T_2 at A and B respectively, T_1 acting towards B and T_2 towards A; the loads in the remaining members will be unaltered, so that T_1 and T_2 may now be considered external forces at the points of application.

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* An analytical method based on the Law of Virtual Work is given in a paper by the Author, *vide* Reference (9), page 220.

T_1 will be equal to $T_2 = T$, as equilibrium exists.

Let U = total internal work of all the members except AB.
y_1 and y_2 will be of opposite senses :

y_1 will be in the direction of T_1, and positive
y_2 ,, ,, opposite direction from T_2, and negative

From the last chapter, and using partial differentials, for there are two independent variables,

$$\frac{\partial U}{\partial T_1} = +y_1 \qquad \frac{\partial U}{\partial T_2} = -y_2 \qquad y_2 - y_1 = \frac{Tl}{AE}$$

$$\therefore -\frac{\partial U}{\partial T_2} - \frac{\partial U}{\partial T_1} = \frac{Tl}{AE} \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{Now as } T_1 = T_2 = T, \frac{\partial U}{\partial T_1} + \frac{\partial U}{\partial T_2} = \frac{dU}{dT} \quad . \quad . \quad (3)$$

$$\left[dU = \frac{\partial U}{\partial T_1} . dT_1 + \frac{\partial U}{\partial T_2} . dT_2\right] \text{on total differentiation}$$

$$\therefore -\frac{dU}{dT} = \frac{Tl}{AE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\text{But } \frac{Tl}{AE} = \frac{d}{dT}\left(\frac{1}{2}\frac{T^2l}{AE}\right) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{1}{2}\frac{T^2l}{AE} = \text{work done in the redundant member.}$$

$$\therefore \frac{dU}{dT} + \frac{d\left(\frac{1}{2}\frac{T^2l}{AE}\right)}{dT} = 0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\text{or } \frac{d\left(U + \frac{1}{2}\frac{T^2l}{AE}\right)}{dT} = 0, \quad \text{or } \frac{d\begin{pmatrix} U_1 = \\ \text{total work done on the members} \\ \text{including the redundant member}\end{pmatrix}}{dT}$$

$$= \frac{dU_1}{dT} = 0, \text{ i.e. a minimum* } \quad . \quad . \quad . \quad (7)$$

---

* The Principle of Least Work is a statement of the practical fact that if an elastic structure is in a state of stable equilibrium under any forces whatever, then the work stored is the smallest amount possible. It is a particular case of Castigliano's Second Theorem,

$\frac{dU_1}{dT} = \lambda$, where λ is a small strain or displacement within the elastic limit.

(See Reference (3), page 192: also *Analysis of Engineering Structures*, by A. J. Pippard and J. F. Baker (Arnold & Co.).)

104. Thus, if an elastic structure is in stable equilibrium under any forces whatsoever the work stored is the least possible amount. To use this method, replace all the redundant members by loads T_1 acting at the required joints for one bar, T_2 acting at the necessary joints for another bar, and so on. The statically determined system which results from the removal of the redundant bars is called the base or principal or perfect system.

Although it is difficult to prove that each partial differential of the total work with respect to one unknown force when there are several such forces each unknown, amounts to zero, it may be taken as true;

$$\text{then } \frac{\partial U_1}{\partial T_1} = \frac{\partial U_1}{\partial T_2} = 0, \text{ and so on.}$$

The number of equations will be the same as the number of unknowns.

105. For one redundant bar, $T_1 = T_2 =$ Force-pair T; so that as before (Chap. VIII),

$$U_1 = \left[\Sigma \tfrac{1}{2}(k_1 T_1 + k_1' T_1 + \Sigma k W)^2 \frac{l}{AE}\right] + \frac{1}{2}\frac{T_1{}^2 l_1}{A_1 E} \quad . \quad (8)$$

where $k_1 T_1$ and $k_1' T_1$ are the loads in a member due to $T = T_1$ acting at one point of application and $T = T_1$ acting at the other : $\Sigma k W$ is the load in a member due to all the external loads acting on the perfect frame.

$k_1 =$ load in a member replacing T_1 by unit load (at one point of application)
$k_1' =$,, ,, at the other

Differentiating Equation (8),

$$\frac{\partial U_1}{\partial T_1} = 0 = \left[\Sigma F_1 \cdot \frac{\partial F_1}{\partial T_1} \cdot \frac{l}{AE}\right] + \frac{T_1 l_1}{A_1 E}$$

where $F_1 = k_1 T_1 + k_1' T_1 + \Sigma k W = K_1 T_1 + \Sigma k W$

and K_1 = load in a member due to a pair of unit loads acting at the points of application, i.e. at the joints of the superfluous bar,

$$\text{or } \left[\Sigma(k_1T_1 + k_1'T_1 + \Sigma kW)(k_1 + k_1')\frac{l}{AE}\right] + \frac{T_1l_1}{A_1E} = 0 \qquad (9)$$

$$\text{or } \Sigma K_1^2T_1\frac{l}{AE} + \Sigma K_1(kW)\frac{l}{AE} + \frac{T_1l_1}{A_1E} = 0 \quad . \quad . \quad (10)$$

$$\text{or } \Sigma K_1^2T_1\frac{l}{AE} + \frac{T_1K_1^2l_1}{A_1E} = -\Sigma K_1(kW)\frac{l}{AE}$$

where K_1 in $\frac{T_1K_1^2l_1}{A_1E}$ = unity

$$\therefore T_1 = -\frac{\Sigma(kW)K_1\cdot\frac{l}{AE}}{\Sigma\frac{K_1^2l}{AE}} = -\frac{y'}{y_1} \quad . \quad . \quad . \quad (10a)^*$$

kW is the stress in any member due to the given loads with the redundant member removed, and K_1 = stress in any member due to a pair of unit forces acting on the structure, in the direction of the redundant member, and at the joints of the member. K_1 = unity for the redundant member. The denominator of the right-hand side of the equation (10a) includes all the members in the frame: the numerator all except the redundant one. In effect the method of solution is as follows: Imagine the superfluous bar is cut and compute the resulting relative displacement y' of the faces of this cut bar. Determine the true stress in the bar, by the principle that it is equal in magnitude to the force pair required to bring these faces into contact. A pair of one-pound forces will move the faces a

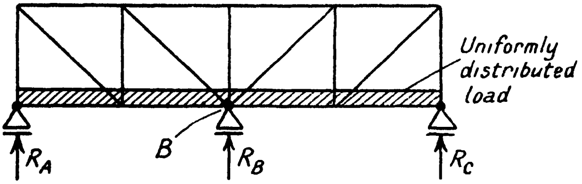

FIG. 121A

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* In equation (10a), (kW) corresponds to F, and K_1 to u in equation (9a), page 182. Therefore (10a) can be written $T_1 = -\Sigma F\,.\,u\,.\,l/AE/\Sigma u^2l/AE$.

distance of y_1, and therefore to move the faces through a distance of y' will require a force pair of $(1 \times y'/y_1)$ lb. $= T_1$ lb.

Equation (10*a*) will also apply for the determination of the redundant reaction of a truss which is simply supported at the ends and is continuous over a third support which is at the same level. (*See* Fig. 121A.)

Let R_B be the redundant force whose value it is desired to find.

$$\text{Then } R_B = -\frac{y'}{y_1} = -\frac{\Sigma K \,.\, (kW) \,.\, \dfrac{l}{AE}}{\Sigma K^2 l/AE} \qquad . \quad (10b)$$

where $\Sigma K^2 l/AE$ is the deflection of the truss at point B, due to unit load acting at B in the direction of R_B, when the redundant support at B is removed. As this support of the actual structure does not move then there is no allowance for the redundant reaction in the denominator of the right-hand side of the equation (10*b*).

kW is the stress in any member due to the original loads with the redundant reaction removed, and K is the stress in any member due to the unit load acting at B.

106. For two redundant bars, replace the bars by T_1 acting at one end and T_1 acting at the other end of 1 redundant bar, and by T_2 acting at the two joint ends of the other bar. T_1 and T_2 are in the directions of their respective bars. The redundant bars have been replaced by force pairs T_1 and T_2 respectively.

k_2 = load in a member due to unit load replacing T_2 at one joint.

k_2' = load in a member due to unit load replacing T_2 at the other joint.

$$\text{Then } U_1 = \left[\frac{1}{2}\Sigma(k_1T_1 + k_1'T_1 + k_2T_2 + k_2'T_2 + \Sigma kW)^2\frac{l}{AE}\right] + \frac{1}{2}\frac{T_1^2 l_1}{A_1E} + \frac{1}{2}\frac{T_2^2 l_2}{A_2E} \qquad . \quad . \quad . \quad (11)$$

$$\text{then } \frac{\partial U_1}{\partial T_1} = 0$$

$$= \left[\Sigma(k_1T_1 + k_1'T_1 + k_2T_2 + k_2'T_2 + \Sigma kW)(k_1 + k_1')\frac{l}{AE}\right] + \frac{T_1 l_1}{A_1E} \qquad . \quad . \quad . \quad (12)$$

Equation (12) can be written

$$\text{as } \Sigma K_1^2 T_1\, l/AE + T_1 K_1^2\, l_1/AE_1 + \Sigma K_1 K_2 T_2 \,.\, l/AE + \Sigma K_1(\Sigma kW)\, l/AE = 0$$

$$\text{or } T_1 \Sigma K_1^2\, l/AE + T_2 \Sigma K_1 \,.\, K_2 \,.\, l/AE + \Sigma K_1(\Sigma kW) l/AE = 0$$

$$\text{(A)} \qquad\qquad \text{(B)} \qquad\qquad\qquad \text{(C)} \qquad . \quad (12a)$$

K_1 has the same definition as in paragraph 105, K_2 is the force in a member of the principal system, due to a unit force pair replacing the redundant force pair T_2.

The factor (A) in equation (12a) includes all members of the base frame, and the one redundant member stressed to T_1. The factors (B) and (C) include only members of the base frame.

$$\frac{\partial U_1}{\partial T_2} = 0$$

$$= \left[\Sigma(k_1 T_1 + k_1' T_1 + k_2 T_2 + k_2' T_2 + \Sigma kW)(k_2 + k_2')\frac{l}{AE}\right] + \frac{T_2 l_2}{A_2 E} \quad . \quad . \quad . \quad (13)$$

If the end of a redundant bar meets at a support point, in this case k_1' will be zero.

Equation (13) can be written

$$T_1 \Sigma K_1 \,.\, K_2 \,.\, l/AE + T_2 \Sigma K_2^2 l/AE + \Sigma K_2(\Sigma kW) l/AE = 0 \quad (13a)$$

A general method of writing the equations (12a) and (13a) is

$$y_A = 0 = y_A' + T_1 y_{AA} + T_2 y_{AB} \quad . \quad . \quad . \quad (12b)$$

$$y_B = 0 = y_B' + T_1 y_{BA} + T_2 y_{BB} \quad . \quad . \quad (13b)$$

Also from Maxwell's Theorem $y_{AB} = y_{BA}$*

The Interpretation of Equation (12b).

The statically determined system which results from the removal of the redundants T_1 and T_2 is called the *base* or *principal* system. If now we imagine the redundants T_1 and T_2 to be entirely removed, and the specified loading applied to the base frame, then there will result a certain relative displacement $y_A' = \Sigma K_1(\Sigma kW) l/AE$ of one of the equal and opposite forces T_1 with respect to the other. If we now imagine the specified loading removed, and the force pairs T_1 and T_2 applied to the base frame in turn, it is found that the relative movement of one of the forces T_1 with respect to the other force T_1 is

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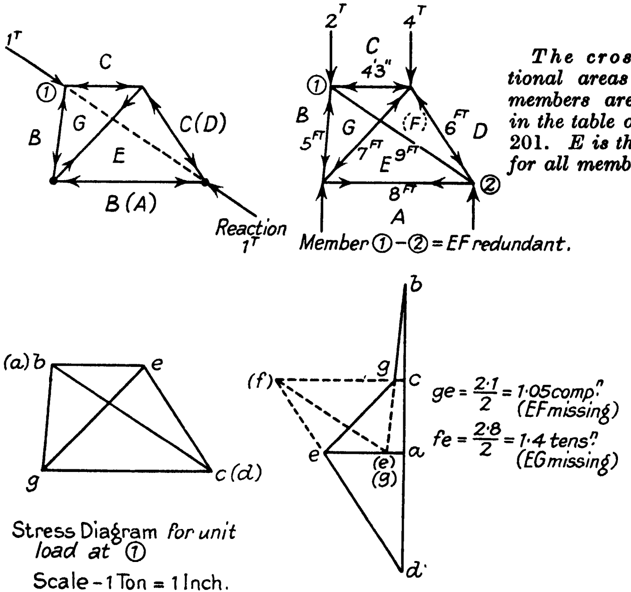
* See page 64: also Reference (9), p. 220.

$$T_1 y_{AA} = T_1 \Sigma K_1^2 \, l/AE$$

(where y_{AA} is the amount due to unit loading, and includes the unit extension of the redundant member) plus a further amount

$$T_2 \, . \, y_{AB} = T_2 \Sigma K_1 K_2 l/AE$$

where y_{AB} is the relative movement of one of the forces T_1 due to the unit force pair acting at the points of application of the force pair T_2. T_1 acts at points A and T_2 at points B. Similarly for Equation (13*b*).


1T
C
①
G
C(D)
B
E
B(A)
Reaction 1T
2T
4T
C
4'3"
①
B
5FT
G
(F)
6FT
D
7FT
9FT
E
8FT
②
A
Member ①-② = EF redundant.
(a)b
e
g
c(d)
b
g
C
(f)
ge = 2·1/2 = 1·05 comp.ⁿ (EF missing)
fe = 2·8/2 = 1·4 tens.ⁿ (EG missing)
e
a
(e)
(g)
d

The cross-sectional areas of the members are given in the table on page 201. E is the same for all members.

Stress Diagram *for unit load at* ①

Scale - 1 Ton = 1 Inch.

Force - Stress Diagram *for given loads and frame with bar EF missing. Dotted lines portion of diagram if EG redundant*

Scale - 2 Tons = ½ Inch.

FIG. 122

Illustrative Problem 31.

Find the forces in the members of the loaded frame given in Fig. 122

To find the forces in the redundant members by superposition, make the frame into two perfect frames, and in this case each perfect frame takes the whole loadings. By means of stress diagrams, find the loads in the members; for any member which occurs in both perfect frames, take the mean of the forces in it due to the two loadings to give the approximate actual force in the member.

Solution is by the method previously detailed

$$K_1 = k_1 \text{ and } k_1' = 0$$

for there is no displacement of the force T_1 acting at the support

TABLE FOR ILLUSTRATIVE PROBLEM 31

Member.	ΣkW Force due to 2^T and 4^T acting on a Perfect Frame.	Force due to Load at Joint (1) $= k_1$	l in.	A sq. in.	$\frac{l}{A}$	$(\Sigma kW). k_1 . \frac{l}{A}$	$k_1^2 \frac{l}{A}$ (including redundant member)
BG . .	– 2·0*	– 0·55	60	3	20	+ 22·0	+ 6·0
CG . .	– 0·2	– 0·90	51	3	17	+ 3·0	+ 13·8
DE . .	– 3·0	– 0·75	72	3	24	+ 54·0	+ 13·5
EA . .	+ 1·7	– 0·48	96	1	96	– 78·5	+ 22·1
EG . .	– 2·2	+ 0·75	84	1	84	– 139·0	+ 47·0
EF . .	(redundant)	(1·0)	108	1	108	—	+ 108·0
					Total .	– 138·5	+ 210·4

$$\text{From equation (10}a\text{) } T_1 = -\frac{\Sigma\Big((kW)\Big)k_1 . \frac{l}{A}}{\Sigma\Big(k_1^2\frac{l}{A}\Big) + \frac{l_1}{A_1}}$$

$$210{\cdot}4\, T_1 = 138{\cdot}5$$

$$T_1 = + {\cdot}66 \text{ ton (tensile).*}$$

† Compressive force – ; tensile force +.

ACTUAL FORCES IN THE MEMBERS, INCLUDING THE REDUNDANT MEMBER

Member.	ΣkW.	k_1T_1	Resultant Force by Principle of Least Work. $F = \Sigma kW + k_1T_1$ Tons.	By Superposition of Two Perfect Frames.* Tons.
BG . .	– 2·0	– 0·36	– 2·36	– 2·9
CG . . .	– 0·2	– 0·60	– 0·80	– 1·45
DE. . .	– 3 0	– 0·50	– 3·50	– 3·9
EA. . .	+ 1·7	– 0·32	+ 1·38	+ 1·05
EG . .	– 2·2	+ 0·50	– 1·70	– 1·1
EF. . .	—	+ 0·66	+ 0·66	+ 1·4

k_1' due to a force T_1 acting at the support is zero for all members.

---

* The positive sign indicates that T_1 acts in the direction of the applied unit load, and vice versa.

† Each perfect frame takes the whole loading, and the resultant force in a member is the mean of the two forces.

The agreement between the forces in the same member depends upon the areas of the members: in one case, known; in the second, unknown.

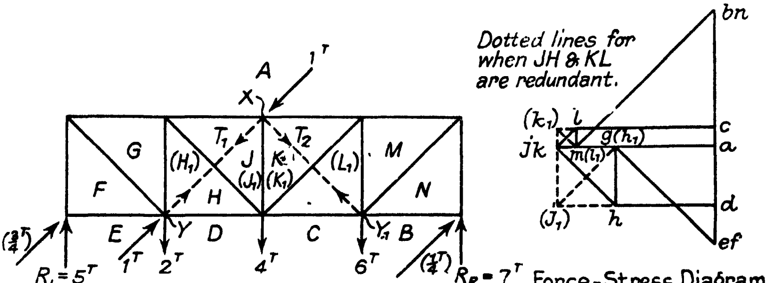
Illustrative Problem 32.

Find the forces in the members of the loaded frame given in Fig. 122A. The calculations are given in the three tables shown on pages 203–204.

The cross-sectional areas of the members are given in the first table on page 203.

E is the same for all members.

Bays: 60 in. long; height 60 in.



Force-Stress Diagram for the perfect frame. See Table for kind of load in a member. Scale - 5 Tons to ½ Inch.

FIG. 122A

107. Work Due to Bending, Using Moments Instead of Forces. In Chapter III it was shown that the internal work stored in a beam between two limits of $x = 0$ and $x = x_1$ due to bending was

$$U = \int_{x=0}^{x=x_1} \frac{M^2 \cdot dx}{2EI}$$

By similar reasoning as for forces, if the total internal work done by bending is a minimum, and this work depends upon some unknown factor, then differentiating the total internal work with respect to the unknown factor, the result must be equal to zero.

108. A direction-fixed-ended beam carries a central load of W tons.

Find the end fixing moments. (Fig. 123, page 204.)

Moment at a section X between the origin and W

$$= M_0 - \frac{W}{2} x$$

where M_0 is the end fixing moment.

TABLE A FOR ILLUSTRATIVE PROBLEM 32

Tensile Forces + ; Compressive Forces –

Member. (*See* Base Frame Diagram.)	Forces due to Specified Loads on the Base Frame. ΣkW. Tons	Force due to force pair of 1 ton acting in direction of force pair T_1 at T_1 joints. K_1 tons.	Force due to force pair of 1 ton acting in direction of force pair T_2 at T_2 joints. K_2 tons.	l in.	A sq. in	l/A
AF . . .	– 5·0	0	0	60	2·0	30
AG . . .	– 5·0	0	0	60	2·5	24
AJ . . .	– 8·0	– 0·70	0	60	3·0	20
AK . . .	– 8·0	0	– 0·70	60	3·0	20
AM . . .	– 7·0	0	0	60	2·5	24
AN . . .	– 7·0	0	0	60	2·0	30
BN . . .	0	0	0	60	1·0	60
CL . . .	+ 7·0	0	– 0·70	60	1·5	40
DH . . .	+ 5·0	– 0·70	0	60	1·5	40
EF . . .	0	0	0	60	1·0	60
FG . . .	+ 7·0	0	0	85	1·0	85
GH . . .	– 3·0	– 0·70	0	60	1·0	60
HJ . . .	+ 4·5	+ 1·00	0	85	1·0	85
JK . . .	0	– 0·70	– 0·70	60	1·0	60
KL . . .	+ 1·5	0	+ 1·00	85	1·0	85
LM . . .	– 1·0	0	– 0·70	60	1·0	60
MN . .	+ 10·0	0	0	85	1·0	85
Redundant T_1 (1) .	—	(1·0)	—	85	2·0	42·5
„ T_2 (2) .	—	—	(1·0)	85	2·0	42·5

Member.	K_1^2	K_2^2	$K_1 \cdot K_2$	$(\Sigma kW) \cdot K_1$	$(\Sigma kW) \cdot K_2$
AF . . .	0	0	0	0	0
AG . . .	0	0	0	0	0
AJ . . .	+ 0·50	0	0	+ 5·6	0
AK . . .	0	+ 0·50	0	0	+ 5·6
AM . .	0	0	0	0	0
AN . . .	0	0	0	0	0
BN . .	0	0	0	0	0
CL . . .	0	+ 0·50	0	0	– 4·9
DH . . .	+ 0·50	0	0	– 3·5	0
EF . . .	0	0	0	0	0
FG . . .	0	0	0	0	0
GH . . .	+ 0·50	0	0	+ 2·1	0
HJ . . .	+ 1·0	0	0	+ 4·5	0
JK . . .	+ 0·50	+ 0·50	+ 0·50	0	0
KL . . .	0	+ 1·00	0	0	+ 1·5
LM . . .	0	+ 0·50	0	0	+ 0·7
MN . . .	0	0	0	0	0
Redundant T_1 (1) .	(1·0)	0	0	—	—
„ T_2 (2) .	0	(1·0)	0	—	—

Member.	$K_1^2 \cdot \frac{l}{A}$	$K_2^2 \cdot \frac{l}{A}$	$K_1 \cdot K_2 \cdot \frac{l}{A}$	$(\Sigma kW) \times K_1 \cdot \frac{l}{A}$	$(\Sigma kW) \times K_2 \cdot \frac{l}{A}$	Total Force in the Member. Tons.
AF . .	0	0	0	0	0	− 5·00
AG . .	0	0	0	0	0	− 5·00
AJ . .	+ 10·0	0	0	+ 112·0	0	− 6·46
AK . .	0	+ 10·0	0	0	+ 112·0	− 7·94
AM . .	0	0	0	0	0	− 7·00
AN . .	0	0	0	0	0	− 7·00
BN . .	0	0	0	0	0	0
CL . .	0	+ 20·0	0	0	− 196·0	+ 7·06
DH . .	+ 20·0	0	0	− 140·0	0	+ 6·54
EF . .	0	0	0	0	0	0
FG . .	0	0	0	0	0	+ 7·00
GH . .	+ 30·0	0	0	+ 126·0	0	− 1·46
HJ . .	+ 85·0	0	0	+ 382·5	0	+ 2·30
JK . .	+ 30·0	+ 30·0	+ 30·0	0	0	+ 1·60
KL . .	0	+ 85·0	0	0	+ 127·5	+ 1·44
LM . .	0	+ 30·0	0	0	+ 42·0	− 0·94
MN . .	0	0	0	0	0	+ 10·00
Redundant T_1 (1)	(42·5)	—	—	—	—	− 2·20
,, T_2 (2)	—	(42·5)	—	—	—	− 0·09
Totals . .	+ 217·5	+ 217·5	+ 30·0	+ 480·5	+ 85·5	—

$$217{\cdot}5T_1 + 30T_2 + 480{\cdot}5 = 0$$
$$217{\cdot}5T_2 + 30T_1 + 85{\cdot}5 = 0$$

On Solution $T_1 = -2{\cdot}20$ tons

$T_2 = -0{\cdot}09$ ton

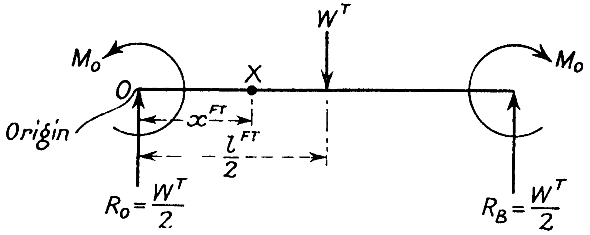


FIG. 123

Total internal work due to bending

$$= U = 2\int_0^{\frac{l}{2}} \frac{\left(M_o - \frac{Wx}{2}\right)^2}{2EI} \cdot dx$$

$$= \frac{1}{EI}\left[M_o^2\frac{l}{2} - M_oW\frac{l^2}{8} + \frac{W^2l^3}{96}\right]$$

The redundant quantity is M_o, and the total internal work must be a minimum for the application of M_o.

$$\text{Thus, } \frac{\partial U}{\partial M_o} = \frac{1}{EI}\left(M_o l - \frac{Wl^2}{8}\right) = 0$$

$$M_o = \frac{Wl}{8} \text{ units } (cf. \text{ Chap. IV})$$

109. **A Continuous Beam** (Fig. 124) of two equal spans is uniformly loaded with w tons per foot run for both spans. Find the value of the fixing moment M_B at the centre support, the supports being at the same levels. For the first span, the moment at a section X distance x from the origin is

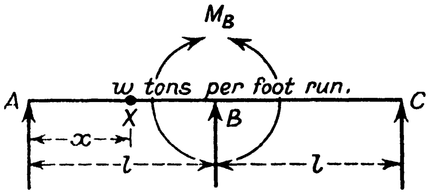


FIG. 124

$$M_X = \frac{M_B x}{l} + \frac{wx^2}{2} - \frac{wlx}{2}$$

Total internal work for the first span,

$$U = \frac{1}{2EI}\int_0^l \left(\frac{M_B x}{l} + \frac{wx^2}{2} - \frac{wlx}{2}\right)^2 \cdot dx$$

$$\text{Let } \frac{M_B}{l} - \frac{wl}{2} = A \text{ and } \frac{w}{2} = B$$

$$U = \frac{1}{2EI}\int_0^l x^2(A + Bx)^2 \cdot dx$$

$$= \frac{1}{2EI}\int_0^l (x^2A^2 + 2ABx^3 + B^2x^4) \cdot dx$$

Integrating and substituting the value $x = l$,

$$U = \frac{1}{2EI}\left(A^2\frac{l^3}{3} + \frac{ABl^4}{2} + \frac{B^2l^5}{5}\right)$$

Substituting for A and B

$$U = \frac{1}{2EI}\left(\frac{M_B{}^2 l}{3} - \frac{M_B w l^3}{3} + \frac{M_B w l^3}{4}\right) \quad . \quad . \quad . \quad (14)$$

U_2 = total internal work for the two spans = $2U$.

M_B is the unknown redundant*; therefore

$$\frac{\partial U_2}{\partial M_B} = \frac{2M_B l}{3} - \frac{wl^3}{12} = 0$$

$$M_B = \frac{wl^3}{12} \times \frac{3}{2l} = \frac{wl^2}{8} \quad (cf.\ \text{Chap. IV.})$$

Illustrative Problem 33.†

Two vertical posts 15 ft. apart and 15 ft. long, made of 5 in. × 3 in. British standard beam sections, are hinged at their bases, and their caps are connected by a beam of the same section rigidly attached to each. If this beam carries a central vertical load of 1 ton, estimate the maximum bending moment on the beam and on the posts. (Fig. 125.)

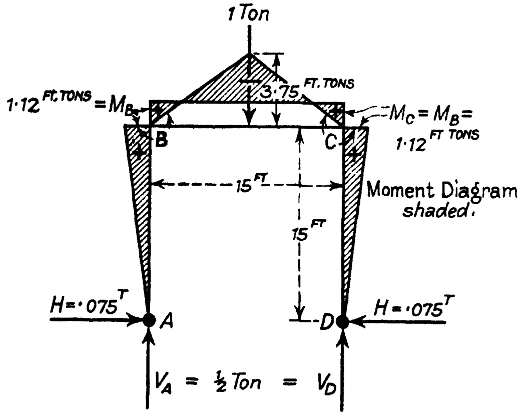


Fig. 125

The structure of Problem 33 is an example of a rigid frame structure, where the members consist of beams and columns. The joints of the columns and beams are assumed rigid, i.e. the rotation of all the bars meeting at a joint is the same.

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* *See* Chapter IV.

† Problem set for solution. Morley, *Theory of Structures*. A general solution of the type of structure given in this problem has been stated by E. H. Bateman in the *Philosophical Magazine*

The bases of the columns may be fixed or hinged. In the example given, the frame is made up of two columns and one beam which is centrally loaded. There will therefore be two couples of equal magnitude, which will act at the horizontal ends of the beam and will oppose the free rotation of the ends, so representing the action of the vertical bars on the beam. The end beam couple M_B can be considered as the only statically indeterminate quantity (*see* Chapter IV). The couple M_B will also act as the couple at the end of the vertical posts bending the bars.

As the beam is rigidly fixed to the posts, the fixing couples at the ends of the beam will cause a horizontal force at each of the base hinges; as there are no horizontal forces in the system, these will be equal and opposite (and their direction is shown in Fig. 125). The vertical reactions at the hinges will be each $= \frac{W}{2} = \frac{1}{2}$ ton.

The diagram of forces and moments is shown in Fig. 125.

Neglect work done due to direct and shear forces.

The total work done by bending on the beam and columns

$$= U = 2\underset{\text{(columns)}}{\int_0^{15} \frac{\left(M_B \frac{x}{15}\right)^2 \cdot dx}{2EI}} + 2\underset{\text{(beam)}}{\int_0^{7\cdot5} \frac{(M_B - \frac{1}{2}x)^2 dx}{2EI}}$$

E and I the same for columns and beam.

$$U = \frac{1}{EI}\left\{\left[\frac{M_B{}^2 x^3}{3 \times 225}\right]_0^{15} + \left[M_B{}^2 x - \frac{M_B x^2}{2} + \frac{x^3}{12}\right]_0^{7\cdot5}\right\}$$

$$= \frac{1}{EI}\left[5M_B{}^2 + 7\cdot5M_B{}^2 - 28M_B + \frac{7\cdot5^3}{12}\right]$$

Let M_B be the unknown redundant,*

$$\text{then } \frac{\partial U}{\partial M_B} = 25M_B - 28 = 0$$

$$M_B = +\ 1\cdot12 \text{ tons-feet}$$

Maximum positive moment on the columns = 1·12 tons-ft

" " " " beam = 1·12 "

---

* *See* Chapter IV, page 78.

Maximum negative moment for the beam

$$= -\tfrac{1}{2} \times 7{\cdot}5 + 1{\cdot}12 = -\ 2{\cdot}63 \text{ tons-ft.}$$

$$H = \frac{1{\cdot}12}{15} \text{ tons} = {\cdot}075 \text{ ton nearly.}$$

109*a*. **The Solution of Statically Indeterminate Structures from the Moment Deflection Method.** In para. 101*a* it was shown that

$$y = \int \frac{M \,.\, m \,.\, dx}{EI}$$

where M = moment at any section of the structure due to the specified loading and m = moment at the section due to an imaginary unit load applied at the point at which it is desired to find y. In the following example it will be shown how the above equation can be applied to the solution of the statically indeterminate problem in general.

EXAMPLE.

Determine the reaction of the centre support (*B*) of a continuous girder *ABC* resting on three rigid supports all at the same level.

Remove the centre support and imagine the simple beam AC acted upon by the specified loads. Calculate the displacement y^1_B at the section B. Imagine now the reaction at the centre support R_B only applied to the simple beam AC. Then R_B will be of such a magnitude that the displacement of B for R_B only on the beam will be equal (but of opposite sense) to the displacement of the simple beam AC under the specified loading.

$$\text{Then } y_B = \int_A^C \frac{M m_B dx}{EI}$$

$$M = M_S + R_B m_B$$

where M_S is the simple beam moment due to the specified loads at any point of AC, and $R_B m_B$ is R_B times the simple beam moment of any point in AC due to unit load applied at B.

$$y_B = 0 = \int_A^C \frac{M_S \,.\, m_B \,.\, dx}{EI} + R_B \int_A^C \frac{m_B{}^2 \,.\, dx}{EI}$$

$$R_B = -\frac{\int_A^C M_S \,.\, m_B \,.\, dx/EI}{\int_A^C m_B{}^2 \,.\, dx/EI} = -\frac{y'_B}{y_{BF}}$$

Compare equation (10*b*).

The general equation is of the form

$$y_B = 0 = y_B' + R_B y_{BB} \text{ (see also Equation (10}a\text{))}$$

Generally, if a beam $ABCD$ is continuous and rests on four rigid supports all at the same level, then suppose the reactions R_B and R_C at the intermediate supports are the redundant ones, and referring also to Equation (12*b*), page 199.

$$y_B = 0 = y_B' + R_B y_{BB} + R_C y_{BC} \qquad (15)$$

and $$y_C = 0 = y_C' + R_B y_{CB} + R_C y_{CC} \qquad (16)$$

In Equation (15) above y_B' and y_{BB} have the same definition as in the previous example and

$$y_{BC} = \int_A^D \frac{m_B \, . \, m_C \, . \, dx}{EI}$$

= deflection at B due to unit load only acting at C on the simple beam AD.

Similar definitions and forms apply to y_C', y_{CB}, and y_{CC}. Also, $y_{CB} = y_{BC}$ from Maxwell's theorem of reciprocal deflections.* The equations can therefore be solved for R_B and R_C.

The following problems illustrate the above method for singly determinate structures.

(*a*) *The Continuous Girder of Two Equal Spans and Carrying a Uniform Load.* (Fig. 126.)

$$EI = \text{Constant}$$

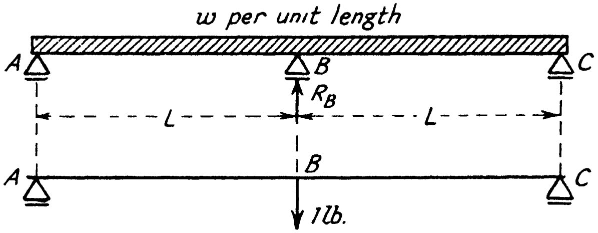

FIG. 126

Treat the centre support as redundant and take the origin at A. Working from $A(x = 0)$ to $x = L$

$$M_S = wLx - \frac{wx^2}{2} \text{ and } m_B = \frac{x}{2}$$

---

* See page 64, and Reference (6), page 219.

$$y_B' = \frac{2}{EI}\int_0^L \left(wL\frac{x^2}{2} - \frac{wx^3}{4}\right)dx = \frac{5wL^4}{24EI}$$

$$y_{BB} = \frac{2}{EI}\int_0^L m_B^2 \, . \, dx = \frac{2}{EI}\int_0^L \frac{x^2}{4} \, . \, dx$$

$$= \frac{L^3}{6EI}$$

$$\therefore \qquad R_B = -\frac{y_B'}{y_{BB}} = -\frac{\frac{5}{24}wL^4}{\frac{L^3}{6}} = -\frac{5}{4}wL.$$

The sign of R_B is negative which indicates that it acts in an upward direction.

$$M_B = -\frac{3}{8}wL \times L + wL \times \frac{L}{2}$$

$$= \frac{wL^2}{8} \text{ (cf. Para. 109.)}$$

(*b*) *Portal Frame.*

A general solution of the portal shown in Fig. 127A by use of the equation (16).

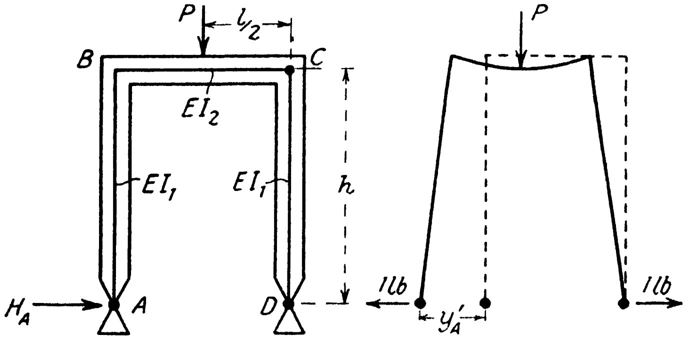

FIG. 127A

The horizontal reaction is treated as redundant, and the effects of longitudinal and shear force strains are neglected.

The fundamental equation is

$$H_A = -\frac{y_A'}{y_{AA}} = \frac{\sum\int\frac{M_S \,.\, m_A \,.\, dx}{EI}}{\sum\int\frac{m_A^2 \,.\, dx}{EI}}$$

$$\sum\int\frac{M_S \,.\, m_A \,.\, dx}{EI} = \frac{2}{EI_2}\int_0^{\frac{l}{2}}\frac{Px}{2} \,.\, h \,.\, dx = \frac{Phl^2}{8EI_2}$$

$$\sum\int\frac{m_A^2 \,.\, dx}{EI} = 2\int_0^h\frac{x^2 \,.\, dx}{EI_1} + \int_0^l\frac{h^2dx}{EI_2} = \frac{2}{3}\frac{h^3}{EI_1} + \frac{h^2l}{EI_2}$$

$$H_A = -\frac{\frac{Phl^2}{8EI_2}}{\frac{2}{3}\frac{h^3}{EI_1} + \frac{h^2l}{EI_2}} = -\frac{\frac{Pl^2}{8I_2}}{\frac{2}{3}\frac{h^2}{I_1} + \frac{hl}{I_2}}$$ when E is the same for all members.

The unit load was applied outwardly: the minus sign shows that H_A acts inwardly.

The moments at the joints are obviously equal to H_Ah in magnitude.

If the properties of the column and beam of Problem 33 are inserted in the equation for H_A, it will be found that its value is $-$ 0·075 tons.

(*c*) *Portal Frame with Side Horizontal Load P at the Top of one Column.* (Fig. 127B.)

Neglect the effect of the axial and shear forces. Let H_A be the redundant quantity.

$$H_A = -\frac{y_A'}{y_{AA}} = -\frac{\sum\int\frac{M_S \,.\, m_A \,.\, dx}{EI}}{\sum\int\frac{m_A^2 \,.\, dx}{EI}}$$

$M_S = 0$ for AB; $M_S = \frac{Ph}{l}x$ for any section distant x from the origin B for beam BC.

$M_S = Px$ for any section distant x from the origin D for the column CD.

$m_A = 1 \times x$ for the columns with origins at A and D.

$m_A = 1 \times h$ at any section of the beam with origin at B.

$$\Sigma\int\frac{M_B\,.\,m_A\,.\,dx}{EI}=\int_0^l\frac{Phx}{l}\,.\,\frac{h\,.\,dx}{EI_2}+\int_0^h\frac{Px^2\,.\,dx}{EI_1}$$

$$=\frac{Ph^2l}{2EI_2}+\frac{Ph^3}{3EI_1}$$

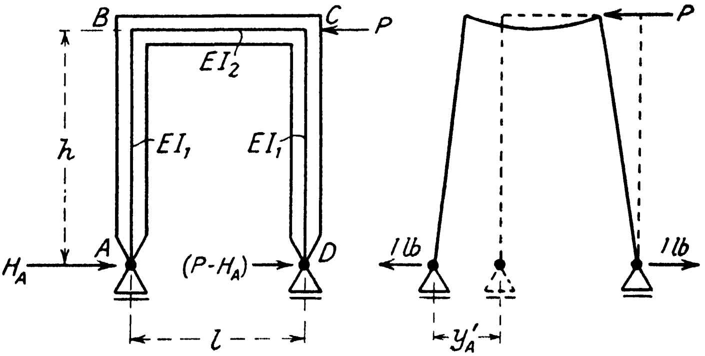

FIG 127B

$$\Sigma\int\frac{m_A{}^2\,.\,dx}{EI}=2\int_0^h\frac{x^2\,.\,dx}{EI_1}+\int_0^l\frac{h^2\,.\,dx}{EI_2}$$

$$=\frac{2}{3}\frac{h^3}{EI_1}+\frac{h^2l}{EI_2}$$

$$\therefore\ H_A=-\frac{\dfrac{Ph^2l}{2EI_2}+\dfrac{Ph^3}{3EI_1}}{\dfrac{h^2l}{EI_2}+\dfrac{2}{3}\dfrac{h^3}{EI_1}}=-\frac{P}{2}$$

This value is also given by the usual approximate formula for P acting at the top of the column, and for no axial shortening of the members.

109*b*. In connection with the solution of rigid frames and continuous structures, special analytical methods of solution have been developed, and the reader who is interested is referred to works dealing with these methods. Well-known ones are the *Slope-deflection* method, the *Moment-distribution* method sponsored by Professor Hardy Cross, the *Slope-distribution* method of Goldberg, and the *Remainder-distribution* method of Bateman. Also in certain cases, solutions can be obtained by the use of the theorem of three moments applied to continuous frames.

Consider the frame given in Problem 2 of the examples at the end of this chapter. As AB is shorter than DC, there will be a displacement of B relative to A and C relative to D. These displacements y will be the same and imagine the movement takes place to the right. The distorted frame can then be imagined opened out as in Fig. 127c.

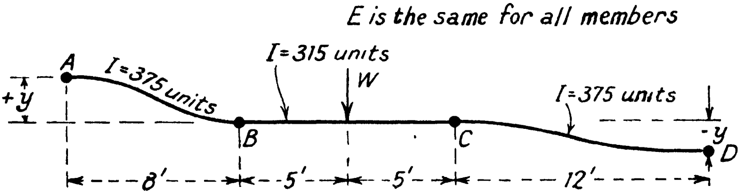

FIG. 127C

As there are hinges at A and D, no moments can occur at these points. Let M_B and M_C be the hogging couples at B and C. (*See* equation (24), Chapter IV.)

Considering members AB and BC,

$$2M_B\left(\frac{8}{375} + \frac{10}{315}\right) + M_C\frac{10}{315} = \frac{375W}{10 \times 315} - \frac{6Ey}{8} \quad . \quad (17)$$

Considering members BC and CD,

$$M_B\left(\frac{10}{315}\right) + 2M_C\left(\frac{10}{315} + \frac{12}{375}\right) = \frac{375W}{10 \times 315} + \frac{6Ey}{12} \quad . \quad (18)$$

If H is taken as the horizontal component of the reactions at the hinges A and D, then $M_B = 8H$ and $M_C = 12H$.

Substituting in equations (17) and (18), eliminating y and solving for H, it is found that $H = 0{\cdot}234$ ton.

109*c*. E. H. Bateman has shown that the strain energy U in an elastic bar AB, of stiffness $I_{AB}/l_{AB} = K_{AB}$, which is bent by any distribution of transverse loading and by end moments M_A and M_B, is given by

$$6EK_{AB}U = (M_A - F_A)^2 - (M_A - F_A)(M_B - F_B) + (M_B - F_B)^2 + C \quad . \quad . \quad . \quad . \quad (19)$$

where E is Young's Modulus, F_A and F_B are the end moments at A and B which would be produced by the transverse loadings if the ends of the bar were fixed in direction, and C is independent of M_A and M_B. The determination of F_A and F_B is simple.

For a concentrated transverse load P, at a point distant x from A,

$$F_A = Px(1-x/l)^2, \quad F_B = Px(1-x/l) \, . \, x/l$$

and for a distributed load

$$F_A = \int_0^l wx(1-x/l)^2 \, . \, dx, \quad F_B = \int_0^l wx(1-x/l) \, . \, x/l \, . \, dx$$

where w is any function of x.

If F_A is taken as acting in a positive direction, then it is a positive number, and F_B acting in a negative direction is a negative number. If M_A and M_B act in the same directions as F_A and F_B, then their signs are the same; if they act in the oppositive direction, then they are of the opposite sign.

Signs of Moments.

End moments and end-fixing moments are all positive when operating in an anti-clockwise rotation. Downward vertical forces acting on horizontal members give a positive fixing moment at the left-hand end of a member, and a negative fixing moment at the right-hand end. Similarly for a vertical member, if the applied force acts horizontally in a left-to-right direction, then the fixing moment is positive at the bottom, and negative at the top of the member.*

The complete solution for the portal frame and loading given in Fig. 127A is now easily derived by the application of Castigliano's theorem of minimum strain energy.

Considering the beam BC, we have, using equation (19),

$$\begin{aligned} 6EI_2/l . U_{BC} = {} & (M_B - F_B)^2 - (M_B - F_B)(+M_C - F_C) \\ & + (+M_C - F_C)^2 + C_{BC} \end{aligned}$$

Let M_B be the redundant we wish to find. Owing to symmetry $M_C = -M_B$ in magnitude and $F_B = -F_C$ as the beam is loaded at the centre. $F_B = \frac{Pl}{8}$

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* Refer also to footnote, page 115.

$$6EI_2/l \,.\, U_{BC} = \left(M_B - \frac{Pl}{8}\right)^2 - \left(M_B - \frac{Pl}{8}\right)\left(-M_B + \frac{Pl}{8}\right)$$
$$+ \left(-M_B + \frac{Pl}{8}\right)^2 + C_{BC}$$

Considering the columns AB and CD, the fixing couple at the end B of column AB is $-M_B$ and at the end C of column CD is $+M_C$: as these members carry no transverse load then

$$F_A = F_B = F_C = F_D = 0$$

Also $\quad M_B = M_C$

There are no fixing couples at the hinges.

Then $\quad 6EI_1/h \,.\, U_{AB} = M_B{}^2 + C_{AB}$

and $\quad 6EI_1/h \,.\, U_{CD} = M_B{}^2 + C_{CD}$

∴ Adding together

$$6E.\, U = M_B{}^2\left(\frac{2h}{I_1}\right) + \frac{l}{I_2}\left(3M_B{}^2 - 6M_B \,.\, \frac{Pl}{8} + \frac{3P^2l^2}{64}\right)$$
$$+ C_{BC} + C_{AB} + C_{CD}$$

Solving for M_B

$$6E \,.\, \frac{\partial U}{\partial M_B} = \frac{4h}{I_1} \,.\, M_B + 6M_B \,.\, \frac{l}{I_2} - 6\frac{Pl}{8} \,.\, \frac{l}{I_2} = 0$$

$$M_B = \frac{\dfrac{3Pl^2}{8I_2}}{\dfrac{2h}{I_1} + \dfrac{3l}{I_2}} \quad \begin{pmatrix} + \text{ for beam } BC \\ - \text{ for col. } BA \end{pmatrix}$$

$M_B = -H_A h$. when considering column AB.

$$\therefore \qquad H_A = -\frac{\dfrac{Pl^2}{8I_2}}{\dfrac{2}{3}\dfrac{h^2}{I_1} + \dfrac{hl}{I_2}} \quad \text{(See equation for } H_A \text{ on page 211.)}$$

Problem. The solution of the Portal Frame loaded as in Fig. 127B.

Referring to this figure. As there are hinges at A and D there will be no moments at these points. There will be fixing couples at B for the column AB and for the beam BC. Let these couples be $+M_B$; there will be equal couples at the end C of beam BC and the top of the column CD: let these couples be $+ M_C$. M_B and M_C may be regarded as the unknown quantities: they are not independent, for it can be easily shown that $M_B + M_C = M = Ph$. Let M_B be the redundant couple it is desired to find. The direction-fixing couples F for the three members are all zero for the loading used.

Strain energy equations for the columns are

$$6E\frac{I_1}{h}U_1 = M_B^2 \text{ and } 6E\frac{I_1}{h}U_2 = M_C^2$$

The strain energy equation for the beam is

$$6E\frac{I_2}{l}U_3 = M_B^2 - M_BM_C + M_C^2.$$

The total strain energy for the portal is

$$U = \frac{hM_B^2}{6EI_1} + \frac{l}{6EI_2}(M_B^2 - M_BM_C + M_C^2) + \frac{h}{6EI_1}M_C^2$$

The equation of equilibrium between the external forces and the terminal couples is

$$M_B + M_C + M = 0 \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where M is the moment about D of the horizontal components of all of the external forces, in the direction of a positive terminal couple. Since M_B and M_C are not independent, only one equation will be required to establish the condition of minimum strain energy, and this is written

$$0 = \frac{\partial U}{\partial M_B} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Also we have from equation (20)

$$\frac{\partial M_C}{\partial M_B} = -1$$

Then equation (21) becomes

$$\frac{\partial U}{\partial M_B} = 2M_B\frac{h}{I} + \frac{l}{I_2}(2M_B - M_C) + M_B\frac{l}{I_2} - 2M_C\frac{l}{I_2} - 2M_C\frac{h}{I_1} = 0$$

Also, $\quad M_C = -M_B - M$

Then $\quad M_B = -\dfrac{M\left(\dfrac{3l}{I_2} + 2\dfrac{h}{I_1}\right)}{\dfrac{6l}{I_2} + 4\dfrac{h}{I_1}}$

Now $$M = +Ph$$

$\therefore$ $$M_B = -Ph/2$$

Considering the column AB.

Let H be the horizontal thrust at the hinge A, which is balanced by an equal and opposite shear force H at the top of the columns. Then $Hh + M_B = 0$.

Then $H = +\frac{P}{2}$ and acts in the opposite direction to P, i.e. towards P, because Hh is positive and this couple has positive sense (and therefore sense of rotation is anti-clockwise). The same procedure is adopted for Portal Frames having fixed column bases: the general solution for such portals when all the members are loaded transversely has been given by Bateman in his paper in the *Philosophical Magazine*, May, 1934. From the general solution, the result for any kind of loading can be easily ascertained.

Solution of Problem (2) *Examples, page* 220, *by the Previous Method.*

There will be no couples at the column bases, but there will be equal and opposite horizontal forces acting inwards equal to H. The terminal couple at the top of column AB will be $8H$ and at the top of column DC it will be $1\frac{1}{2} \times 8H = 12H$.

Let $8H = M$ which is the terminal couple at B of the beam BC: the terminal couple at C of the beam BC will be $-3/2M$.

As the beam BC is centrally loaded, the direction-fixing couples for B and C will be $+F$ and $-F$ respectively.

The strain energy for the whole system can be written

$$6EU = M^2 \times \frac{8}{375} + \frac{10}{315}\left\{(M-F)^2 - (M-F)\left(-\frac{3M}{2} + F\right) + \left(-\frac{3M}{2} + F\right)^2\right\} + \frac{12}{375}\left(\frac{3M}{2}\right)^2 + C$$

$$= \frac{M^2}{375}(8 + 27) + \frac{10}{315}\left\{\frac{19}{4}M^2 - \frac{15}{2}MF + 3F^2\right\} + C$$

Cancelling $6E$ and differentiating,

$$\frac{\partial U}{\partial M} = \frac{70}{375}M + \frac{10}{315}\left(\frac{19}{2}M - \frac{15}{2}F\right) = 0$$

$$M = \frac{75}{153{\cdot}8}F\,;\ F = \frac{15}{4}\text{ tons-ft.}\quad \therefore\ M = 1{\cdot}83\text{ tons-ft.}$$

$$8H = 1{\cdot}83\text{ tons-ft.}\quad \therefore\ H = 0{\cdot}23\text{ ton}$$

and acts from left to right at base of Column *AB*.

(*Cf.* 0·234 by continuous beam method.)

For the solution of portals and continuous frames having built-in or fixed column bases, the student is referred to the papers by E. H. Bateman, which are noted in the references at the end of the chapter.

109*d*. **Mechanical Solution.*** Experiments on models of structures to determine the redundants of reactions or stresses for the corresponding full-scale structure. Only the outline of the method can be given, by showing the application to the continuous beam of Fig. 127D.

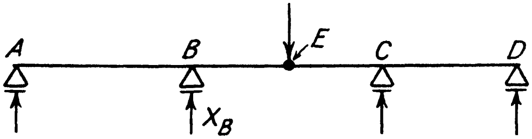


FIG. 127D

In this method the fundamental structure is not the simple structure with all redundants removed: it is the structure obtained by the removal of the redundant it is desired to find, and no other. Let it be supposed that in the above girder it is required to find the value of the reaction X_B for a unit load at any point *E* on the beam. The support at *B* is removed and the girder *ACD* is considered and is our base system. Let Δ be the deflection at any point on the beam, then

$$X_B = R_B = -\frac{\Delta_{BE}}{\Delta_{BB}}$$

where in general Δ_{BE} is the deflection at *B* due to unit load at *E* on the girder *ACD*, and Δ_{BB} is the vertical deflection at *B* due to unit load at *B*.

For solution, a model of the beam is made to scale, and by means of suitable arrangements it is supported and hinged at the corresponding points A, C, and D. The value of I of the beam is proportional to that of the actual girder. The model is displaced at the corresponding point B by an amount y_{BB} in the direction of X_B: the amount of displacement of the model at the corresponding point E in the direction of the actual unit load is then measured. This is equal to y_{EB} which is equal to y_{BE}.

---

* A full discussion of this solution is given in the two papers of reference (9), page 220, and in the paper of reference (8), page 220.

Also refer to Chapters XV and XVI for modern work on statically indeterminate structures and the mechanical solution of these structures.

It has been shown that the model ratio of deflections

$$\frac{y_{EB}}{y_{BB}} = \frac{y_{BE}}{y_{BB}}\text{; is equal to } \frac{\Delta_{BE}}{\Delta_{BB}}$$

for the full-scale structure. Thus, in general, the ratio of model displacements is equal to a force ratio for the actual structure; and if P_E is the load at a point E on the full-scale structure and X_B is an unknown redundant (which may be a couple), then

$$-\frac{y_{EB}}{y_{BB}} = \frac{X_B}{P_E}.$$

Good results have been obtained by the use of relatively simple and easily constructed models in celluloid. Cardboard has been used for model making, but as it is not a homogeneous material, it is not recommended for use in cases where the full-scale structure is of homogeneous material. Care must be taken in the design of model members to eliminate, so far as possible, axial and shear force strain effects where these are neglected in the analytical discussion. The reader is referred to papers dealing with this subject, a few of which are given in the references which follow.

REFERENCES

(1) *Principle of Least Work*, Martin. (Published by *Engineering*.)

(2) *Higher Problems in Structures*, Andrews. Other methods for finding stresses in imperfect frames and moments for portals, etc.

(3) *Mechanics of Internal Work*, Church (Wiley.) Complete theory and many examples.

(4) *Principle of Virtual Velocities and Its Application to the Theory of Elastic Structures*, E. H. Lamb, D.Sc (Paper No. 10, *Proceedings Institute of Civil Engineers*, 1923.)

(5) *Aeroplane Structures*, Pippard and Pritchard. (Longmans, Green & Co.) Appendix: Principle of Least Work.

(6) *Statically Indeterminate Structures*, Parcel and Maney. (John Wiley & Sons.) Includes, amongst many methods, discussion of Slope-deflection Method.

(7) Papers in *Philosophical Magazine*, E. H. Bateman. December, 1933, May, 1934, January, 1935, on "The Strain Energy Method in Elastic Network Analysis."

(8) *Proc. Am. Con. Institute,* Vol. XVIII, pp. 58–82, Professor Begg. Solution of Indeterminate Problems, mechanically.

(9) *Structural Engineer,* August and October, 1930, H. W. Coultas and V. H. Lawton. Discussion of mechanical solution of Statically Indeterminate Structures.

(10) "Analysis of Continuous Frames by Distributing Fixed-end Moments," Hardy Cross. *Transactions of American Society Civil Engineers,* Vol. 96, pp. 1–156.

(11) *Continuous Frames of Reinforced Concrete,* Cross and Morgan. (J. Wiley & Sons.)

(12) "Wind Stresses by Slope-deflection and Converging Approximations," J. E. Goldberg. *Am. Soc. C.E. Proceedings,* 1933.

(13) "The Stress Analysis of Continuous Frames." A comparison of various methods. E. H. Bateman. A Paper in which is given the Remainder Distribution Method. *Structural Engineer,* October and November, 1936.

(14) *Theory of Modern Steel Structures,* Vol. II, "Indeterminate," L. E. Grinter (Macmillan.)

(15) Chapters XV and XVI, pages 374 and 482.

EXAMPLES

1. A horizontal beam of span l is rigidly connected to two columns of length h, which are hinged at their lower ends. The moment of inertia of the section of the beam is I_b and of the columns is I_c. The beam carries a uniformly-distributed load of w tons per foot run. Neglecting the effect of thrust in the columns, determine the bending moment diagrams for the beam and columns. (U. of B.)

2. The frame *ABCD* (Fig. 128A) has rigid joints at *B* and *C*, and is hinged at *A* and *D* to fixed supports. *W* is a load of 3 tons applied at the centre of *BC*. The moment of inertia of the cross-sections of *AB* and *CD* is 375 in inch units, and that of *BC* is 315 in inch units. Find the bending moments at *B* and *C*, and the horizontal thrusts at *A* and *D*. Draw the bending moment diagram for *AB* and *BC*. (U. of L.)

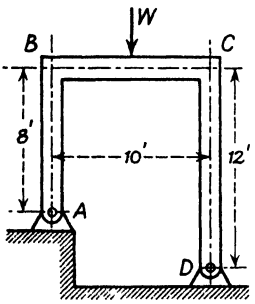


FIG. 128A

3. The lattice girder shown in Fig. 128B is loaded at the joint *H* with a load of 50 tons. Find the forces in the members due to the loading. The ratio of length to area of cross-section is the same for every member.

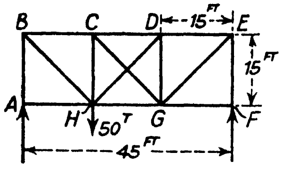


FIG. 128B

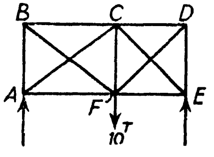


FIG. 128C

4. If the bases of the columns of the frames in Questions 1 and 2 are rigidly fixed, draw the moment diagrams for the columns and beam for both frames under the respective loadings.

5. Find the forces in the members of the lattice frame shown in Fig. 128c. The ratio of length to area of the cross-section is the same for every member. $AF = 12$ ft., $FE = 10$ ft., $DE = 10$ ft.

6. A beam is continuous over two spans of 20 and 30 ft. It is simply supported at the ends and the supports are at the same level. There are loads of 5 tons and 6 tons at distances of 10 and 35 ft. from the left-hand support. Find the fixing moment at the centre support by the principle of least work. (Check the result by the theorem of three moments.)

$E = 12{,}000$ tons/sq. in., and $I = 144$ in. units for both spans.

7. A beam having the supports at the same height and simply supported at the ends, is continuous over three spans of 20, 30, and 20 ft. There are loads at the middle points of the first, second, and third spans of 5, 6, and 4 tons respectively. $E = 12{,}000$ tons/sq. in., and $I = 144$ in. units are constant. Find the fixing moments at the central supports by the principle of least work.

8. A beam of length l is fixed rigidly at its ends. It carries a load of W tons at a distance nl (where $n < 1$) from the left support. Find the general expression for the fixing moments at the supports, by the principle of least work.

EI is a constant.

NOTE. When moment diagrams are drawn, they are usually placed on the tension sides of the various members.

CHAPTER X

BEAMS AND FRAMES WITH LIVE LOADS

110. **Moving Loads.** The determination of stresses in bridges and structures subjected to rolling loads is an important factor in bridge design. It is often facilitated by the use of "Influence Lines" and diagrams. Such lines and diagrams will be considered for bending moment and for shear. Fig. 129 gives a few typical examples of moving loads which bridges may have to carry.

111. **Definition.** An influence line for any given section P of a structure is such a line that its ordinate (to the beam as base) at any point X gives the bending moment, shear, or similar quantity at P when a load is placed at X. In the case of bending moment, shear force diagrams, etc., for dead loads, the ordinate at a section X gives the particular quantity for this section X only; whereas as regards an influence line for one particular section, the ordinate at any point on the beam gives the value of the moment, shear, etc., at the particular section, *and each section along the structure has its own influence line.*

112. The unit influence line will be developed by considering a load of 1 ton crossing over a beam or frame; and from this unit influence line the moment, shear, etc., for a number of moving loads or distributed loads can be simply ascertained.

113. **Influence Lines for Simply-supported Beams.** AB is a simple beam I—**Unit Influence Line of Bending Moment** (Fig. 130).

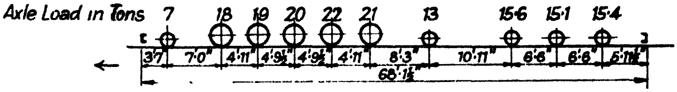
To construct the unit influence line of moment for the point P when a load of 1 ton is placed at any point on the structure AB,

$$AP = a \qquad PB = b$$

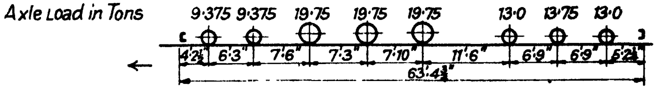
(*a*) Let the load of 1 ton be at any section X between A and P distant x from A and $l - x$ from B.

$$R_A = \frac{1 \times (l - x)}{l}\ ; \ R_B = \frac{1 \times x}{l}$$

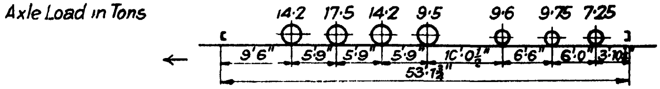
Railway Bridges.



L & N.E.Rly. (G.N.R.) Train of Locomotives as above.

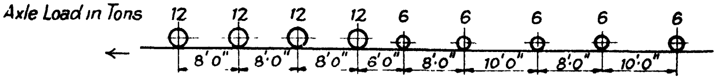


L M.&S.Rly. (L.N.W.R) "Sir Gilbert Claughton" Class.

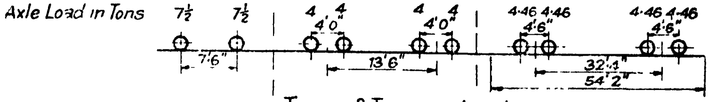


L.M.&S.Rly. (L.N.W.R.) 4 cylinder 8 wheels coupled Goods Engine.

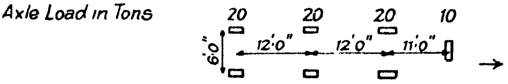
Highway Bridges.



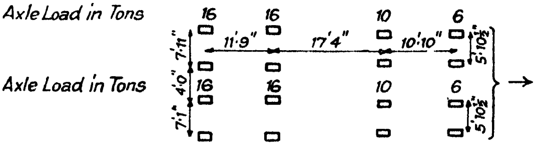
Light Railway.



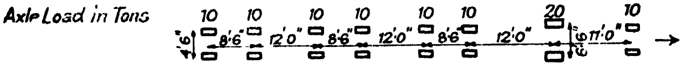
Types of Tramcar Loads



30 Ton Engine and 40 Ton Trailer.
(2 sets taken)



Two 16 Ton Tractors each with a 32 Ton Trailer.



30 Ton Tractor with 3 - 20 Ton Trailers.

FIG. 129

Retaining the ordinary signs of moment,

$$\text{The moment at } P = M_P = -R_Bb = -\frac{xb}{l}$$

R_B is proportional to x and, therefore, M_P is proportional to x, and is a maximum when $x = a$.

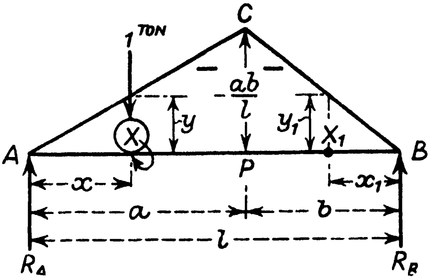


FIG. 130

Maximum value of M_P

$$= -\frac{ab}{l}$$

When the 1 ton is at A,

$R_B = 0 \qquad M_P = 0.$

At P erect an ordinate PC equal to $\frac{ab}{l}$ tons-ft. to scale.

Join C to A; then AC is the unit influence moment line for the point P with the load of 1 ton in any position between A and P.

Consider the load at X,

$$\frac{y}{x} = -\frac{ab}{la}\text{, then } y = -\frac{xb}{l}$$

which agrees with the equation $M_P = -R_Bb = -\frac{xb}{l}$

(*b*) Similarly for the load at any section between P and B, considering B as origin and a section X_1 distant x_1 from B. Join C to B, and the unit influence moment line for the section P is completed for the load at any point on the beam.

y_1, the ordinate at X_1 is equal to $-\frac{x_1a}{l}$ tons-ft. to scale

$$= M_P = -R_Aa = -\frac{ax_1}{l}$$

Thus, the ordinate of the unit moment diagram for the load of 1 ton at that ordinate gives the value to scale of the moment at the section considered.

(*c*) If a load of a value greater than 1 ton crosses the span, draw the unit influence moment diagram and multiply the ordinates by the value of the load.

Let the load $= W$ tons.

With 1 ton at $X, M_P = -\frac{xb}{l}$ tons-ft. to scale

With W tons at X, $M_P = -\frac{Wxb}{l}$ tons-ft.

For $\frac{Wx}{l} =$ reaction at $B = R_B$ and $M_P = -R_Bb$

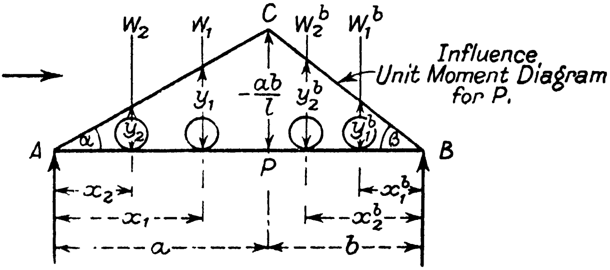


FIG. 131

114. The Moment at a Section P when a Number of Loads Cross the Beam. (Fig. 131.) Construct the unit moment influence line for P. $PC = -\frac{ab}{l}$.

Let some of the loads be between A and P, and distant x_1, x_2, x_3, etc., from A, the loads being W_1, W_2, W_3, etc.

Let some of the loads be between B and P, and distant x_1^b, x_2^b, x_3^b from B, the loads being W_1^b, W_2^b, W_3^b, etc.

Let the ordinates to the unit line be y_1, y_2, y_3, etc., and y_1^b, y_2^b, y_3^b, etc.

Considering Fig. 131,

$$M_P = W_1y_1 + W_2y_2 + W_1^by_1^b + W_2^by_2^b,$$

due regard being paid to the sign of the moment.

Let the angle $CAP = \alpha$, and angle $CBP = \beta$.

Maximum unit ordinate

$$= -\frac{ab}{l};\ y_1 = x \tan \alpha;\quad \therefore\ W_1y_1 = \tan \alpha\ W_1x_1, \text{ etc.}$$

$M_P = \tan \alpha(W_1x_1 + W_2x_2 + \ldots) + \tan \beta(W_1^bx_1^b + W_2^bx_2^b + \ldots)$

Therefore, for any number of loads on the beams,

$$M_P = \tan \alpha\, \Sigma Wx + \tan \beta \Sigma W^b x^b,$$

where W represents the loads between A and P
and W^b ,, ,, ,, P and B,

and where M_P will be of the negative sense in the case of a simple beam.

The Maximum Value of M_P. M_P will be a maximum when the loads are in such a position that

$$\tan \alpha \Sigma Wx + \tan \beta \Sigma W^b x^b \text{ is a maximum.}$$

It will always occur when one of the loads is at the section. To find by trial, place the loads on the beam with one of the loads at P; calculate the moment. Move on the loads to bring another on to P; again find the moment. The maximum value can thus be found. Professors Lea and Andrews have shown that to obtain a maximum moment, place a load on P, so that if considered as part of ΣW, then $b\Sigma W - a\Sigma W^b$ is positive, and if a part of ΣW^b, then $b\Sigma W - a\Sigma W^b$ is negative.

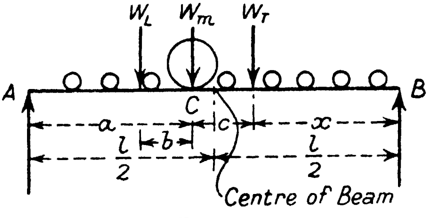


Fig. 132

115. **To Find the Section Having the Greatest Possible Moment under any given Load for a System of Concentrated Loads on a Beam.*** (Fig. 132.) Let W_T be the sum of *all* the loads on the beam, and acting at the centre of gravity of the system, distant x from the support B. It has been mentioned that a maximum moment for a section occurs with one of the loads at the section. In Fig. 132, let C be the point of maximum possible moment under W_m the given load.

Let W_L be the sum of the loads, including W_m on the portion AC of the beam. W_L will act at the centre of gravity of these loads, and let it be at a distance b from C.

Let c be the distance between W_T and W_m.

$$\text{Now } R_A = \frac{W_T x}{l}$$

---

* Or find the position of any given load so that the bending moment under it is the maximum possible for this particular load.

The moment at $C = M_m$,

$$\text{therefore, } M_m = -\frac{W_T xa}{l} + W_L b \quad . \quad . \quad . \quad (1)$$

By hypothesis this is the maximum possible moment: it is of negative sign.

For the given loads, b is a constant; but both x and a are variables: therefore, M_m is a maximum when xa is a maximum.

Also for the given set of loads, as c and l are constant, then

$$(l-c) = (x+a) \text{ is a constant,}$$
$$\text{i.e. } x + a = K = \text{constant}$$
$$\therefore a = K - x$$

Let $z = ax = x(K-x)$

$$\frac{dz}{dx} = K - 2x = 0 \text{ for a maximum}$$
$$\therefore 2x = K = x + a$$

For maximum moment, $x + a = 2x$

$$\therefore x = a$$

Hence, for a maximum value of M_m in Equation (1), the wheel under which the maximum moment occurs and the centre of gravity of all the loads must be at equal distances from the supports: this requires the *centre of the beam to be midway between the load under which the maximum moment occurs and the centre of gravity of the loads.* The maximum moment at the section under the given load is

$$M_m = -\frac{W_T a^2}{l} + W_L b \quad . \quad . \quad (2)$$

116. In determining the greatest possible maximum moment for a given set of loads, it is usually necessary to calculate the maximum value for several sections (usually near to the centre of the beam). On comparing these maximum values, the amount and position of the maximum possible moment is obtained, and also the position of the loads causing this moment.

117. **Important General Cases. Case I.** Two equal loads of W tons and distance d ft. apart. (Fig. 133.) Using the Rule given in paragraph 115, and referring to Fig. 133, the maximum possible moment is at two sections X, both distant $\frac{d}{4}$ from, and at either side of, the centre

$$= M_r = -\frac{2W}{l}\left(\frac{l}{2} - \frac{d}{4}\right)^2$$

Case II. Two unequal loads W_1 and W_2 distant d ft. apart. Let W_1 be greater than W_2. From the example No. 34 it will be noticed that the greatest moment occurs under the greater load. (Fig. 134.)

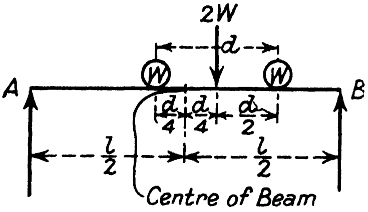


FIG. 133

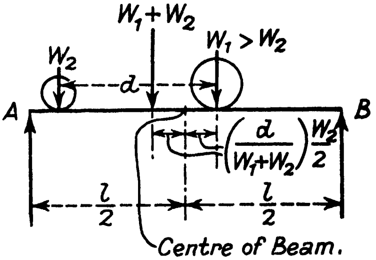


FIG. 134

The greatest possible maximum moment occurs at the section under W_1 distant $\frac{d}{2}\left(\frac{W_2}{W_1 + W_2}\right)$ from the centre and is

$$M_x = -\frac{W_1 + W_2}{l}\left(\frac{l}{2} - \frac{W_2}{W_1 + W_2}\frac{d}{2}\right)^2$$

$$= -\frac{W_1 + W_2}{4l}\left(l - \frac{W_2 d}{W_1 + W_2}\right)^2$$

Illustrative Problem 34. (Fig. 135.)

A beam is 20 ft. long, and loads of 2 tons and 4 tons, 4 ft. apart, move from left to right. Find the maximum moment at the centre of the beam.

(a) 4^T load leading

$M_c(4^T \text{ at } C) = 4 \times 5 + 2 \times 3 = 26$ tons-ft.
$M_c(2^T \text{ at } C) = 2 \times 5 + 4 \times 3 = 22$ „

therefore, max. when 4^T at centre,

$$\left.\begin{aligned} \text{i.e. } \Sigma W - \Sigma W^b &= +6 - 0 \\ \Sigma W - \Sigma W^b &= 2 - 4 = -2 \end{aligned}\right\} \text{using rule given}$$

(b) 2^T load leading

M_c (2^T at C 4^T at 6) $= 2 \times 5 + 4 \times 3 = 22$ tons-ft.
(2^T at 14 4^T at C) $= 2 \times 3 + 4 \times 5 = 26$ „

In both cases a maximum when 4^T at the centre.

Note. The greatest moment occurring under W_2 will be when W_2 is at a distance of $\frac{d}{2}\left(1 - \frac{W_2}{W_1 + W_2}\right) = \frac{d}{2}\left(\frac{W_1}{W_1 + W_2}\right)$ from the centre.

Its value will be $\frac{W_1 + W_2}{4l}\left(l - \frac{W_1 d}{W_1 + W_2}\right)^2$

(B) Loads of 2^T, 2^T and 4^T at 4 ft. centres cross the beam.

Find the maximum moment at the centre and the load under which it occurs.

M_c (4 tons at centre) $= 4 \times 5 + 2 \times 3 + 2 \times 1 = 28$ tons-ft.

$M_c =$ (Mid 2^T at centre) $= 4 \times 3 + 2 \times 5 + 2 \times 3 = 28$,,

$M_c =$ (Last 2^T at centre) $= 4 \times 1 + 2 \times 3 + 2 \times 5 = 20$,,

Conditions for maximum,

$$b\Sigma W - a\Sigma W^b \text{ to change sign.}$$

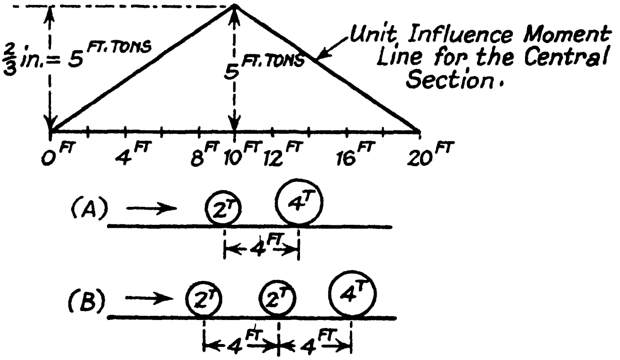


FIG. 135

4^T at centre $8 - 0 = 8 +$

Mid 2^T ,, $4 - 4 = 0$

Last 2^T ,, $2 - 6 = -4$

A maximum moment for the centre when either 4^T at C or middle 2^T at C.

In the previous problem for the 2-ton plus 4-ton loads crossing the span, find the section having the greatest moment and also the value of the greatest moment.

(a) The 2-ton and 4-ton loads only. The C. of G. of the two loads is at

$$\frac{4}{6} \times 2 = \frac{4}{3} \text{ ft. from the 4-ton load.}$$

The section having the greatest moment will be at

$$10 + \frac{4}{3 \times 2} = 10\tfrac{2}{3} \text{ ft. from the left support,}$$

with the 4^T load leading from left to right

or ,, 2^T ,, right to left

(If the loads are reversible then with the 4^T load leading from right to left, the section of greatest moment will be at 9⅓ ft. from the left support.)

$$\text{Max. moment} = -\frac{6}{80}\left(20 - \frac{2}{6} \times 4\right)^2$$

$$= -\frac{6}{80} \times \frac{56^2}{9} = 26{\cdot}13 \text{ tons-ft.}$$

118. **Simple Beam with a Uniformly-distributed Load Moving on the Beam.** (Fig. 136.) Let the length of the load of w tons per ft.-run be l_1 and let l_1 be less than l the length of the span.

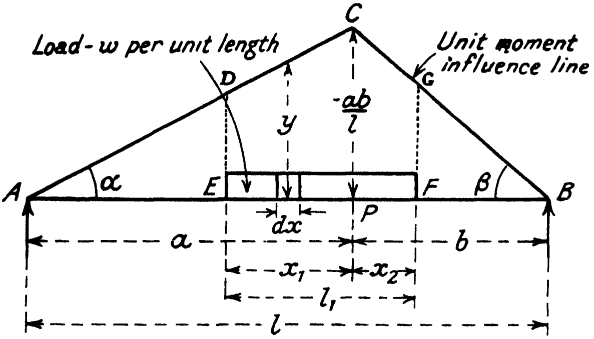


FIG. 136

The unit influence moment line for a section P with a maximum ordinate $= -\frac{ab}{l}$ is shown in Fig. 136.

The load on an elemental length $dx = w \,.\, dx$, so that moment at P due to $w \,.\, dx = dM_P = w \,.\, y \,.\, dx$, as y is the ordinate of the unit diagram for $w \,.\, dx$. Now $y \,.\, dx$ is the area of the unit influence diagram above dx; hence total $M_P = w\Sigma y \,.\, dx$ $= w \times$ area of unit influence diagram above the length of the uniformly-distributed moving load, i.e.

$$M_P = w \times \text{area } CDEFG \text{ to scale}$$

118*a*. **Maximum Moment.** (*a*) For a rolling load as long as, or longer than, the length of the span, the maximum moment for any section will obviously occur when the whole span is covered.

(*b*) For a rolling load of length l_1 less than the length l of the span.

$$M_P = w \times \text{area } CDEFG\text{: for a maximum } \frac{dM_P}{dx} = 0$$

Accordingly M_P must be expressed in such a form that it can be differentiated—

$$CP = -\frac{ab}{l}; \ \tan \alpha = \frac{ab}{l} \times \frac{1}{a} = \frac{b}{l}$$

$$\tan \beta = \frac{a}{l}$$

$$x_1 = EP, \ x_2 = PF$$

$$AE = (a - x_1), \ FB = (b - x_2)$$

Hence $ED = AE \tan \alpha = (a - x_1)\frac{b}{l}, \ FG = (b - x_2)\frac{a}{l}$

Then $M_P = w[\text{area } CDEP + \text{area } CPFG]$

$$= w\left[x_1\frac{(ED + CP)}{2} + x_2\left(\frac{CP + FG}{2}\right)\right]$$

$$= w\left[\frac{x_1}{2}(a - x_1)\frac{b}{l} + \frac{ab}{l}\right] + \frac{x_2}{2}\left[(b - x_2)\frac{a}{l} + \frac{ab}{l}\right]$$

Substituting $x_2 = (l_1 - x_1)$, it can be shown that

$$M_P = \frac{w}{l}\left[abl_1 - \frac{bx_1^2}{2} - \frac{a(l_1 - x_1)^2}{2}\right]$$

$$\frac{dM_P}{dx_1} = -\left(\frac{w}{l}\right)\frac{2bx_1}{2} + \left(\frac{w}{l}\right)\frac{2a(l_1 - x_1)}{2} = 0$$

Hence M_P is a maximum when $bx_1 = a(l_1 - x_1) = ax_2$

$$\text{i.e. when } \frac{x_1}{x_2} = \frac{a}{b}$$

so that the maximum moment occurs at the section, when the section point divides the load in the same ratio as it divides the span.

119. **Equivalent Uniformly-distributed Load for Moments—For Concentrated Loads Moving Over a Span.** For a single concentrated load moving over a span, the maximum bending moment at any section occurs when the load is at that section, and this maximum moment is equal to $\frac{Wx(l - x)}{l}$

where W tons is the moving concentrated load and x is the distance of the section from the origin ;

$$\text{that is, } M_{max} \text{ for any section} = \frac{W}{l}\left[xl - x^2\right]$$

This is an equation where M_{max} depends on x^2 ; and a curve plotting M_{max} as ordinates against x as abscissae will be a parabola (as in Fig. 137).

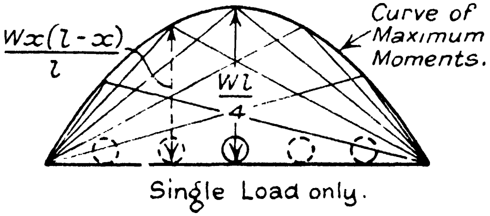


FIG. 137

The maximum ordinate will be at the centre of the span

$$= \frac{Wl}{4}$$

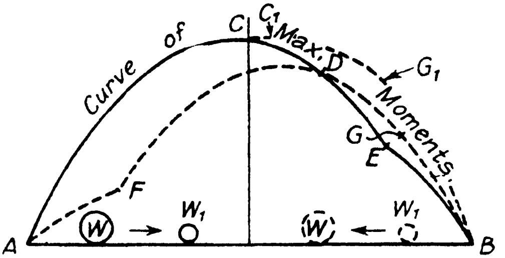
it is also equal to the ordinate of a moment parabola for an equivalent uniformly-distributed load w_e, thus,

$$\frac{w_e l^2}{8} = \frac{Wl}{4} \quad \text{i.e.} \quad w_e = \frac{2W}{l}$$

w_e is the equivalent uniformly-distributed load for W. A maximum moment for a uniformly-distributed load always occurs for any section when the beam is totally covered. For any system of concentrated rolling loads, find the maximum moment for various positions of the loads on the beam. Draw (a) a polygon on the length of the beam as base to enclose the maximum moments for different load positions when all or the greater part of the loads are on the beams.

Then to find the equivalent uniformly-distributed load w_e, take the maximum height (at the centre) in moment units of the enclosing parabola and equate to $\frac{w_e l^2}{8}$

Draw (*b*), a polygon to enclose all the maximum moments and to include cases when only a few of the loads are on the beams, i.e. near to the end supports: then the maximum ordinate of the enclosing parabola in this case will in most cases be greater than that of the case (*a*). Equate this maximum moment ordinate to $\frac{w_{e1}l^2}{8}$ *



Loads of unequal Magnitude.

FIG. 138

The student is requested to find the equivalent uniformly-distributed load for various lengths of beams using the loads given in the loading diagram (Fig. 129).

120. EXAMPLE.

Two rolling concentrated loads crossing a span. To draw the curve of maximum moments (Fig. 138)--

(*a*) Construct the unit moment influence line for a number of sections. Find the moment for these sections with the first load leading, say, from left to right and with the first load at the section. Construct a curve having as ordinates the moment at the section when the load is at that section. Now let the loads move from right to left with the other or second load leading. Construct a curve having as ordinate at a section the moment at that section with the second load at the section.

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* *Examples given in*—

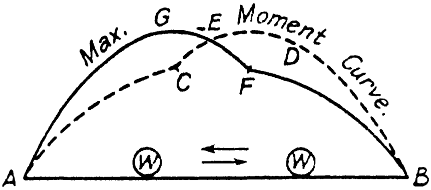
(*a*) Arrol's *Bridge and Structural Engineers' Handbook*, by Adam Hunter, M.I.C.E. (Spon.) In paragraph on Moving loads—graphical construction.

(*b*) *Structural Engineering*, by J. Husband and W. Harby. (Longmans, Green & Co.)

Also consult Papers by Professor F. C. Lea on "Influence Lines" in the *Proceedings Institute of Civil Engineers*. (*See* references at the end of the chapter.)

Now draw an enveloping parabola to enclose the maximum moments which occur at the sections with either of the loads at that section. This curve will be of the parabolic form : take its maximum ordinate and equate to $\frac{w_e l^2}{8}$ where w_e is the equivalent uniformly-distributed load.

For two loads of equal intensity, the curve of maximum moment will be as shown in curve (*AGFB*), Fig. 139. Again, draw the enveloping parabola to enclose the curve of maximum moments for the sections.



Loads of Same Value.

FIG. 139

NOTE ON FIG. 138. *AFDGB* is the curve of moments for the sections when the smaller load is at the section and leading from left to right. *ACDEB* is the curve of moments when the larger load is at the section and going from right to left. Then, obviously, *ACDGB* is the curve of maximum moments for all sections. If the loads were reversible, then the curve of maximum moments would have to be symmetrical about a vertical line through the centre of the beam, and would be ACC_1G_1B shown dotted in Fig. 138.

NOTE ON FIG. 139. *AGEDB* is the curve of maximum moments for all sections. It is a special case of the preceding one.

121. **Unit Influence Line of Shear Force for any Section of a Simple Beam.** The shear force at any section of a beam is the algebraic sum of the external forces to the right or left of the section.

Forces acting upwards and downwards to the left of a section are negative and positive respectively ; forces acting upwards and downwards to the right of a section are respectively positive and negative.

Let *AB* (Fig. 140) be a simple beam ; it is required to construct the influence line of shear for any section *P* when unit load crosses the beam.

With the unit load between *A* and *P*, the shear at *P* is equal to $+ R_B$;

With the load between *P* and *B*, the shear at *P* is equal to $- R_A$, because in their respective cases they are the only forces to the right or left of the section.

With 1 ton at A, $R_A = -1$ ton $\quad R_B = 0$

With 1 ton at B, $R_B = +1$ ton $\quad R_A = 0$

At A and B, erect ordinates $AD \downarrow = -1$ ton and $BC \uparrow = +1$ ton to scale.

Join A to C, and B to D.

With unit load at any section X between A and P, and distant x from A, y is the ordinate of the diagram ABC.

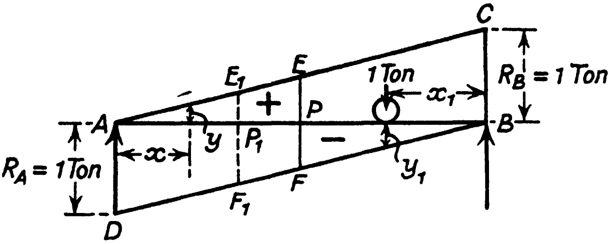


FIG. 140

Then $\dfrac{y}{x} = \dfrac{1}{l}$

i.e. $\quad y = \dfrac{x}{l} = R_B$ when the load is at X.

And similarly for any section between A and P. Therefore, AEP is the unit influence diagram and positive for the shear at P when the load is between A and P.

Let the load of 1 ton just move to the right of P, then the single force to the left of P is $R_A \uparrow$ and of negative sense.

Let the load be at any section X_1 distant x_1 from B and between P and B, and let y_1 be the ordinate to the R_A diagram ADB,

then $\dfrac{y_1}{x_1} = -\dfrac{1}{l} : \quad y_1 = -\dfrac{x_1}{l}$

$y_1 = -R_A$ with the load of 1 ton at X_1;

therefore, PFB is the unit influence diagram and negative for shear at P when the load is between P and B. The complete unit influence shear diagram for P is $AEPFB$ when unit load crosses the span. Similarly for any other section P_1 the unit influence shear diagram is $AE_1\,P_1\,F_1\,B$. If a load of W tons crosses the span, construct the unit diagram as shown for the particular section, and multiply the ordinates of this diagram by W.

122. If a number of loads cross the span, then the shear at a particular section will be equal to

$$\Sigma W_1 y_1 - \Sigma W_2 y_2 \quad . \quad . \quad . \quad . \qquad (3)$$

where W_1 and y_1 represent loads giving positive shear at the sections and ordinates of the positive unit shear diagram respectively, and W_2, y_2 representing loads giving negative shear and negative ordinates respectively.

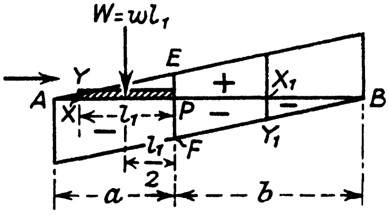


FIG. 141

The sign of the shear force will depend on the magnitudes of the quantities in Equation (3).

123. **The Position of the Loads to Give Maximum Shear at Any Point** is found by trial; that is, take different positions of the loads, and the maximum shear is easily ascertained by finding the largest sum total of shear; it occurs with a load at the section, and usually one of the largest.

124. **The Shear at a Section when a Uniformly-distributed Load Moves Over a Beam** (Fig. 141). Construct the unit influence shear diagram for the section. As in the case of the moment diagram, the shear at the section $= w \,.\, dx \,.\, y = w \,.\, dA$

$$dA = \text{small area of the unit diagram};$$

therefore, the area of the unit shear force diagram between the right limits × w gives the shear at a section when a uniformly-distributed load crosses the beam.

125. **Case I. Length of Rolling Load Greater than the Span.** Let the load cross in the direction A to B. It is obvious that the shear of positive sense will be a maximum when the portion AP of the beam only is covered (Fig. 141), and that the shear at P of negative sense will be a maximum when only the portion PB of the beam is covered. If AP and part of PB is covered, then the shear at P will be

+ ($w \times$ area AEP) − ($w \times$ part area of PFB). The sign of the resultant shear will depend on the magnitudes of the areas.

Case II. Length of Rolling Load Less than the Span.

Let $AP = a$; $PB = b$, and $a < b$.

Let the length of the moving load be $l_1 < a$, and moving from left to right.

Then maximum positive shear at P is when the beginning of the load is at P, and maximum negative shear at P_1 when the end of the load is at P. Also vice versa for the load moving from right to left.

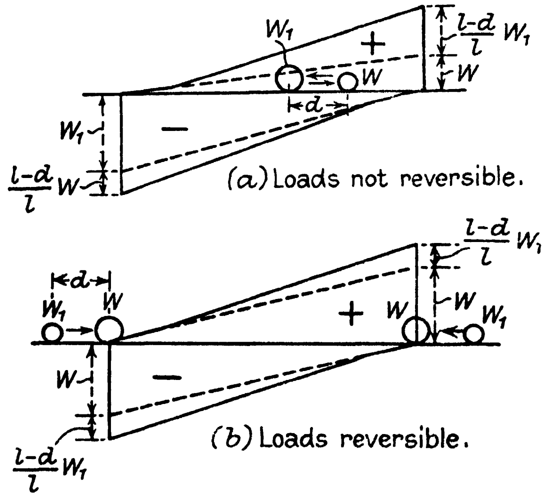


Fig. 142

Referring to Fig. 141, the maximum positive shear at P is

$$= w \times \text{Area } PXYE$$

$$= w\left[\frac{l_1(PE + XY)}{2}\right]$$

126. **Maximum Shears** (Fig. 142). (1) For a single load moving over a bridge, the maximum negative or positive shear for a section occurs when the load is at that section. For two loads rolling over a bridge, the maximum positive or negative shear can be ascertained by the use of the unit influence shear

diagram for the section : it will occur when one of the loads is at the section. A diagram can be plotted giving the maximum shear at the section. (*See* Fig. 142.)

(2) For a uniformly-distributed rolling load over the span and longer than the span, the maximum negative or positive shear occurs at the span, when either one or other of the portions of the beam made by the section are wholly covered. It has been shown that the shear is the corresponding area of the unit influence shear diagram to scale. In this case, therefore, maximum shear is a function of x^2, and consequently the curve of maximum shears plotted against x is a semi-parabola as illustrated in Fig. 143.

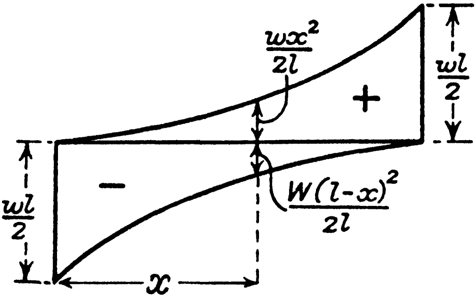


FIG. 143

127. **Equivalent Uniformly-distributed Load for Shear for any Section.** As discussed previously, find the maximum shear at a section by placing the loads at different positions on the beam with a load at the section. From the magnitudes of the loads and the corresponding ordinates of the unit influence shear diagram for the section, the maximum shear at the section can be found.

Let $-S_X$ be the maximum negative shear for a section X, then the negative portion of the unit influence shear diagram must be covered by a uniformly-distributed load of value w_e to cause the shear $-S_X$. (*See* Fig. 144.)

$$\text{i.e.}\quad -S_{x\ max} = \left(\frac{l-x}{l}\right)\left(\frac{l-x}{2}\right)w_e = w_e\frac{(l-x)^2}{2l}$$

$$w_e = -S_{x\ max}\cdot\frac{2l}{(l-x)^2}$$

Similarly for maximum positive shear,

$$w_e = +S_{x\ max} \cdot \frac{2l}{x^2}$$

The maximum possible shear obviously occurs at the ends of the structure, and, therefore, the maximum equivalent uniformly-distributed load will occur for end shear.

128. **Influence Diagrams for Perfect Frames of the Warren and N-girder Types which have Parallel Flanges.** Influence diagrams are specially applicable to frames, as the loads are generally applied at the joints; the stresses in the members are ascertained by a consideration of moments and shear forces.

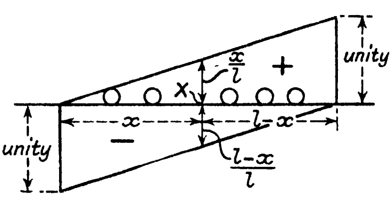


Fig. 144

Influence Lines for a Warren Girder, of "Through" Type, single members as shown in Fig. 145. Let a unit load = 1 ton cross from A to B.

(1) To ascertain the stress in a *member* of the *top* flange, say CE, take a section line (1) – (1); then the force in CE is found by taking moments about the bottom joint D, i.e.

$$\text{Force in } CE \times p = \text{moment at } D = M_D$$

Therefore it is required to find the influence moment diagram for a joint in the lower boom. Take R on side of section line away from the load, then with unit load between A and D,

$$M_D = R_B \, . \, BD$$

also with unit load between D and B,

$$M_D = R_A \, . \, AD$$

R_A and R_B are found in the usual way by the method of moments.

The influence moment diagrams for the joints on the bottom flange are the same as those for the different sections of a simply supported beam. The ordinate at D of the unit influence moment diagram for D is $\frac{AD \, . \, BD}{AB}$

The maximum stress in $CE = M_D(\text{Max})/p$

(2) The stress or force in a *member of the bottom boom* such as DF is found by taking moments about a joint (here E) in the top flange. (*See* Figs. 145 and 146.)

Force in $DF \times p = M_E$.

$\therefore$ Force in $DF = M_E/p$, so that the maximum force in DF depends on the maximum moment at E.

(*a*) With unit load between F and B, distant x_b from B,

$$M_E = R_A \,.\, AE_1 = R_A l_1 ;\; R_A = \frac{x_b}{l}$$

so that $M_E = \frac{x_b l_1}{l}$: this is a linear equation in x (each term being of the first degree, it is the equation of a straight line).

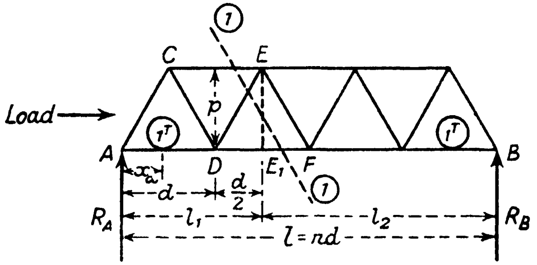


FIG. 145

Let d = length of one bay, then $BF = n_1 d$ and $l = nd$, where n = number of bays in the bottom boom and n_1 = number of bays from F to B.

When the 1 ton load is at F, $x_b = BF = n_1 d$.

$$\therefore\; M_E = \frac{n_1 \,.\, d \,.\, l_1}{nd} = \frac{n_1 l_1}{n}$$

At F erect an ordinate $FG = \frac{n_1 l_1}{n}$ tons-ft. to scale.

When the 1 ton load is at B, $x_b = 0$ and $M_F = 0$. Join G to B, then GB is a portion of the unit influence moment line for joint E.

(*b*) With the unit load between A and D, distant x_a from A (*see* Fig. 145)

$$M_E = R_B \,.\, BE = R_B l_2 ;\; R_B = x_a/l \text{ and } M_E = \frac{x_a \,.\, l_2}{l}$$

which is again the equation of a straight line.

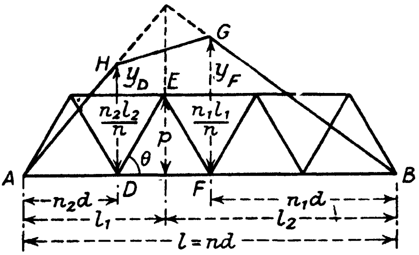
When $x_a = 0$; $M_E = 0$.

When $x_a = n_2 d$ (where n_2 = number of bays from A to D)

$$M_E = \frac{n_2 d \,.\, l_2}{nd} = \frac{n_2 l_2}{n}$$

At D erect an ordinate DH to scale $= n_2 l_2/n$ tons-ft. Join H to A, then AH is a portion of the unit influence moment line for joint E with the load of 1 ton between A and D.

(*c*) To find the Influence Line for Moment (or Shear) for a section in the bay, when the load is in the bay and transferred to the main girder at the bay points by cross girders.



AHGB is the unit influence moment line for the joint E.

FIG. 146

In Figs. 145 and 146, when the load is in the bay DF, the load will be applied at points D and F. It is required to construct the influence line for moment (or shear) for any section X_1 in bay DF. Let y_D = ordinate of unit influence moment (or shear) diagram for section X_1 when unit load is at D; it has been shown that $y_D = n_2 l_2/n$; let y_F = ordinate of same diagram for unit load at F, then $y_F = n_1 l_1/n$. Let the 1-ton load be between D and F, distant x from D, and let $DF = d$ = length of bay.

The load transmitted at F

$$= \text{Reaction at } F = R_F = x/d$$

also the load transmitted at D

$$= \text{Reaction at } D = R_D = (d - x)/d$$

Let the moment (or shear) at X_1 be represented by y_{x1}.

Then $$y_{x_1} = \frac{x}{d}(y_F) + \left(\frac{d-x}{d}\right)y_D = y_D + \frac{x}{d}(y_F - y_D)$$

as y_D and y_F are ordinates of the unit influence diagram for X_1.

(Proof: If 1 ton is at D, $x = 0$ and hence $y_{x_1} = y_D$, and if 1 ton is at F, $x = d$ and $y_{x_1} = y_D + \frac{d}{d}(y_F - y_D) = y_F$.)

Accordingly, the influence line for a section in the bay, with the load in the bay DF, is found by joining the tops of ordinates y_D and y_F. The complete unit influence moment line for joint E is thus $AHGB$. Force in $DF = M_E/p$.

129. **Maximum Moment About a Joint.** For a single load crossing the frame, the maximum moment will occur when the load is at the maximum ordinate of the unit influence moment diagram. With a uniformly-distributed load of length longer than the span, the moment is a maximum when the whole frame is covered; for a load of length less than the span, by trial the position of the load to give the maximum area of moment diagram above it can be readily ascertained. For a joint in the bottom flange of a Warren girder, the position of the load will be the same as that obtained by using the form in paragraph 118 for simple beams. The rule for maximum moment for lower joints due to irregular loading is the same as for simple beams.

Maximum moment due to a number of irregular loads for the upper joints of a Warren girder. Place the loads on the span so that one load is at F, Fig. 146; find the moment for this case. Move the loads along a little to the right and re-work; by trial, the position of the loads for maximum moment can be found. The equivalent, uniformly distributed, can be found for both cases in the same way as for a simple beam.

130. **Stresses in the Diagonal Members.** The stresses in the diagonal members depend on the shear force in the particular bay. For example, consider the bay DF (Fig. 147). To find the stresses in the diagonals DE and EF. With the load of 1 ton between F and B, the only force to the left of F is R_A: so the shear force in the bay DF is $-R_A$. With the load of 1 ton between A and D, the only force to the right of D is R_B and the shear force in the bay DF is then $+R_B$.

$$AD = n_2 \text{ bays}$$
$$FB = n_1 \quad ,,$$
$$\text{Length of bay} = d$$

With the load at F, $R_A = -\frac{1}{l} \times n_1 d$ tons

,, ,, D, $R_B = +\frac{1}{l} \times n_2 d$ tons

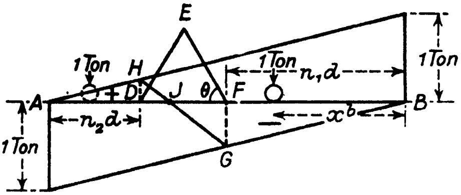


FIG. 147

At D and F, erect ordinates to scale.

$$DH = +\frac{n_2 d}{l} \; ; FG = -\frac{n_1 d}{l}$$

Join A to H and B to G, then AH and GB are portions of the unit shear influence line for the bay DF.

Join H to G, and by the previous theory in paragraph 128 (*c*), HG is the influence line for the load between D and F, that is, when it is in the bay.

HG cuts the bay DF in the point J.

$$\text{The force in } DE = \frac{\text{shear force in bay } DF}{\sin \theta}$$

$$\text{The force in } EF = \frac{\text{shear force in bay } DF}{\sin \theta}$$

If shear in bay DF $\pm$, force in DE $\pm$, force in EF $\mp$ where tensile force $+$ and compressive force $-$.

Thus, the kind of force in the diagonals varies according to the position of the load; that is, negative shear in a bay, the left-hand diagonal is a strut, the right hand a tie, and vice versa for positive shear.

The maximum load in the diagonals will depend on the maximum shear in the bay. For a single concentrated load, the maximum shears will occur at the bay limits D and F. For a uniformly-distributed load longer than the span l, the maximum positive shear in the bay DF will occur when the portion AJ of the span is covered, and maximum negative when the portion JB is covered. For a uniformly-distributed load less than the length of the span l, and moving from left to right maximum positive shear occurs when the front of the load is at J, and maximum negative shear when the end of it is at J.

131. Maximum Shear in a Bay Due to a Uniformly-distributed Load Longer than the Span. (*See* Fig. 147.)

By similar triangles

$$\frac{JF}{JD} = \frac{FG}{HD} = \frac{y_F}{y_D} = \frac{n_1 d}{n_2 d}$$

so that
$$\frac{JF}{JD + JF} = \frac{n_1 d}{n_1 d + n_2 d}.$$

But $(n_1 d + n_2 d) = (l - d)$; and if number of bays $= n$, $l = nd$, so that the expression $\frac{n_1 d}{(l - d)} = \frac{n_1}{(n - 1)}$.

Also $JD + JF = d$; hence $\frac{JF}{d} = \frac{n_1}{(n-1)}$

$$\therefore \quad JF = \left(\frac{n_1}{n-1}\right)d.$$

It has been indicated that the maximum negative shear occurs when the length $JB = FB + JF$ is covered by a uniformly-distributed load; so that maximum negative shear occurs for a continuous loading in a bay, e.g. the $(n_1 + 1)^{th}$ bay from the right-hand support when a length $d\left(n_1 + \frac{n_1}{n-1}\right)$ is covered, i.e. when $\left(n_1 + \frac{n_1}{n-1}\right)$ bays are covered by the load.

Similarly for maximum positive shear in the same bay: working from the left-hand support, a length of

$$d\left(n_2 + \frac{n_2}{n-1}\right) = \left(n_2 + \frac{n_2}{n-1}\right)$$

bays must be covered to give maximum positive shear.

132. **Maximum Shear in a Bay Due to Irregular Loads.** Consider bay *DF* in Fig. 147. The loads move from left to right.

(*a*) Let W_L be the resultant of all the loads in the frame to the left of the bay, W_B the resultant of all the loads in the bay, and W_R the resultant of all the loads to the right of the bay.

Let y_L, y_B, y_R be the ordinates of the unit influence shear diagram at the positions of W_L, W_B, and W_R respectively.

Then the resulting shear $= (W_L y_L + W_B y_B + W_R y_R)$, regard being paid to the signs.

For this position of the loads, this equation gives the total shear.

(*b*) Move on the loads until the first heavy load passes *D* and comes into W_B: find the resulting shear for this new position of the loads.

(*c*) Continue thus until the maximum is found. Professors Lea and Andrews have shown that in the limit and when the loads are very close together

$$W_B = \frac{W_L + W_B + W_R}{n}$$

i.e. the maximum shear in any bay occurs when W_B, the sum of the loads in the bay, is equal to the total load $(W_L + W_B + W_R)$ divided by the number of bays. As a general rule, the maximum shear occurs in a bay when the first heavy load passes the general point *D*. (*See* Fig. 147.)

133. **Framed Girders with Vertical Posts of the N or Pratt Type.** (Fig. 148.) The moment influence lines for any joint will be the same as for the lower flange joints of the Warren girder; and for shear in any bay, the influence line will be the same as for a bay of the Warren girder. The maximum load in an upright member will be equal to the maximum shear force; the maximum load in a diagonal will be equal to

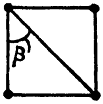


FIG. 148

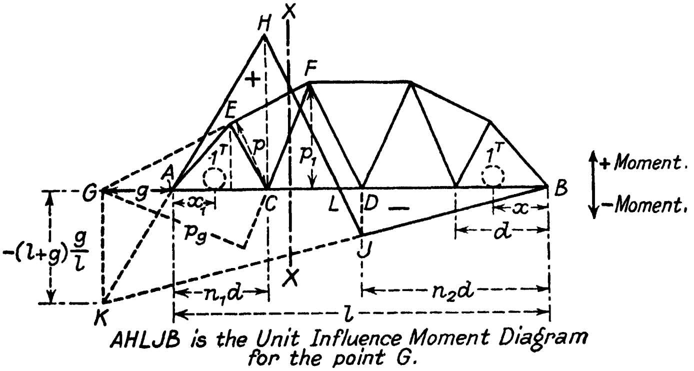
$$\frac{\text{Maximum shear force}}{\cos\beta}$$ where β = angle diagonal makes with the vertical

For shear in the bay $\mp$, load in the left-hand vertical $\mp$, load in the diagonal $\pm$ where compressive forces $-$, and tensile forces $+$.

134. Frames or Trusses with Curved Flanges. (Fig. 149.) The frame in Fig. 149 is a curved flanged frame ; it is required to find the forces in the members *EF*, *CD*, and *FC* as being typical of members in the top and bottom flanges and the diagonals.

$$\text{Force in } CD = \frac{\text{moment at } F}{p_1} = \frac{M_F}{p_1}$$

$$\text{Force in } EF = \frac{\text{moment at } C}{p} = \frac{M_C}{p}$$



AHLJB is the Unit Influence Moment Diagram for the point G.

FIG. 149

The moment influence lines for *C* and *F* are as for the bottom and top joints of the Warren girder considered previously.

FORCE IN *FC*. Produce *FE* to cut *BA* produced in *G*. From *G* drop a perpendicular to *FC* produced

$$\text{Then the force in } FC = \frac{\text{moment about } G}{p_g} = \frac{M_G}{p_g}$$

(1) Referring to Fig. 149, let a unit load be between *D* and *B* distant *x* from *B*. Let *XX* be a dividing line cutting the three members *EF*, *FC*, and *CD*. Then the left-hand half of the girder is in equilibrium under the external loads, including the reaction and the forces in the members *EF*, *FC*, and *CD* acting as external forces. Let the unit load be between *D* and *B* distant *x* from *B* ; then the only external load to the left of *XX* is $R_A = \dfrac{1 \times x}{l}$

Moments about G,

$$M_G = -\frac{x}{l} \times AG = -\frac{x}{l} \times g, \text{ negative}$$

as it will cause a compressive stress in FC. This is a linear equation in x, and, therefore, the unit influence moment line for unit load between D and B is a straight line BJ,

$$\text{where } JD = -\frac{BDg}{l} = -\frac{n_2dg}{l} = -\frac{n_2g}{n}$$

$n_2 =$ the number of bays in BD,

$d =$ length of one bay,

$n =$ total number of bays.

Let the unit load be between A and C, distant x_1 from A. Considering the portion of the frame to the right of XX,

the only external load will be $R_B = \frac{x_1}{l}$ tons.

Moment about G,

$$M_G = R_B(AB + AG) = R_B(l + g)$$

$$= \frac{x_1}{l}(l + g) \text{ and is positive,}$$

as it will cause a tensile stress in FC,

$$M_G = x_1\left(1 + \frac{g}{l}\right)$$

This is again a linear relation in x_1, so that the unit influence line when the load is between A and C is a straight line AH, the ordinate CH being equal to

$$AC\left(1 + \frac{g}{l}\right) = n_1d\left(1 + \frac{g}{l}\right) \text{ where } n_1 = \text{number of bays in } AC.$$

The unit influence moment line for the load crossing the bay is found by joining H to J.

HA and BJ produced meet in one point K, which is vertically below G. Join K to G.

KG, considered as an ordinate of BJ, produced

$$= -\frac{(l + g)g}{l} \text{ where } x = l + g$$

KG, considered as an ordinate of HA produced,

$$= -\frac{(l+g)g}{l} \text{ where } x_1 = -g$$

Therefore, K is vertically below G.

Thus, to construct this particular influence line of moment, set down from G an ordinate GK to scale

$$= -\frac{(l+g)g}{l}$$

Join KA and produce to meet a vertical through C, to give the ordinate $+CH$.

Join K to B; from D drop an ordinate DJ to meet KB in J. Join H to J. The unit influence moment line for the point G will then be $AHJB$.

MAXIMUM LOAD IN CF. For a single isolated load, this will occur when it is situated at the maximum ordinate of the unit diagram, that is, at the bay points, to give a maximum tensile or compressive force; at D a maximum compressive force will occur; at C a maximum tensile force.

For a uniformly-distributed load of length greater than the span, the maximum occurs when AL or LB is covered: for length of load less than the span and moving from left to right, maximum positive moment occurs when the front of the load is at the point L, and maximum negative moment when the rear of the load is at the point L in the bay CD.

Tests for maximum moments about C and F are as previously determined for parallel flange trusses and simple beams.

135. In dealing with the forces in the members of the flanges, care must be taken that the kind of force in the member is correctly determined. This can easily be ascertained by the consideration of the turning direction of the moments of the known external loads about a point. A force in a member acting towards a joint indicates a compressive force, and a force acting from a joint indicates a tensile force.

REFERENCES TO PAPERS AND WORKS ON INFLUENCE LINES

Proceedings of the Institution of Civil Engineers, Prof. F. C. Lea; Vol. clxxxv, p. 288; Vol. clxxxv, p. 277—"Influence Lines for Continuous Girders and Lattice Girders" Vol. clxi, p. 284, "Notes on Equivalent Uniformly-distributed Loads."

Higher Problems in the Theory and Design of Structures, Ewart S. Andrews. (Chapman & Hall.) Influence lines for beams and frames.

Influence Lines, Sprague. (Scott, Greenwood & Son.) Including also notes on impact.

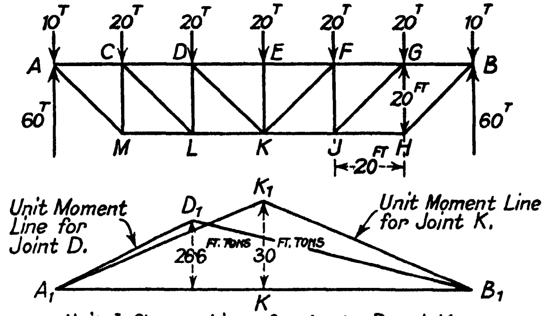
Theory of Structures, Timoshenko and Young.

Illustrative Problem 35.

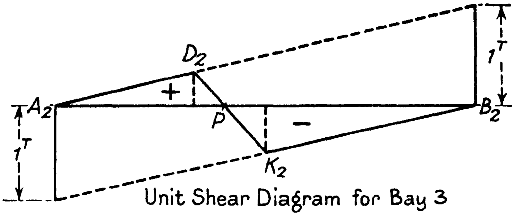
A symmetrical N-type deck bridge has six equal panels and spans 120 ft. The depth of each main girder is 20 ft. If on each truss there acts a uniform dead load of 1 ton per foot run and a uniform moving load of 1 ton per foot run, determine the forces for which the members *DE*, *DK*, *LK* should be designed. Assume the dead load is concentrated at the top chord panel joints. (U. of L.)

Considering the forces in the members due to dead load only. Referring to Fig. 150,

$$R_A = 60 \text{ tons} = R_B.$$



Unit Influence Lines for Joints *D* and *K*.



Unit Shear Diagram for Bay 3

FIG. 150

Force in *LK*, considering the equilibrium of the portion of the girder to the left of the dividing line and taking moments about *D*.

$$(\text{Force in } LK) \times (-20) = 20 \times 20 + 10 \times 40 - 60 \times 40$$

(Force in *LK*) = 80 tons and is positive, giving a tensile force.

For the force in *DE*,

Let a dividing line cut *DE*, *KE*, and *KJ*.

Taking moments about *K*,

$$(\text{Force in } DE) \times 20 = 400 + 800 + 600 - 3600$$

(Force in *DE*) = 90 tons, and is compressive.

The shear in bay 3 = $-60+50=-10$ tons.

Considering the forces to the left of the bay, *EK* takes this, and, therefore, the force in the diagonal *DK* will be

$$+\frac{10}{\cos 45^\circ}=10\sqrt{2}=14{\cdot}14 \text{ tons, and is tensile.}$$

Maximum forces in the members due to the live load of 1 ton per foot run.

The maximum ordinate of the unit influence moment line for joint *D* will be

$$-\frac{40\times 80}{120}\text{ tons-ft.}=-26{\cdot}6\text{ tons-ft.}$$

The maximum moment at *D* will, therefore, be

$$1\times\frac{-26{\cdot}6}{2}\times 120=-1600\text{ tons-ft.}$$

Force in *LK* due to live load $=\dfrac{-1600}{-20}=80$ tons tensile force.

The maximum ordinate of the unit influence line for the joint *K* is $-\dfrac{60\times 60}{120}=-30$ tons-ft.

Maximum moment at *K* will, therefore, be

$$-\frac{30}{2}\times 120\times 1=-1800\text{ tons-ft.}$$

Force in *DE* due to the live load,

$$=\frac{-1800}{20}=-90\text{ tons, and is compressive.}$$

The maximum positive shear in the third bay from the left-hand support will be (*see* paragraph 131) when $\left(2+\frac{2}{5}\right)$ bays from the left-hand support are covered: maximum negative shear when $\left(3+\frac{3}{5}\right)$ bays from the right-hand support are covered.

Maximum positive shear in bay 3 therefore

$$=1\times\frac{1}{3}\times\left(2+\frac{2}{5}\right)\frac{20}{2}=+8\text{ tons.}$$

[$\frac{1}{3}$ = ordinate unit load influence line with 1 ton at *D*.]

Maximum negative shear in bay 3,

$$= 1 \times \frac{1}{2} \times \left(3 + \frac{3}{5}\right)\frac{20}{2} = -\ 18 \text{ tons.}$$

[$\frac{1}{2}$ = ordinate unit load influence line 1 ton at E.]

$$\begin{array}{l}\text{Max. compressive force in } DK \\ \quad\text{due to the live load}\end{array} = -\frac{8}{\cos 45} = -\ 11{\cdot}3^{\text{T}}$$

Maximum negative shear causes a tensile force in DK.

$$\begin{array}{l}\text{Max. tensile force in } DK \\ \quad\text{due to live load}\end{array} = \frac{18}{\cos 45^\circ} = 11{\cdot}3 \times 2{\cdot}25$$

$$= 25{\cdot}43 \text{ tons.}$$

The members, therefore, should be designed to carry the following loads.

$$\left.\begin{array}{ll} LK,\ +\ 80 + \ 80 & = +\ 160 \text{ tons} \\ DE,\ -\ 90 - 90 & = -\ 180 \text{ tons} \\ DK,\ +\ 14{\cdot}14 + 25{\cdot}4 & = +\ 39{\cdot}54 \text{ tons} \end{array}\right\}$$

Minimum tensile stress in $DK = 14{\cdot}14 - 11{\cdot}3 = 2{\cdot}84$ tons.

If the resultant load in a diagonal is changed from a tensile force to a thrust, then the bay can be counterbraced by a member along the other diagonal to take a tensile force, assuming the first diagonal cannot take thrust loads.

REFERENCES

Design of Modern Steel Structures, L. E. Grinter. (Macmillan.)
Kinetic Theory of Engineering Structures, Molitor. (McGraw-Hill.)
Code of Practice for Simply Supported Steel Bridges. (Institutions of Civil and Structural Engineers.)
See also those listed at the end of Chapter VII, page 175.

EXAMPLES

1. A girder 40 ft. long is supported at its ends, and two isolated loads of 10 tons and 4 tons travel slowly across it from opposite ends at the same speed, passing each other at the centre of the span. Draw the diagram of maximum bending moment and shearing force for the girder. (U. of L.)

2. A load of w lb. per foot run moves from left to right over a Pratt girder which is divided into N panels of length l. Show that the maximum shear S at the nth panel point from the left is given by the equation—

$$S = \frac{wn^2l}{2(N-1)}$$ (U. of B.)

3. A Pratt truss, 104 ft. span, depth 13 ft., is divided into 8 equal panels. It carries a uniform dead load of 0·6 ton per foot run and a uniform live load of 1½ tons per foot run, both supported at the joints of the bottom chord. Determine the maximum stresses in all the members of the third and fourth

panels from one support, and state whether these panels require to be counterbraced.

4. A Howe trussed girder of 120 ft. span is divided into 10 bays of 12 ft. each. The depth of the girder is 16 ft., and it is subjected to a dead load of 3 cwt. per foot run. Find the maximum shear in each panel when a live load of 2 cwt. per foot run rolls over the bridge, and the number of bays that require counterbracing. Width of bridge = 15 ft. (U. of B.)

5. Railway bridges have a great variety of wheel loads moving over them at different times. Show carefully how you would determine "the equivalent load per foot run" for a given span for any type of load—

(*a*) for end shear,

(*b*) for bending moment at all points of the girder. (U. of B.)

6. An *N* girder of 120 ft. span has 6 equal bays in the lower boom. A rolling load of 2 tons per foot run passes over the girder. Determine (1) the maximum force in the top boom of the second bay from the left; (2) the maximum force in the diagonal of the same bay. (U. of B.)

7. An *N* girder 160 ft. span is loaded on the top boom and is also supported at the ends of the top boom. It has 8 panels, each 20 ft. long. The depth of the girder is 25 ft. There is a dead load of 0·5 ton per foot run and a travelling load longer than the span of 1 ton per foot run. Determine the maximum forces in the vertical members 40 ft. and 60 ft. respectively from the left abutment, and also in the three members cut by a vertical section 50 ft. from the left abutment (i.e. the three members connecting the two vertical members). Neglect impact. (U. of L.)

8. A lattice girder of the Warren type has 4 equal bays of 20 ft. in the lower boom and, consequently, 3 equal bays in the upper boom. The diagonals of the girder are inclined at 60° to the horizontal. Draw the influence diagrams for the three members cut by a vertical section 35 ft. from the left abutment; and determine in each case the maximum force produced in the member when a uniformly-distributed load of 2 tons per foot run, 80 ft. long, passes over the girder. (I.S.E.)

9. Two loads of 10 and 15 tons respectively at 6 ft. apart roll over a girder of 30 ft. span. Draw to scale a diagram showing the variation of maximum bending moment that occurs under the load of 15 tons. Determine the maximum shearing force at the centre and also at the end of the girder. (U. of L.)

10. A locomotive, having wheel loads as shown in Fig. 151, passes over a bridge of 50 ft. span. Determine (*a*) the bending moment at the centre of the girder when the load of 18 tons is at the centre, (*b*) the maximum shearing force at the abutments due to these loads.

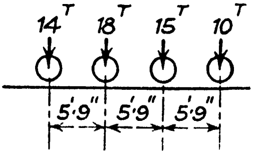


FIG. 151

11. Show clearly how you would find the equivalent load per foot run for bending moment for the wheel loads shown in Fig. 151 for any span. (U. of L.)

12. An N girder has a span of 150 ft. and has 10 equal bays in the lower boom. The girder is 20 ft. deep. Show how to find the influence diagrams for (1) any boom member, (2) any diagonal. A load consisting of a locomotive having 16 tons on each of 5 axles spaced 6 ft. apart rolls over the girder. Neglecting any impact factor, find the maximum forces in the members cut by a vertical section 40 ft. from one abutment.

13. The floor of a single line railway bridge is carried on cross girders which rest on the main girder at the panel points as shown in Fig. 152. The span of the main girder is 100 ft. and there are five panels each 20 ft. The girder is 15 ft. at the centre and 10 ft. deep at ED. A uniform load of 2 tons per foot runs over the bridge from G to H. Show that the forces due to the moving load in the two members AB and AC are a maximum when the front of the load is 71·1 ft. from G, and hence determine these forces in magnitude and kind. (U. of L.)

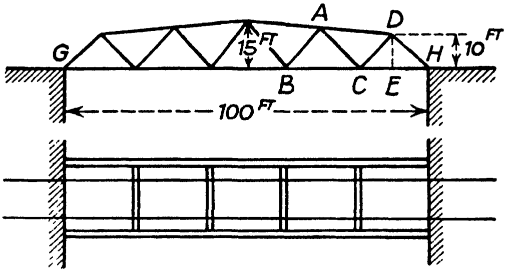


FIG. 152

14. The student is requested to take the frames mentioned in the problems and to carry out exercises for himself to test any results and formula given in the text, using any of the types of moving loads given in Fig. 129.

CHAPTER XI

THREE-PINNED METAL ARCHES, RIBS, AND SUSPENSION BRIDGES

136. AN arch rib may be looked upon as a curved girder, either a solid rib or braced, supported at its ends and carrying transverse loads which are frequently all vertical. The arch, as a whole, is subjected to thrust, because the transverse loading at any section normal to the axis of the girder is at an angle to the normal face, and the force can, therefore, be split up into a tangential or shear force across the face, and a force or thrust normal to the section. The line of resultant thrust is called the " linear arch."

For an arch carrying vertical loads, it can easily be drawn, when, in addition to the vertical loads, the horizontal component of the thrust at the abutments is known. The vertical components of the reactions at the abutments are determined algebraically or graphically as for a straight beam, and they are not affected by the horizontal thrust if the abutments are at the same level, as is evident if moments about an abutment are considered.

Arches are of three kinds—

(*a*) Three-pinned, having hinges at the crown and the abutments.

(*b*) Two-pinned, having hinges at the abutments only.

(*c*) Fixed or solid arches, having no hinges at all.

The former is a statically determinate structure ; the two latter are statically indeterminate. In this book the theory of the three-hinged arch only will be considered.

137. The general condition of loading for an arch is as illustrated in Fig. 153.

Let the external loads act vertically downwards. At the abutments there are, in general, (*a*) a fixed moment M_0 resisting bending ; (*b*) a reaction R_0 due to the thrust of the arch, and which can be resolved into V_0 acting vertically upwards, and H_0 the horizontal component acting towards the other abutment.

The straining actions at any normal section C are as in Fig. 153, and they are resolved into (*a*) a bending moment M, and (*b*) a shearing force S (as in a straight beam); in addition, a thrust P normal to the section. These three actions are statically equivalent to a single thrust T acting at some point D on the normal face C produced, and where

$$C_1D = \frac{M}{P}$$

D lies on the linear arch; C_1 is a point on face C lying on the axis of the arch, the axis passing through the centroid of the section. For continuous loading, the linear arch will be a curve having the direction of the resulting thrusts as tangents to the curve.

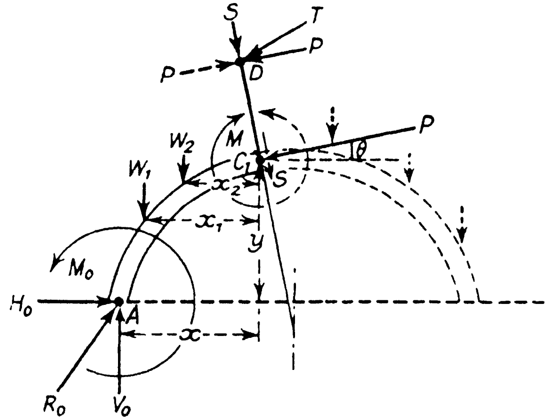


FIG. 153

The straining action may then be specified by the normal thrust P, the radial shearing force S, and the bending moment M; or simply by the linear arch. When the straining actions are known, the stress intensities in the rib can be determined. As in straight beams, the shearing force may be neglected as producing little effect on the stresses. If the curvature of the rib is not great, it is usually sufficient to calculate the bending stresses as for a straight beam. The uniform compression arising from the thrust P is added algebraically to the bending stresses.

138. The general equations for all types of arches are—

Considering the equilibrium of the portion AC_1 in Fig. 153,

y = height of the axis of the arch at C_1 above the supports.

Resolving the forces horizontally, θ = slope of tangent to arch axis at C.

$$P \cos\theta - S \sin\theta - H_o = 0 \quad . \quad . \quad . \quad . \quad (1)$$

Resolving the forces vertically,

$$P \sin\theta + S \cos\theta + (\Sigma_A^{C_1} W_1) - V_o = 0 \quad . \quad . \quad (2)$$

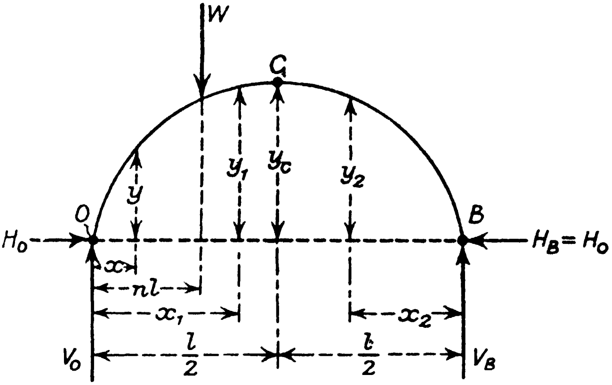


FIG. 154

Moments about C_1 (the signs being the same as for straight beams)

$$M_o - M - V_o x + H_o y + (\Sigma_A^{C_1} W_1 x_1) = 0 \quad . \quad . \quad (3)$$

The Three-hinged Arch. Thus there are six unknowns in these three equations: the general case, therefore, requires three more equations which are developed from the strains produced. But for the three-hinged arch, i.e. the arch having hinges at the abutments and the crown, and which is statically determinate, three of the unknowns in the general equations are easily found. M_o is zero, i.e. there is no moment at the springings or abutments; and also the moment at the crown hinge is zero. The horizontal thrust H_o is found from these facts and, consequently, the line of thrust or linear arch. The line of thrust passes through the hinges, and V_o is found as for a simply-supported beam. Therefore, for the three-hinged arch,

$$M = -V_o x + (\Sigma_{A}^{C_1} W_1 x_1) + H_o y \quad . \quad . \quad . \quad . \qquad (4)$$

$$= \begin{pmatrix}\text{Moment at the section as for} \\ \text{a simply-supported beam}\end{pmatrix} + H_o y \quad . \quad . \qquad (5)$$

$$= M_s + H_o y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

M_s will always be of negative sign for downward loads.

139. A few cases of loading will now be considered for the three-hinged arch.

A single dead load between the abutment and the crown.

Abutment hinges are at same height.

W is distant *nl* from the origin *O*, and between *O* and *C*. Referring to Fig. 154, page 256.

$$V_o = \frac{W}{l}(l - nl) = W(1 - n) \quad . \quad . \quad . \quad . \qquad (7)$$

$$V_B = \frac{W}{l} \,.\, nl \quad = Wn \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Considering moments about the crown hinge *C*,

y_c = height crown hinge above the origin.

$$-V_o \frac{l}{2} + W\left(\frac{l}{2} - nl\right) + H_o y_c = 0 \quad . \quad . \qquad (9)$$

$$-W(1 - n)\frac{l}{2} + W\left(\frac{1}{2} - n\right) l + H_o y_c = 0 \quad .$$

$$H_o = \frac{Wl}{y_c}\left(\frac{1}{2} - \frac{n}{2} - \frac{1}{2} + n\right)$$

$$\therefore H_o = \frac{Wnl}{2y_c} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Similarly if *W* be between *C* and *B*,

$$H_o = \frac{W}{2y_c}(1 - n)l$$

and $\frac{Wnl}{2}$ = moment at *C*, as if the arch was a simply-supported beam.

$$\text{If } W \text{ at } C,\ n = \frac{1}{2}$$

$$\frac{Wnl}{2} = \frac{Wl}{2} \times \frac{1}{2} = \frac{Wl}{4} = \text{moment at the centre of a simply-supported beam with } W \text{ at the centre.}$$

$$\text{Then } H_o = \frac{Wl}{4y_c} \text{ and is a maximum.}$$

The moment at any section X between A (the origin) and the load point,

$$M_X = -V_o x + H_o y = -W(1-n)x + \frac{Wln}{2} \cdot \frac{y}{y_c} \quad . \quad . \quad (11)$$

Moment at any section X_1 between the load point and the crown hinge $= M_{X1} = -V_o x_1 + W(x_1 - nl) + H_o y_1$

$$= -W(1-n)x_1 + W(x_1 - nl) + \frac{Wln}{2} \cdot \frac{y_1}{y_c}$$

$$= \frac{Wln}{2}\left(\frac{2x_1}{l} - 2 + \frac{y_1}{y_c}\right) \quad . \quad . \quad . \quad (12)$$

Moment at any section X_2 on the unloaded half

$= M_{X2} = H_o y_2 - V_B x_2$ where x_2 is measured from B

$$= \frac{Wnl}{2} \cdot \frac{y_2}{y_c} - Wnx_2 \quad . \quad . \quad . \quad (13)$$

140. **Graphical Solution of This Problem.** Referring to Fig. 155, as there is no load on BC and as the moment at C is zero, the thrust at B must pass through C to meet the line of action of W in Z.

R_o, R_B, and W are in equilibrium, and the line of the reaction R_o at A must, when produced, meet W in Z.

The vector diagram to the right represents $ef = W$ to scale acting downwards.

$$fh = V_B \text{ and } he = V_o.$$

From f and e draw fo and eo parallel to R_B and R_o respectively to meet in O; then $fo = R_B$ to scale and $eo = R_o$ to scale. O is the pole of the diagram.

Then Oh is horizontal and to scale

= horizontal thrust at A and $B = H_o$.

Now triangle AZB represents the moment diagram to scale for a simple beam, and is of negative sign.

The curved figure *ACBA* represents to the same scale the moment diagram for H_0y and is of the positive sense.

y_c is common to both diagrams.

Thus the resultant moment diagram to scale for the arch is the shaded diagram *AZCXBCA*. For the loaded half, the resultant diagram is negative, and, for the unloaded half, positive.

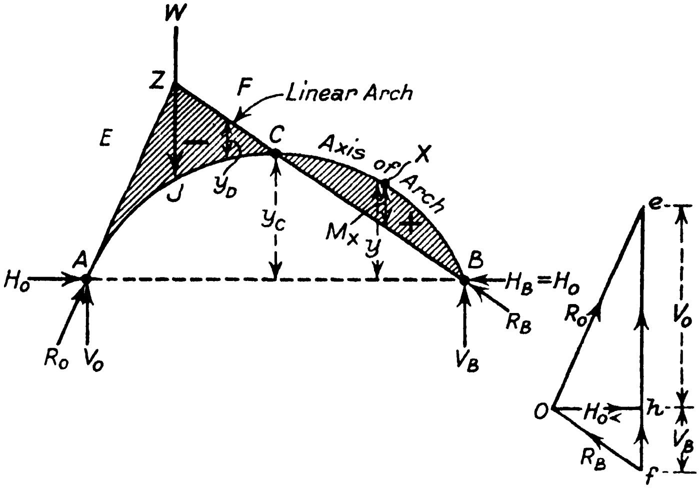


FIG. 155

If H_0 in tons and any ordinate y_D of the shaded diagram is measured in feet to the same scale as y_c, then the moment at the section corresponding to $y_D = H_0 \times y_D$ *tons-ft.*, due regard being paid to the sign.

Thus the resultant moment at any section is represented by the ordinate of the diagram formed by the linear arch and the axis of the arch.

141. (*a*) The Graphical and (*b*) the Algebraic Analysis of the Three-hinged Arch Carrying Several Vertical Concentrated Loads. (Fig. 156.)

(*a*) THE GRAPHICAL METHOD. Calculate H_0 by the method stated, and also calculate V_0 and V_B by the usual methods as for a simple beam.

Referring to Fig. 156, draw the load diagram of the vector polygon to scale; draw the pole line *Oh* horizontal equal to H_0

to scale, h being the point on the load line splitting the total loads into the two vertical reactions. Join O to the different points on the load line. The outer radii will be R_o and R_B in magnitude and direction.

Draw the funicular polygon as for a simple beam and thus obtain the linear arch, which will pass through C.

The resultant moment diagram is again represented by the diagram formed by the linear arch and the axis of the arch.

If there are only a few loads on the arch, the linear arch can be easily found by constructing the simple beam moment diagram by finding the moment at the different load points and erecting ordinates to a scale equivalent to y_c, where y_c will represent the moment at C.

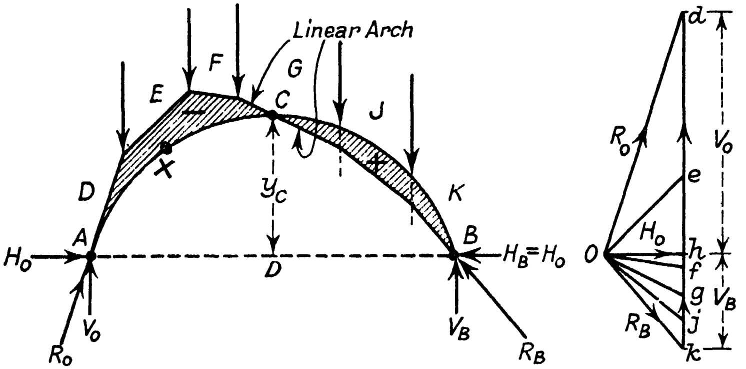


FIG. 156

(*b*) THE ALGEBRAIC SOLUTION. Let M_{SX} represent the simple bending moment at any section X due to the vertical loads acting on a straight horizontal beam.

Then the actual moment at $X = M_X$

$$= M_{SX} + H_o y \text{ from equation (6)}$$

Let M_{SC} be the simple moment at the centre D (*see* Fig. 156) of a span AB; then since the bending moment at C is zero

$$0 = M_{SC} + H_o y_c$$

$$\text{or } H_o = -\frac{M_{SC}}{y_C}$$

Hence for any other section

$$M_X = M_{SX} - M_{SC}\frac{y}{y_C}$$

due attention being paid to the sign of the simple moments; they are of the negative sign for downward loads.

The normal thrust at any radial cross-section X (Fig. 156) may be found by multiplying the resultant thrust (represented by *eo*) by the cosine of the angle between the tangent to the arch axis at X, and the direction of the thrust (*eo*); the transverse or radial shearing force may be obtained by multiplying the resultant thrust (*eo*) by the sine of its inclination to the tangent of the arch axis at X.

Algebraically, the resultant thrust may be obtained by compounding the constant horizontal thrust H_0 with the vertical shearing force determined as for a straight horizontal beam.

142. **Uniformly-distributed Load of Length Equal to the Span on a Three-hinged Arch.** The moment diagram for a simple beam is a parabola, and it has its maximum ordinate at C.

The resultant moment at C is zero, and therefore the resultant moment diagram for the whole arch of any curve will be of the same sign throughout. If the axis of the arch is a parabola, then obviously the resultant moment for the whole arch is zero; because the curve of the simple beam moment line is parabolic and it passes through C, therefore the two curves coincide.

Illustrative Problem 36.

The equation of the axis of a three-hinged arch is $y = x - \frac{x^2}{40}$, the origin being the left-hand support. The span and rise are 40 ft. and 10 ft. respectively. The left-hand half is loaded with a uniformly-distributed load of 1 ton per foot run. Find the reactions at the supports, and the normal thrusts at sections 10 ft. and 30 ft. from the left-hand support. What are the positions of maximum moment. Draw the linear arch.

The linear arch is drawn in Fig. 157. Note $y_c = 10$ ft. to scale $= 1$ in., and for the linear arch diagram being the moment diagram for the simple beam, $M_c = 100$ tons-ft. $= 1$ in.

$$\text{Therefore, } \frac{M_c}{y_c} = \frac{100}{10}$$

$$M_c = 10y_c \text{ or } M = 10y.$$

A formula required for finding the slope of the linear arch.

Now $V_0 = 15$ tons. $V_B = 5$ tons.

Taking moments about C and equating to zero,

$$\left(H_o \times 10\right) + \frac{20^2}{2} - \left(15 \times 20\right) = 0$$

$$H_o = 10 \quad \text{tons}$$

$$R_o = \sqrt{15^2 + 10^2} \quad = 18 \ \text{tons}$$

$$R_B = \sqrt{5^2 + 10^2} \quad = 11{\cdot}2 \ \text{,,}$$

The linear arch is constructed by finding the simple moments about a number of sections (i.e. the effect of H_o is not taken into account) and plotting the moments to such a scale that at C the linear arch passes through C.

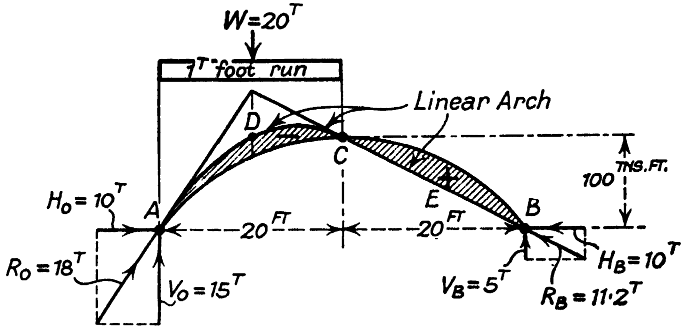


FIG. 157

The resultant moment diagram is the diagram enclosed between the linear arch and the axis of the arch. The linear arch is a straight line from C to B.

The resultant moment for any section X between A and C is

$$M_x = -15x + \frac{x^2}{2} + 10y$$

$$= -15x + \frac{x^2}{2} + 10\left(x - \frac{x^2}{40}\right) = -5x + \frac{x^2}{4}$$

This is a maximum when $\frac{dM_x}{dx} = 0$

i.e. $0 = -5 + \frac{x}{2}$ or $x = 10$

that is, the section of maximum negative moment is 10 ft. from A.

The maximum resultant moment is – 25 tons-ft., and agrees with the amount in the diagram.

Between C and B, the resultant moment equation for any section X_1 distant x_1 from B is

$$M_{x1} = -5x_1 + 10y_1$$

$$= -5x_1 + 10\left(x_1 - \frac{x_1^2}{40}\right) = 5x_1 - \frac{x_1^2}{4}$$

The maximum positive moment occurs at a section X_1 distant 10 ft. from B, for

$$\frac{dM_{x1}}{dx_1} = 5 - \frac{x_1}{2} = 0 \quad \text{from which } x_1 = 10 \text{ ft.}$$

And the maximum moment at this section is

$$= +25 \text{ tons-ft.}$$

To Find the Normal Thrusts at the Sections Distant 10 ft. and 30 ft. from A.

The resultant thrust at any section

$$= \sqrt{(\text{vertical shear at the section})^2 + H_o^2}$$

The direction of the resultant thrust is found from the slope of the linear arch; the direction of the normal thrust from the slope of the axis of the arch.

For the Section 10 ft. from A.

Equation of the linear arch is

$$(+M) = +15x - \frac{x^2}{2}$$

due regard when considering it being paid to the sign.

In terms of the y ordinate of the axis of the arch,

$$(+10y) = 15x - \frac{x^2}{2}$$

The slope at any section is

$$\frac{dy}{dx} = 1{\cdot}5 - \frac{x}{10}$$

The slope at $x = 10 = +{\cdot}5$.

The equation of the axis of the arch

$$y = x - \frac{x^2}{40}$$

The slope of the axis of the arch,

$$\frac{dy}{dx} = 1 - \frac{x}{20}$$

At $x = 10$ the slope is $1 - \cdot 5 = + \cdot 5$,

$$\text{therefore, } \theta_1 = 27^\circ \quad (\tan 27^\circ ; = \cdot 5),$$

that is, at $x = 10$, the slopes of the linear arch and the axis are the same; therefore, the resultant thrust is equal to the normal thrust.

$$\text{Normal thrust} = \sqrt{(-15 + 10)^2 + 10^2} = 11{\cdot}2 \text{ tons}$$

or Normal thrust $= 10 \cos 27 + (15 - 10) \sin 27^\circ = 11{\cdot}2$ tons using formula given in paragraph 140.

The radial shear force will be zero.

NORMAL THRUST AND RADIAL SHEAR AT *A*. The direction of the resultant thrust $(R_A) = \text{angle } \theta = \tan^{-1} 1{\cdot}5 =$ nearly 57°. The inclination of the axis of the arch is

$$= \text{angle } \theta_1 = \tan^{-1} 1 = 45^\circ$$

$$\theta - \theta_1 = 12^\circ.$$

Normal thrust at $A = R_A \cdot \cos 12^\circ = 17{\cdot}6$ tons
Radial shear at $A = R_A \cdot \sin 12^\circ = 3{\cdot}75$ „
where $R_A = 18$ tons.

SECTION 30 FT. FROM *A*. This is a section in the right-hand half of the arch, and where the resultant thrust is the same for all sections. Take *B* as origin and x to the left as positive.

Resultant thrust $= R_B = 11{\cdot}2$ tons

and its direction is $\theta^\circ = \tan^{-1} \dfrac{10}{20}$

$$\therefore \theta^\circ = 27^\circ.$$

The slope of the axis of the arch is also 27° at this section, therefore, resultant thrust = normal thrust = 11·2 tons. [Normal thrust equal to $10 \cos 27 + 5 \sin 27 = 11{\cdot}2$ tons.]

At *B*, the slope of the arch, considering *B* as origin and x to the left as positive is the same as at *A*, $\theta_1 = 45^\circ$.

$$\theta_1 - \theta = 18^\circ$$

Normal thrust at $B = 11{\cdot}2 \times \cos 18^\circ = 10{\cdot}65$ tons
Radial shear at $B = 11{\cdot}2 \times \sin 18^\circ = 3{\cdot}46$ „

TAKE A SECTION *X* 5 FT. FROM *A*

The slope of the linear arch $= 1{\cdot}0 \quad \therefore \theta = 45^\circ$
The slope of the axis of the arch $= \cdot 75 \quad \therefore \theta_1 = 36^\circ 50'$

$$\text{Normal thrust} = 10\cos 36°50' + 10\sin 36°50'$$
$$= 14 \text{ tons.}$$
$$\textit{or}\ \text{resultant thrust} = \sqrt{10^2 + 10^2} = 14{\cdot}14.$$
$$\therefore \text{Normal thrust} = 14{\cdot}14 \times \cos(45° - 36°50')$$
$$= 14 \text{ tons.}$$

143. **Three-hinged Spandril-braced Arch with Dead Loading.** (Fig. 158.) A frame having its bottom boom curved or arched, and hinged at the supports and the crown, is called a three-hinged spandril-braced arch; it is illustrated in Fig. 158. Calculate R_A and R_B as for a three-hinged ribbed arch, and proceed to find the stresses in the members as for ordinary frames, that is, by the force-stress diagram, method of sections, etc.

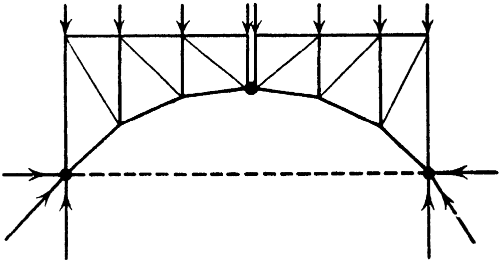


FIG. 158

144. **Influence Lines for a Three-hinged Ribbed Arch.** (Fig. 159.) **Of Horizontal Thrust.** Let a concentrated load of 1 ton move across the span, and let it be at any time distant nl from A (Fig. 159) and between AC. As before, the reactions are made up of

V_o and H_o (the horizontal thrust) and
V_B and H_o.

Referring to Equations (7) to (10) and substituting 1 for W,

$$V_o = (1 - n) : V_B = n$$

$$H_o = \frac{nl}{2y_c}, \text{ that is, } H_o \text{ depends upon } n \quad . \quad (14)$$

For unit load between C and B, and distant $l(1 - n)$ from B, by similar working as before,

$$H_o = \frac{l(1 - n)}{2y_c} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Again a linear relation between H_o and n,

$$\begin{aligned} &\text{when } n = 0 \text{ from } (14) && H_o = 0 \\ &n = \tfrac{1}{2} \quad ,, \quad (14) && H_o = \frac{l}{4y_o} \quad . \quad . \quad . \quad (16) \\ &n = 1 \text{ from } (15) && H_o = 0 \\ &n = \tfrac{1}{2} \quad ,, \quad (15) && H_o = \frac{l}{4y_o} \quad . \quad . \quad . \quad (16a) \end{aligned}$$

$\frac{l}{4y_o}$ is a maximum value for H_o, and it occurs when the unit load is at C.

Thus join A and B to C, and let y_o represent

$$H_o = \frac{l}{4y_o} \text{ tons,}$$

and the unit influence line for the horizontal thrust at the abutments is obtained; that is, the ordinate of the diagram to scale gives the horizontal thrust at the abutments when unit load is at that ordinate: let it be y.

For a number of loads, $H_o = \Sigma Wy$, where y is the ordinate of the unit diagram at the position of the load.

For a uniformly-distributed load, H_o is represented by the area of the unit diagram to scale underneath the length of the load over the arch.

For the uniformly-distributed load of 1 ton per foot covering half the span AC in Problem 36,

$$H_o = \frac{40}{4 \times 10} \times \frac{20}{2} \times 1 = 10 \text{ tons.}$$

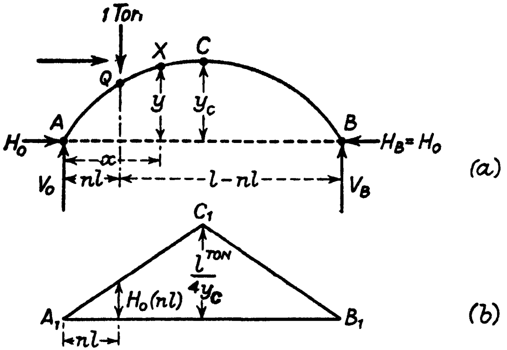
145. **Influence Diagram of Bending Moment.** Let the unit load be at any section Q between A and C, and distant nl from A, Fig. 159.

To draw the unit influence moment diagram for any section X distant x from A, it has been shown that the moment at any section X

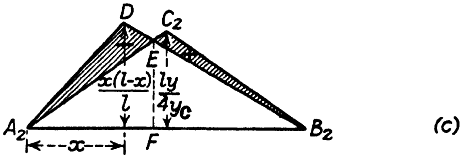
$$\begin{aligned} = M_x = &\text{ Negative moment for a freely-supported beam} \\ &+ \text{Positive moment due to the horizontal thrust,} \end{aligned}$$

therefore, the resulting influence diagram of moment will be the difference of the influence moment diagram for a simple beam and the moment diagram due to the horizontal thrust.

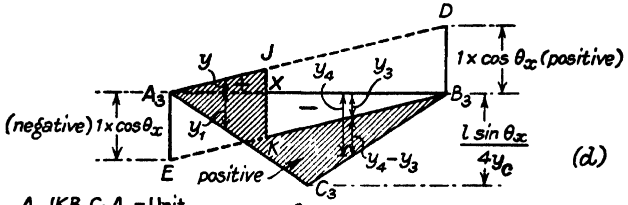
The horizontal thrust is a maximum when the load is at C,



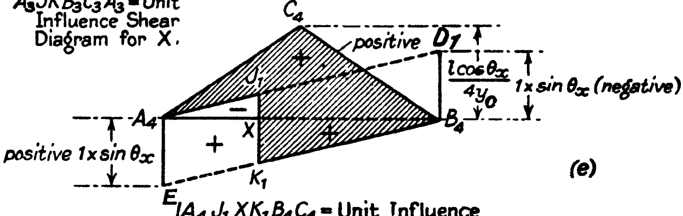
Unit Influence Diagram - Horizontal Thrust.



Unit Influence Moment Diagram for Section X.


D
J
y
X
y_4
y_3
$1 \times \cos\theta_x$ (positive)
A_3
B_3
(negative) $1 \times \cos\theta_x$
y_1
K
$\frac{l \sin\theta_x}{4y_c}$
$y_4 - y_3$
E
positive
C_3
(d)

$A_3JKB_3C_3A_3$ = Unit Influence Shear Diagram for X.


C_4
positive
D_1
J_1
$\frac{l \cos\theta_x}{4y_0}$
$1 \times \sin\theta_x$ (negative)
A_4
X
B_4
positive $1 \times \sin\theta_x$
K_1
E
(e)

$1A_4J_1XK_1B_4C_4$ = Unit Influence Normal Thrust Diagram for X.

FIG. 159

therefore, the maximum moment at X due to the horizontal thrust

$$= \frac{l}{4y_c} \times y \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

where y is the rise of the arch at the section X. The moment due to the horizontal thrust at X is $H_0 y$ where H_0 varies with the position of the load and y is a constant. Thus the horizontal thrust moment diagram is found by erecting a central ordinate on the horizontal span to scale $= \frac{ly}{4y_c}$ (Fig. 159), and joining C_2 to A_2 and B_2.

The maximum moment due to a simple beam is when the load is at X and $= \frac{x(l-x)}{l}$ and must be drawn to the same scale as for $H_0 y$.

The resulting unit influence moment diagram for the section X is shown shaded in Fig. 159, and it consists of two triangles, A_2DE negative and C_2B_2E positive.

MAXIMUM MOMENT. With a single concentrated load, the maximum negative moment is when the load is at D, and the maximum positive moment when the load is at C. With a uniformly-distributed load longer than the span, maximum positive and negative moments occur when B_2F and A_2F are covered respectively (Fig. 159 (*c*)).

146. **Shear and Thrust Influence Lines. Radial Shear.** If θ_x is the inclination of the tangent to the arch at the point X (between A and C), then the shear at X will be the resultant force normal to the arch axis at that point.

With the load between A and X,

$$S_x = V_B \cos\theta_x + H_0 \sin\theta_x . \quad . \qquad (18)$$

Load between X and B,

$$S_x = -V_0 \cos\theta_x + H_0 \sin\theta_x \quad . \qquad (19)$$

The signs are due to considering the forces to the right and left of a section.

For unit load on the arch, (18) and (19) are as for a simply-supported beam with unit load multiplied by $\cos\theta_x$, together with a load $H_0 \sin\theta_x$ added. The unit influence shear diagram is shown in Fig. 159 (*d*).

Diagram A_3DB_3 is positive,
,, A_3EB_3 is negative,
,, $A_3C_3B_3$ is positive.

$A_3C_3B_3$ is the unit influence shear diagram for $H_o \sin\theta_x$. H_o is a maximum when the load is at the centre of the arch and for unit load $= \dfrac{l}{4y_c}$ tons.

Thus CC_3 to scale $= H_o \sin\theta_x = \dfrac{l \sin\theta_x}{4y_c}$

and the triangle $A_3C_3B_3$ is the unit shear diagram for $H_o \sin\theta_x$.

A_3JXKB_3 is the unit shear diagram for the section X as for a simple beam.

With the unit load between A and X, the radial shear at X (*see* (*d*) Fig. 159) is $S_x = +y + y_1$.

With the unit load between X and B, $S_x = +y_4 - y_3$.

With a uniformly-distributed load, the shear (S_x) will be the area of the diagram under the load to scale, due respect being paid to signs.

147. **Influence Line for Normal Thrust.** The normal thrust at a section X, between A and C, will be the resultant force tangential to the axis of the arch at the section. θ_x is the direction of the tangent to the arch rib at X.

For the load between A and X, the thrust at X

$$= T_x = H_o \cos\theta_x - V_B \,.\, \sin\theta_x \quad . \quad . \quad . \quad (20)$$

Between X and B

$$T_x = H_o \cos\theta_x + V_o \sin\theta_x \quad . \quad . \quad . \quad (21)$$

By similar reasoning to that for shear, the following is the construction for the unit normal thrust influence line. (*See* (*e*) Fig. 159.)

Make B_4D_1 and A_4E_1 equal to ($1 \sin\theta$).

Join D_1A_4 to meet the vertical through X in J_1
,, B_4E_1 ,, ,, ,, in K_1

At the centre of the span, set up an ordinate CC_4

$$= +\frac{l \cos\theta_x}{4y_c} \text{ and join } C_4 \text{ to } A_4B_4$$

then the shaded portion, $A_4J_1XK_1B_4C_4$, is the unit normal thrust influence diagram for the section X.

To construct the unit radial shear and normal thrust diagrams for a section between *B* and *C*, use *B* as the origin and take x to the left as positive, and work as given in paragraphs 146 and 147. Or, by similarity, the diagrams for a section distant x from *B*, will be the "reflections" of those for a section distant x from *A*.

148. **Suspension Bridges.** A HANGING CABLE AND ITS RELATION TO THE LINEAR ARCH. If it is assumed that the cable has no resistance to bending, the form of the centre line of a hanging chain or cable carrying vertical loads is that of the funicular polygon for the loads and end-supporting forces, the horizontal pole distance from the load line in the vector polygon representing the horizontal tension in the cable. In all cases, each vertical load is balanced by the tensions in the two segments of the cable meeting in its line of action. The horizontal tension which evidently cannot vary throughout the cable, since no forces having horizontal components are applied except at the ends, fixes the precise outline of the cable centre line and supplies the remaining condition to fix the pole *O*. The horizontal distance from the pole to the load line represents the horizontal component to scale of all the tensions in the various segments. Thus the hanging cable is the "linear arch" itself.

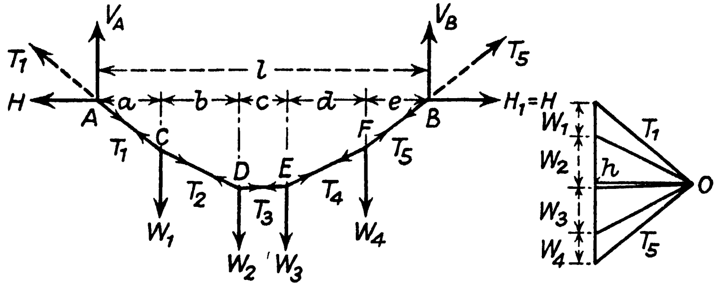


FIG. 160

An arch supports vertical loads by material exposed to thrust, and for an arch we have seen that the funicular polygon represents the line of thrust, or the linear arch.

In the case of the three-hinged arch, the resultant moment is zero at the crown hinge. In this case, the linear arch may pass outside the axis of the arch ; while in the case of the flexible cable, the axis of the cable and the line of resistance must coincide. The cable is in stable equilibrium; the arch is in unstable equilibrium.

149. Let a chain be loaded as in Fig. 160, and let the supports be the same height. A and B are the support points at the same level.

At the supports the reactions will be T_1 and T_5 respectively, which can be divided into V_A and V_B vertically, H and H_1 horizontally.

As all the loading is vertical, then $H = H_1$ and acts in opposite directions outwards.

The horizontal component of the tension in each member AC, CD, etc., is also $= H$.

Point C is in equilibrium under W_1, T_1, and T_2, and these three forces form the sides of a triangle.

Point D is in equilibrium under W_2, T_2, and T_3, and these forces form the sides of a triangle and similarly for the other points.

The shape of the cable will, therefore, be the funicular polygon which is constructed from the vector diagram by the usual methods.

Taking moments about B,

$$V_A l = W_1(l - a) + W_2(l - a - b) + \dots \qquad (22)$$

V_A can be found and subsequently V_B.

Let the vertical component of the stress T_n in the n^{th} member from the origin A be T_v.

$$T_v = V_A - W_1 - W_2 \dots - W_{n-1} \qquad (23)$$

Horizontal component of T_n will be H.

$$\therefore\ T_n = \sqrt{T_v^2 + H^2} = H \sec\theta. \qquad (24)$$

Let the slope of the member be $\tan\theta$,

$$\text{then } \tan\theta = \frac{T_v}{H} \qquad (25)$$

$$\text{and } T_n = H \sec\theta \qquad (26)$$

Let y_c be the maximum depth of the cable below the supports at any point C and at x_c from A then

$$Hy_c + W_1(x_c - a) + W_2(x_c - a - b) + \dots = V_A x_c \qquad (27)$$

$$\text{or } Hy_c + \left(\begin{array}{c}\Sigma \text{ moments of the external loads}\\ \text{to the left of } C \text{ about } C\end{array}\right) = V_A x_c \qquad (28)$$

150. **Consider a Hanging Cable on which the Load Carried is a Continuous One.**

Let this be w per unit length of horizontal span.

$$\text{Total load} = \int_0^l w\,.\,dx$$

Let the slope of the cable at some section X distant x from the left-hand support be $\frac{dy}{dx}$

$$\text{then } \frac{dy}{dx} = \frac{V_A - \int_0^x w\,.\,dx}{H} = \tan\theta \quad . \quad . \quad . \quad (28a)$$

Differentiating,

$$H\frac{d^2y}{dx^2} = -w$$

$$\frac{d^2y}{dx^2} = -\frac{1}{r}\sec^3\theta. \quad \text{For } \frac{1}{r} = -\frac{\frac{d^2y}{dx^2}}{\sec^3\theta}*$$

θ = angle tangent to the curve makes with the horizontal or x axis

therefore, $\frac{H}{r}\sec^3\theta = w$,

$$\text{and } H = wr\cos^3\theta \quad . \quad . \quad . \quad (29)$$

$$,,\ T = wr\cos^2\theta \quad . \quad . \quad . \quad (30)$$

At the vertex of the curve where $r = r_v$,

$$w = w_v \text{ and } \cos\theta = 1,$$

$$T = H = w_v r_v \quad . \quad . \quad . \quad . \quad (31)$$

151. **Hanging Cable and Uniformly-distributed Load.** When the load is uniformly distributed over the span of the cable, as approximately in some suspension bridge cables, and in telegraph and trolley wires, which are tightly stretched and loaded by their own weight, the form of the curve in which the cable hangs is parabolic.

If the uniform loads are applied at short intervals, then the funicular polygon will be circumscribed by the parabola corresponding to continuous loading; i.e. the points of application of the load will lie on a parabola, which the cable would follow under continuous loading.

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* **Radius of curvature.** *See* **Mathematical textbooks.**

If w is a constant,

we have $H\dfrac{d^2y}{dx^2} = -w$ from Equation (28a)

$$\text{Integrating twice, } y = -\frac{wx^2}{2H} + \frac{wlx}{2H} \quad . \quad . \quad (32)$$

which is the equation to a parabola having the left support as origin, and, therefore, the form of the cable will be a parabola. y will be positive downwards.

Let the maximum depth of the cable having supports at the same level be y_c, and this will be at $x = \dfrac{l}{2}$.

$$y_c = +\frac{wl^2}{8H} \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$H = \frac{wl^2}{8y_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Note $\dfrac{wl^2}{8}$ is the moment as for a simple beam,

$$\text{and } y = \frac{4y_c}{l^2}(l - x)x \; . \quad . \quad . \quad . \quad . \quad (35)$$

and is positive downwards for all values of x.

The greatest tension in the cable will occur at the support points where the slope is a maximum.

$$T_{max} = \sqrt{H^2 + \left(\frac{wl}{2}\right)^2}$$

$$= \frac{wl^2}{8y_c}\sqrt{1 + \left(\frac{4y_c}{l}\right)^2} \quad . \quad . \quad (36)$$

The load carried by a suspension bridge cable, including the stiffening truss (paragraph 153), is nearly uniform in intensity in reference to a horizontal line; and so nearly so, that it is assumed to be exactly uniform.

Length and dip of cable or chain having parabolic form.*

$y = \dfrac{4y_c}{l^2}(l - x)x$, where l = horizontal span.

$$L = \text{length of chain} = l\left(1 + \frac{8y_c^2}{3l^2} - \frac{32y_c^4}{5l^4} + \dots\right) \quad . \quad . \quad . \quad (37)$$

---

* Working from the left support as origin.

For very flat curves,

$$L \triangleq l\left(1 + \frac{8y_c^2}{3l^2}\right) \quad . \quad . \quad . \quad (38)$$

$\frac{y_c}{l}$ generally between ·2 and ·05.

The approximate greatest length of span*

$$l = \frac{\dfrac{2ty_c}{3\rho l}}{1 + \dfrac{8y_c^2}{3l^2}} \quad . \quad . \quad . \quad . \quad (39)$$

where ρ = weight of 1 cu. in. of steel = ·2836 lb.
t = 60,000 lb./sq. in. for steel wire.
= 30,000 lb./sq. in. for nickel steel eyebar cable.

152. **Anchorages.** Loads in the anchorage cables and on the piers. Assume the side cables in a straight line from the tops of the supports to the anchorages.

(1) Cable over a fixed pulley or roller on the top of the support,

T_1 = tension in the anchorage cable, making an angle β with the horizontal;

T = tension in the hanging cable at the support and at an angle α to the horizontal.

In this case, $T = T_1$.

Horizontal pressure on the tower = H_1

$$H_1 = T \cos \alpha - T \cos \beta \quad . \quad . \quad (40)$$

If $\alpha = \beta$, $H_1 = 0$.

The vertical load will be

$$P_v = T \sin \alpha + T \sin \beta \quad . \quad . \quad (41)$$

(2) To avoid horizontal pressures on the piers, movable saddles over which the cables pass are placed on the top, free to move over rollers.

In this case, $H_1 = 0$, neglecting friction, therefore

$$T \cos \alpha = T_1 \cos \beta \quad . \quad . \quad . \quad (42)$$

If α and β are known, T_1 can be found.

$$\text{If } \alpha = \beta, T = T_1 \quad . \quad . \quad . \quad (43)$$

---

* From *Suspension Bridges*, by Burr (Wiley).

Vertical pressure on the tower

$$= P_v = T \sin \alpha + T_1 \sin \beta. \qquad (44)$$

Illustrative Problem 37.

A cable of span 100 ft. and a maximum dip of 10 ft. carries a uniformly-distributed load of ½ ton per foot of span. Find the maximum and minimum tensions in the cable.

The maximum tensions will be at the support points $= T_{max}$; the minimum tension will be the horizontal component of all the tensions $= H$.

$$\text{Now } H = \frac{wl^2}{8y_c} \text{ (equation 34)}$$

$$= \frac{\cdot 5 \times 100 \times 100}{8 \times 10} = 62{\cdot}5 \text{ tons}$$

$$T_{max} = \sqrt{(62{\cdot}5)^2 + \frac{w^2l^2}{4}}$$

$$T_{max} = 67{\cdot}25 \text{ tons.}$$

Find the alteration in these tensions due to a rise in temperature of 30° F if the coefficient of expansion is ·000006.

Now length of cable in a flat parabolic curve is

$$L = l + \frac{8}{3}\frac{y_c^2}{l}$$

$$l = \text{span} : y_c = \text{dip};$$

$$L \text{ at normal temperature} = 100 + \frac{8}{3} \times \frac{100}{100}$$

$$= 102{\cdot}7 \text{ ft. nearly;}$$

L after rise of temperature of 30° F.

$$= 102{\cdot}7(1 + \cdot 000006 \times 30)$$

$$= 102{\cdot}718 \text{ ft.}$$

$$\therefore \frac{8}{3} \times \frac{y_c^2}{100} = 2{\cdot}718$$

$$y_c = \sqrt{\frac{2{\cdot}718 \times 300}{8}} = 10{\cdot}1 \text{ ft.}$$

$$H \text{ becomes } \frac{\frac{1}{2} \times 100^2}{80{\cdot}8} = 61\tfrac{3}{4} \text{ tons}$$

$$T_{max} = 66\tfrac{1}{2} \text{ tons.}$$

Illustrative Problem 38.

A chain having a span of 99 ft. carries a uniformly-distributed load of $\frac{1}{4 \cdot 5}$ tons per foot of span. The left-hand and right-hand support are 4 ft. and 16 ft. respectively above the lowest point of the centre line of the chain Find the tension at the supports and at the lowest point.

Let the lowest point be a horizontal distance x from A and x_1 from B.

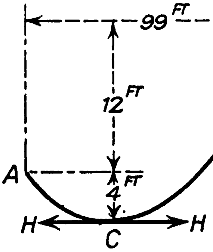


FIG. 161

$$\text{Then } H = \frac{wx^2}{2 \times 4} = \frac{wx_1^2}{2 \times 16}$$

$$\therefore\ x^2 = \frac{x_1^2}{4}$$

$$x = \frac{x_1}{2}$$

$$\text{therefore } x = \frac{99}{3} = 33 \text{ ft.}$$

$$H = \frac{1}{4 \cdot 5} \times \frac{33^2}{8} = 30 \cdot 25 \text{ tons.}$$

Let V_A = vertical reaction at A

V_B = „ „ B

$$V_A = \frac{33}{4 \cdot 5} \text{ tons} = 7 \cdot 33 \text{ tons}$$

$$T_A = \sqrt{30 \cdot 25^2 + 7 \cdot 33^2} = 31 \cdot 2 \text{ tons}$$

$$V_B = \frac{66}{4 \cdot 5} = 14 \cdot 66 \text{ tons}$$

$$\therefore\ T_B = \sqrt{30 \cdot 25^2 + 14 \cdot 66^2} = 33 \cdot 6 \text{ tons}$$

153. **Stiffened Suspension Bridges.** To make suitable for heavy traffic, the suspension bridge requires stiffening to resist changes of shape in the roadway.

This is done by—

(1) Carrying the roadway on a girder hinged at the two ends of the span.

(2) Carrying the roadway on two girders, each taking half the span, hinged together and at the ends of the span.

(3) Replacing the cable by two stiff suspension girders hinged together midway between the piers. These virtually

form a three-pinned arch, which is statically determinate. The determination of the reactions and stresses is exactly analogous to those for the three-hinged ribbed arch.

154. **Three-hinged Stiffening Truss.*** Fig. 162(*a*). The function of the stiffening truss is to distribute to the cable, uniformly along a straight line, any load whatever applied to the structure, so that the parabolic form of the cable is preserved under all conditions of loading.

Thus the chain keeps its parabolic form, assuming the live load is evenly distributed along the whole length of cable.

Working from the left support of the chain as origin, the dip at any section distant x horizontal from the origin is

$$y = \frac{4y_c}{l^2}(l-x)x \quad . \quad . \quad . \quad . \quad (45)$$

General case. $w_o =$ dead load weight per foot-run

$S =$ live „ „

$$S + w_o = -H\frac{d^2y}{dx^2} \text{ from Equation (28}a\text{)}$$

$$S + w_o = H \cdot \frac{8y_c}{l^2} \qquad \text{(Equation (34))}$$

Forces on the stiffening truss : its weight, the reactions due to the moving load, and the upward pulls of the hangers.

The loads in the hangers produce moments M_H in the trusses opposite to those in general caused by the vertical loads on the truss.

∴ Resultant moment at any section of the girder

$$= M = M_s - M_H \quad . \quad . \quad . \quad (46)$$

Now the whole weight of the girder is taken by the cable, and therefore, neither moment nor shear in the girder is due to its own dead weight : thus M_s is simply due to the live load acting on a simple beam. M_H will be the moment due to the evenly-distributed hanger load equivalent to the live load.

Now let $W =$ live load ;

and $w_e =$ uniformly-distributed load on the hangers due to the live load ;

$$\therefore\ w_e l = W \text{ and } w_e = \frac{W}{l}$$

---

* Case (2), para. 153.

Now the reaction acting downwards at the end hinge of the stiffening truss assumed simple will be $= \frac{w_e l}{2}$ which is also the reaction acting upwards at the cable support due to the live load.

For the stiffening truss, therefore, the moment at any section X distant x from the left-hand hinge is

$$-M_s + \frac{w_e lx}{2} - \frac{w_e x^2}{2} = M \quad . \quad . \quad . \quad . \quad (47)$$

$M_s =$ moment in truss as a simple beam due to live load.

For the cable, section distant x from the support point

$$-\tfrac{1}{2}w_e lx + \tfrac{1}{2}w_e x^2 . - Hy = 0 \quad . \quad . \quad . \quad . \quad (48)$$

Neglecting the small hanger loads represented by $\frac{1}{2}w_e x^2$, we get

$$M = -M_s + w_e \frac{lx}{2} \quad . \quad . \quad . \quad . \quad . \quad (49)$$

and $$w_e \frac{lx}{2} = Hy \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

$$\therefore M = -M_s + Hy \quad . \quad . \quad . \quad . \quad . \quad (51)$$

For a three-hinged stiffening girder, the moment at the centre hinge must be zero : i.e.

$$M_s = Hy \quad . \quad . \quad . \quad . \quad . \quad (52)$$

At C (Fig. 162, page 280), $M_x = M_c = 0$.

$M_{sc} =$ simple moment at C. Then

$$Hy_c = M_{sc} \text{ and } H = \frac{M_{sc}}{y_c} \quad . \quad . \quad (53)$$

155. **Single Concentrated Load on the Stiffening Girder.** (Fig. 162.) Let W be at a distance nl from E and between E and C. Then

$$H = \frac{Wln}{2y_c} \quad . \quad . \quad . \quad . \quad . \quad (54)$$

If W between C and F,

$$H = \frac{W(1-n)l}{2y_c} \quad . \quad . \quad . \quad . \quad (55)$$

H is a maximum when $n = \frac{1}{2}$,
i.e. when W is at C and

$$H = \frac{Wl}{4.y_c} \quad . \quad . \quad . \quad . \quad . \quad (56)$$

The shear at any section of the stiffening girders is

$S_x =$ Positive or negative shear at the section of the girder as for a simple beam (S_{sx}) plus the vertical component of the cable tension at the corresponding section of the cable.

$$S_x = \pm S_{sx} + H \tan \theta_x \quad . \quad . \quad (57)$$

where θ_x is the inclination of the axis of the cable at the section.

$$S_x = \pm S_{sx} + H\frac{dy}{dx} \quad . \quad . \quad . \quad (58)$$

$$y = \frac{4y_c}{l^2}(l - x)x \quad . \quad . \quad . \quad . \quad (59)$$

$$\frac{dy}{dx} = \frac{4y_c}{l^2}(l - 2x) \quad . \quad . \quad . \quad . \quad (60)$$

For any section when the load is between E and C, and at a distance nl from E, using Equation 60 for $\frac{dy}{dx}$

$$S_x = \pm S_{sx} + \frac{Wnl}{2y_c}\left(l - 2x\right)\frac{4y_c}{l^2}$$

$$= \pm S_{sx} + \frac{2Wn(l - 2x)}{l} \quad . \quad . \quad . \quad (61)$$

When the load is between C and F, and at a distance $l(1 - n)$ from E,

$$S_x = \pm S_{sx} + \frac{W(1 - n)l}{2y_c}\left(l - 2x\right)\frac{4y_c}{l^2}$$

$$= \pm S_{sx} + \frac{2W(1 - n)(l - 2x)}{l} \quad . \quad . \quad . \quad (62)$$

156. **Influence Lines of Moment and Shear for any Section of the Stiffening Girders.** Let unit load cross the girders. Then the unit moment influence line for this case is exactly the same as for the three-hinged ribbed arch.

157. **Unit Shear Influence Line** (Fig. 162). It has been shown that at any section X distant x from E, the resultant shear

$$S_x = \pm S_{sx} + H\frac{dy}{dx}$$

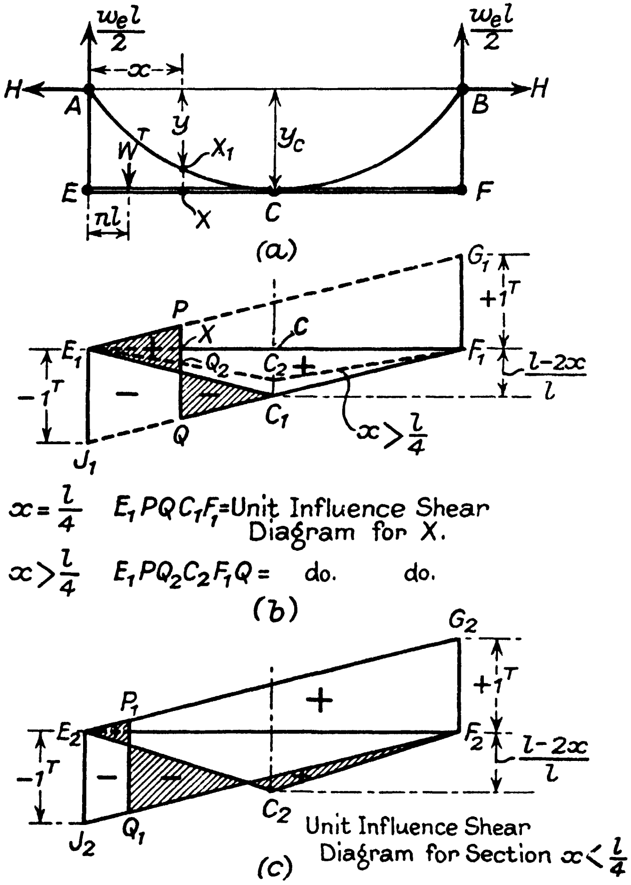


FIG. 162

The influence shear diagram for the section X will, therefore, be a combination of two diagrams—

(1) The unit negative or positive influence diagram as for a simple beam.

(2) The unit positive diagram for $H \cdot \frac{dy}{dx}$.

For the unit load between E and C and distant nl from E,

$$H \cdot \frac{dy}{dx} = \frac{2n(l - 2x)}{l}$$

$H \cdot \frac{dy}{dx}$ is thus proportional to n as x is a constant, being the distance of the section X from E.

$H \frac{dy}{dx}$ is a maximum when the unit load is at $C \quad = \frac{(l - 2x)}{l}$

(when $n = 0$, $H = 0$.)

Similarly for the load between C and F,

$$H \frac{dy}{dx} = \frac{2\,(1 - n)\,(l - 2x)}{l}$$ and $H\left(\frac{dy}{dx}\right)$ is proportional to n

when $n = 1$, $H \frac{dy}{dx} = 0$

when $n = \frac{1}{2}$, $H \cdot \frac{dy}{dx}$ is a maximum

$$= \frac{(l - 2x)}{l} \quad . \quad . \quad . \quad . \quad . \quad (63)$$

The unit positive diagram for $H \frac{dy}{dx}$ will, therefore, be a triangle with the span as base and the maximum ordinate at the centre

$$= \frac{(l - 2x)}{l} \text{ tons}$$

To construct the complete unit influence shear line. (*See* Fig. 162 (*b*).)

First draw the unit shear influence line $E_1G_1F_1J_1$ as for a simply-supported beam.

$$E_1J_1 = -1\,; \; F_1G_1 = +1$$

Diagram $E_1G_1F_1$ is positive.
Diagram $E_1J_1F_1$ is negative
At the centre C set down

$$CC_1 = \frac{l - 2x}{l}$$

and join C_1 to E_1 and F_1 : this triangle is of positive area.

The difference of the two diagrams shown shaded gives the influence diagram required for the section X.

Note. If x less than $\frac{l}{4}$ F_1Q comes above F_1C_1. (Fig. 162 (*c*).)

If $x = \frac{l}{4}$, F_1Q and F_1C_1 coincide,

i.e. no shear at X when the load is between C and F.

If x greater than $\frac{l}{4}$, F_1Q below F_1C_1. (*See* Fig. 162 (*b*).)

and the shear wholly negative for load between X and F.

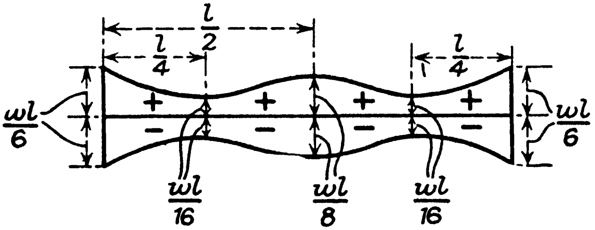


FIG. 163

Curves of maximum shear due to uniformly-distributed load, longer than the span, are given in Fig. 163.

158. Maximum Shear. The maximum shear force curves for a single rolling load may be deduced from Fig. 162 (*b*) and (*c*).

The maximum shearing force curves for a uniformly-distributed load may be easily found, also from the areas in the influence diagrams, the loaded lengths for the different maximum values being the projections of the areas of like sign in Fig. 162 (*b*). The positive and negative areas are equal for any value of x,* and so the positive and negative curves are similar. When $x = 0$ $\left(\text{a particular case of } x < \frac{l}{4}\right)$ in Fig. 162, the loaded lengths are $\frac{l}{3}$ for maximum negative and $\frac{2l}{3}$ for maximum positive shear at the origin and both shears

$$= \frac{1}{2} \times \frac{wl}{3} = \frac{wl}{6}$$

---

* *See* Andrews, *Problems in Higher Structures.*

The student is requested to prove the maximum shears given in Fig. 163.

Illustrative Problem 39.*

A suspension cable of 100 ft. span and 10 ft. dip is stiffened by a three-hinged girder. The dead load is $\frac{1}{4}$ ton per foot run. Determine the maximum tension in the cable and the maximum bending moment in the girder due to a concentrated load of 5 tons crossing the span, assuming that the whole of the dead load is carried by the cable without stressing the girder. Find the bending moment in the girder at $\frac{1}{10}$ span from either pier when the concentrated load is 25 ft. from the left-hand pier.

Tensions in the cable due to the dead load:

$$\text{Horizontal tension} = \frac{1}{4} \times \frac{100^2}{8 \times 10}$$

$$= 31{\cdot}25 \text{ tons.}$$

The vertical reaction at the support is

$$12{\cdot}5 \text{ tons} = \frac{wl}{2} = \frac{100}{4 \times 2}$$

Due to the live load of 5 tons on the girder, H is a maximum when the load is at the centre of the hinged girder, i.e. at 50 ft. from the support points.

H_{max} due to the live load

$$= \frac{5 \times 100}{4 \times 10} = 12{\cdot}5 \text{ tons.}$$

Vertical reaction at the support due to the live load is 2·5 tons.

Maximum tension in the cable

$$= \sqrt{(12{\cdot}5)^2 + (31{\cdot}25)^2} + \sqrt{(2{\cdot}5)^2 + 12{\cdot}5^2} = 46{\cdot}3 \text{ tons.}$$

MAXIMUM MOMENT IN THE GIRDER DUE TO THE LIVE LOAD. Draw the unit influence moment line for a section X distance x from the left-hand end of the girder. (Fig. 164.)

y = dip of the cable at the section X

y_c = maximum dip = 10 ft.

$$y = \frac{4y_c}{l^2} x(l-x) \quad \text{(equation for the cable)}$$

$$\left(\frac{ly}{4y_c} \text{ reduces to } \frac{x(l-x)}{l}\right)$$

---

* Problem given for solution in Morley's *Theory of Structures.*

To find the sections at which the largest negative and positive moments occur; the maximum negative moment for any section occurs when the load is at X, i.e. $nl = x$; and the maximum positive moment when $nl = \frac{l}{2}$

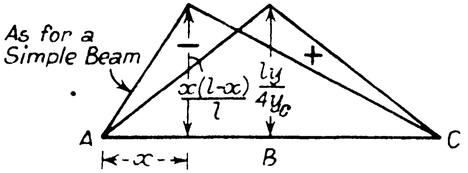
Unit negative moment at X (Fig. 164).

$$(= -M_s + Hy)$$

$$= -\frac{x(l-x)}{l} + \left[\frac{x(l-x)}{l} \times \frac{2x}{l}\right] \text{ (using equation (54) for } H)$$

$$-M_x = \frac{-xl}{l} + \frac{x^2}{l} + \frac{2x^2l}{l^2} - \frac{2x^3}{l^2}$$

$$\text{or } -M_xl^2 = (-xl^2 + x^2l + 2x^2l - 2x^3) \quad . \quad . \quad . \quad (64)$$



Note - $\frac{ly}{4y_c}$ reduces to $\frac{x(l-x)}{l}$ for Parabolic Cable

FIG. 164

To find the section having the biggest negative moment

$$\frac{d(-M_xl^2)}{dx} = -l^2 + 6xl - 6x^2 = 0$$

$$x = ·5l \pm ·289l$$

$$= ·211l \text{ or } ·789l.$$

Substituting in (64) the biggest negative moment = ·096l, per unit load.

The maximum positive moment for any section = $Hy - M_s$

$$= +M_x = \frac{x(l-x)}{l} - \frac{x}{2} = \frac{2xl - 2x^2 - xl}{2l}$$

$$= \frac{xl - 2x^2}{2l} \quad . \quad . \quad . \quad (65)$$

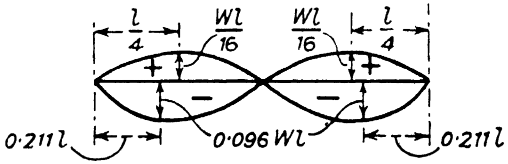
To find the section having the largest positive moment

$$\frac{d(+M_x 2l)}{dx} = l - 4x = 0$$

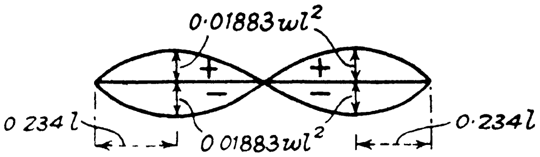
$$x = \frac{l}{4}$$

and also for $x = \frac{3l}{4}$ by similarity.

Substituting in (65) the biggest positive moment per unit load $= \frac{l}{16}$



(a) Single Load W



(b) Uniformly Distributed Load w

Fig. 165

Maximum negative moment due to the 5-tons load
$= \cdot096 \times 500 = -48$ tons-ft.,
for sections 21·1 and 78·9 ft. from the left-hand end of the girder.

Maximum positive moment

$$= \frac{500}{16} = +\ 31{\cdot}2 \text{ tons-ft.,}$$

for sections 25 and 75 ft. from the left-hand end of the girder.

The curves of maximum positive and negative moment for any section due to a load W are shown in Fig. 165 (*a*) and the curves for maximum positive and negative moments due to a uniformly-distributed load of w tons longer than the span, for any section, are given in Fig. 165 (*b*).*

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* These curves of maximum positive, and negative moment (Fig. 165 (*a*) and (*b*)) also apply to the *parabolic* three-hinged arch.

The student is requested to show that the biggest positive and negative moments for the girder which are of the same value $\cdot 01883\, wl^2$, are those for the sections $\cdot 234l$ and $\cdot 766l$ from the origin (the left-hand hinge).

Second Part of the Problem. (Fig. 166.)

Moment at 10-foot section when 5 tons at 25 ft. from O

$$= \left(-\frac{9 \times 75}{90} + 4\cdot 5\right) 5 = -15 \text{ tons-ft.}$$

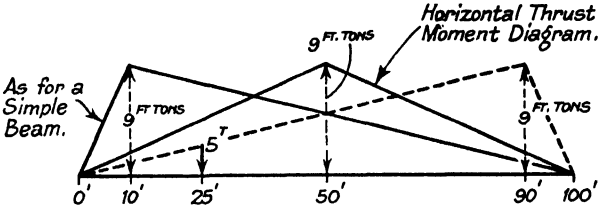

FIG. 166

Moment at 90-foot section when 5 tons at 25 ft. from O

$$= \left(+ 4\cdot 5 - \frac{9}{90} \times 25\right) 5 = +10 \text{ tons-ft.}$$

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EXAMPLES

1. A three-hinged segmental arch of 100-ft. span, rise 30 ft., is loaded with a load of 30 tons at a point 20 ft. from the centre of the arch. Draw the bending moment diagram for the arch, and find—

(1) The horizontal thrust in the arch ;
(2) The resultant thrusts at the abutments ;
(3) The maximum bending moment and the point at which it occurs.

(U. of B.)

2. The axis of an arch rib is parabolic ; it has three hinges, the span is 90 ft., and the rise at the centre 16 ft. There is a concentrated load of 9 tons at a point 30 ft. from the crown of the arch, measured horizontally. Find—

(*a*) The axial thrust, and the normal shear at the point of application of the load.

(*b*) The maximum bending moments—positive and negative. (U. of L.)

3. In a suspension bridge consisting of a cable supported on towers and stiffened by girders hinged at the centre, find an expression for the maximum bending moment due to a single concentrated load on the bridge at one-quarter the span. (I.C.E.)

4. A three-pinned segmental arch has a span of 150 ft. and a rise of 30 ft. A load of 10 tons rolls over the arch. Determine—

(1) the maximum horizontal thrust in the arch ;

(2) the maximum bending moment at a point 50 ft. from one abutment, and the resultant thrust and radial shear in the arch at the abutment when this maximum bending moment occurs. (U. of L.)

5. Explain what you understand by "link polygon" and "reciprocal figures." Six equal weights of 1 ton, placed at equal horizontal distances of 6 ft. apart in one plane, are supported by a link polygon of seven bars. The end bars are inclined at 60° to the vertical. Find by a graphical construction the forces in the bars, and mark the magnitude of the stresses on the corresponding bars of your sketch.

6. A footbridge, 8 ft. wide, has to be supported across a river 75 ft. wide by means of two cables of uniform cross-section. The dip of the cables at the centre is 9 ft., and the maximum load on the platform is to be taken as 140 lb. per square foot of platform area. The working stress in the cables is not to exceed 5 tons per square inch, and the steel from which the cables are made weighs 0·28 lb. per cubic inch. Determine a suitable cross-sectional area for these cables.

7. Each chain of a suspension bridge of 120 ft. span has a dip of 12 ft. and carries a dead load of 600 lb. per horizontal foot run. The chain is stiffened by a horizontal three-pinned girder. A uniform travelling load longer than the span traverses the girder and is 300 lb. per foot run per chain. Determine the maximum positive and negative bending moments in each half of the girder, and the positions of the load at which these occur. Find also the maximum pull in the chain. (U. of L.)

8. Question 7 (*cont.*). Determine the maximum shear forces in each half of the girder and the positions of the load at which these occur.

9. A three-pinned segmental arch has a span of 120 ft. and a rise of 30 ft. The arch is loaded with 2 tons per horizontal foot run over the whole span, and with an additional load of 1 ton per foot run over half the span only from one abutment. Find the maximum bending moment in the arch rib and the horizontal thrust. (U. of L.)

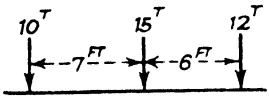


Fig. 167

10. A three-pinned parabolic arch has a span of 120 ft. and a rise of 40 ft. The loads shown in the diagram (Fig. 167) rest on the arch with the load of 15 tons over the central pin. Draw the diagram of bending moment for the arch ; determine the horizontal thrust and the maximum bending moment. (U. of B.)

11. Let the loads in Question 10 move over the span. Find—

The maximum bending moment at a section distant 50 ft. from the left-hand support, and the resultant thrust at this section when the maximum moment occurs.

12. A bridge has 6 segmental three-pinned arches 7 ft. apart, the span being 150 ft. and the rise 30 ft. The dead load is 200 lb. per square foot, and the live load equivalent to 400 lb. per square foot. What is the horizontal thrust when the span is half covered by the live load. If the plate girder ribs are 4 ft. deep over angles, flange plates 18 in. × 1 in., angles 4 in. × ½ in., web plate ½ in., calculate approximately the maximum stress in the rib. (*See* Chapter VI for combination of bending and direct stresses.) (I.S.E.)

13. Taking the same bridge as in Question 7, determine the maximum positive and negative bending moments in each half of the girder when two loads of 2 tons each, 6 ft. apart, cross the girder. Also find the maximum shear forces in each half of the girder.

14. A suspension bridge has a span of 600 ft. and a dip of 50 ft., and carries by means of two cables a total load of 100 tons uniformly distributed along the length of the platform. Assuming the hanging rods to be very numerous, determine the tension in each cable at the lowest point and at the piers. The cables are attached to saddles resting upon rollers on the tops of the piers, and the anchor cables make an angle of 30° with the vertical. Determine the maximum stress in the anchor cables and the total vertical pressure on each of the piers.

15. A three-hinged arched rib has hinges at the abutments A and B and at the crown C. C and B are respectively 35 ft. and 60 ft. from the vertical through A, and respectively 11 ft. and 5 ft. higher than A.

Construct the influence line for the horizontal thrust H for unit load crossing the span. Using this line, determine the magnitude of H produced by a uniformly distributed load of 1000 lb. per ft. extending over the whole of the span.

Note. The reactions at the hinges A and B will consist of the vertical reactions R_A and R_B as for a simply supported beam and a thrust H_1 in the directions A to B and B to A. Show that the horizontal component H of $H_1 = R_A \cdot \frac{a}{f}$ for the unit load between C and B where $a = 35$ ft. in the problem and f = vertical rise from line AB to hinge C. Similarly $H = R_B \cdot \frac{b}{f}$ for unit load between C and A. Continue problem as for case where abutment hinges at the same level.

CHAPTER XII

PRINCIPAL STRESSES

159. **Oblique Stresses.** Referring to Fig. 168, let a small prism of cross-section a sq. in. be under an intensity of tensile stress p in the direction of its length. This is normal to the section $A'B'$. It is required to find the stress intensity normal and tangential to AB, a section at an angle θ to $A'B'$.

Area $AB = A'B' \sec\theta = a \sec\theta$.

P = Total pull on the prism, normal to $A'B'$

P_n = Total pull on the section AB

P_t = Total shear force on the section AB.

$$P_n = P\cos\theta$$

$$p_n = \frac{P_n}{a\sec\theta} = \frac{P}{a}\cdot\frac{\cos\theta}{\sec\theta} = p\cos^2\theta \qquad . \qquad . \qquad (1)$$

$$P_t = P\sin\theta$$

$$p_t = \frac{P}{a}\cdot\frac{\sin\theta}{\sec\theta} = p\sin\theta\,.\cos\theta \qquad . \qquad . \qquad . \qquad (2)$$

p_t is a maximum when $\sin\theta = \cos\theta$, i.e. when $\theta = 45°$.

Then $p_t = p\cos^2\theta$,

and $p_n = p_t = \frac{p}{2}$. (3)

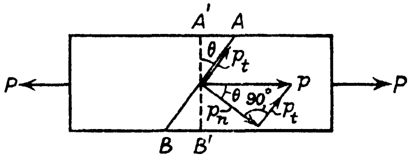


FIG. 168

When p_t is a maximum, θ is 45°; that is, the maximum shear stress occurs on planes inclined at an angle of 45° to the plane $A'B'$, the plane for which p is a normal.

Illustrative Problem 40.

Construct the polar diagram for the stresses on a plane oblique to a direct stress p. (*See* paragraph 159.)

θ = angle between direction of p and the normal to the plane;

or = angle between the plane of direct stress p and the other plane.

Tangential stress $= p_t = p \sin\theta \, . \cos\theta$

Normal stress $= p_n = p \cos^2\theta$

As shown in Fig. 169, take the plane as vertical on which p is normal. From a centre O, draw radiating lines at angles of 15, 30, 45, 60, etc., to 180° to this vertical; and along these lines to scale mark off the corresponding values of p_t: the vectors p_n are drawn normal to the radiating lines; join up the ends of the vectors required by a smooth curve and the required diagrams are obtained, which are given in Fig. 169.

Table for the Problem, (Fig. 169)

θ	$\cos\theta$	$\cos^2\theta$	$\sin\theta$	$\sin\theta \times \cos\theta$
0	1·0	1·0	0	0
15	·97	94	26	25
30	·87	·76	50	·44
45	71	·50	71	50
60	·50	·25	·87	·44
75	·26	·068	·97	25
90	0	0	1·0	0
105	–·26	·068	·97	–·25
120	–·50	·25	·87	–·44
135	–·71	·50	71	–·50
150	–·87	·76	·50	–·44
165	–·97	·94	·26	–·25
180	1·0	1·0	0	0

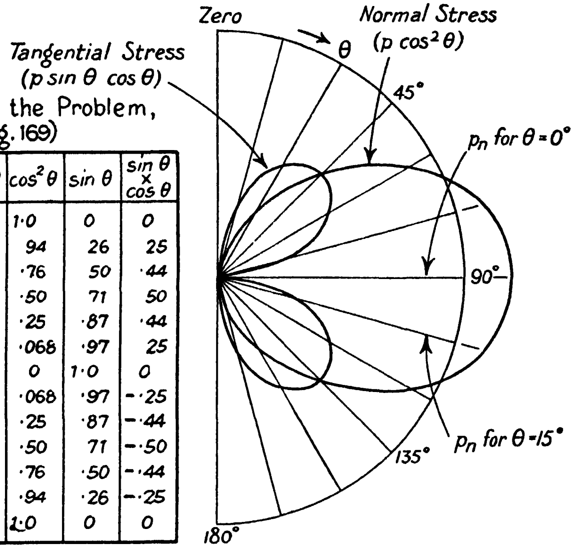


FIG. 169

160. **Principal Stresses.** If a body is under a complex system of stresses, these stresses may be resolved into three simple normal tensile or compressive stresses in planes at right angles to each other. These simple stresses are called Principal Stresses and the planes are called Principal Planes. The direction of the principal stresses are called the Axes of Stress.

In many cases, one of the principal stresses is zero or negligibly small, and consequently there are only two principal stresses to be considered, and these will act in the same plane.

161. EXAMPLE. (Figs. 170, A and B.)

Let us consider principal planes and stresses when complementary shear stresses are accompanied by a normal stress on the plane of one shear stress. (For Complementary Shear Stresses, *see* para. 57, Chapter V.)

Fig. 170A shows the forces acting on a rectangular block *ADEC* of unit thickness perpendicular to the plane of the paper, and of indefinitely small dimensions parallel to the figure (unless the stresses are uniform). Let α be the inclination of a principal plane *BC* to the plane *AC* which has a normal unit stress p and a unit shear stress of q acting on it, and let R be the unit stress wholly normal on *BC*. The face *AD* has only the unit stress q acting tangentially to it.

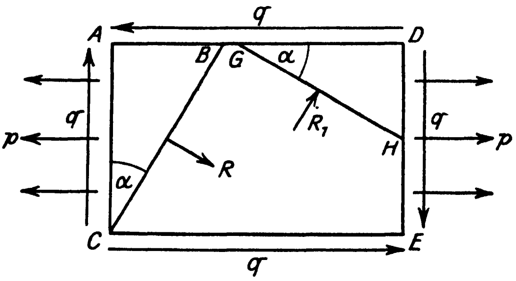


FIG. 170A

Problem.

To find the principal stresses and planes.

Consider the equilibrium of the wedge *ABC* (Fig. 170B).

Let p and q act on a plane, cutting the plane of the paper in *AC*, and let *BC* represent similarly one of the planes of principal stress. Let the intensity of this principal stress be R.

Then if *AB* is at right angles to *AC*, we have acting along *AB* (since for every shear stress there is an equal and opposite shear stress at right angles) a shear stress of intensity q.

Let the width of the wedge at right angles to the plane of the paper be unity.

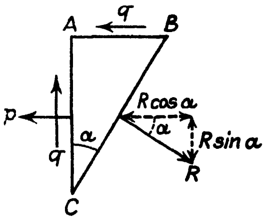


FIG. 170B

Resolving along AB and AC,

$$p \,.\, AC + q \,.\, AB = R \,.\, BC \cos a \quad . \quad . \quad . \quad (4)$$

$$qAC = R \,.\, BC \sin a \quad . \quad . \quad . \quad (5)$$

From (4),

$$p \cdot \frac{AC}{BC} + q \cdot \frac{AB}{BC} = R \cos a$$

$$p \cos a + q \sin a = R \cos a \quad . \quad . \quad . \quad (6)$$

$$\text{or } p + q \tan a = R$$

From (5),

$$q \cdot \frac{AC}{BC} = R \sin a$$

$$q \cos a = R \sin a \quad . \quad . \quad . \quad (7)$$

$$\text{or } q \cot a = R$$

From (6) and (7),

$$q \tan a \times q \cot a = R(R - p)$$

$$\text{or } R(R - p) = q^2 \quad . \quad . \quad . \quad . \quad (8)$$

$$R^2 - pR - q^2 = 0$$

$$R = \frac{p}{2} \pm \frac{\sqrt{p^2 + 4q^2}}{2}$$

$$= \frac{p}{2}\left(1 \pm \sqrt{1 + \frac{4q^2}{p^2}}\right) \quad . \quad . \quad (9)$$

The negative sign of the root corresponds to the second principal stress R_1 and the plus sign to the maximum principal stress R. R_1 is at right angles to R, and acts on a plane GH at right angles to the plane BC (see Fig. 170A).

The direction of the plane on which the stress R acts is—
From (6),

$$R \cos a - p \cos a = q \sin a$$

From (7),

$$R \sin a = q \cos a$$

$$\therefore \ R = q \frac{\cos a}{\sin a}$$

$$q \frac{\cos^2 a}{\sin a} - p \cos a = q \sin a$$

$$q = \frac{p \cos a \sin a}{\cos^2 a - \sin^2 a} = p \cdot \frac{\frac{1}{2} \sin 2a}{\cos 2a}$$

$$\text{Thus } \tan 2a = \frac{2q}{p} \qquad . \quad . \quad . \quad (10)$$

The planes of maximum shear are inclined at an angle of 45° to the principal planes, and the intensity of the maximum shear stress is

$$q_{max} = \sqrt{\frac{p^2}{4} + q^2} \qquad . \quad . \quad . \quad (11)$$

For the shear stress on a plane at 45° to the principal plane of R will be

$$\frac{R}{2} = \frac{p}{4}\left(1 + \sqrt{1 + \frac{4q^2}{p^2}}\right) \text{ from Equation (9)}$$

Similarly, the shear stress on a plane at 45° to the plane of R_1

$$= \frac{p}{4}\left(1 - \sqrt{1 + \frac{4q^2}{p^2}}\right)$$

The shear stress due to R_1 is of the opposite sign of that due to R,

$$\text{therefore } q_{max} = \frac{p}{4}\left(1 + \sqrt{1 + \frac{4q^2}{p^2}}\right) - \frac{p}{4}\left(1 - \sqrt{1 + \frac{4q^2}{p^2}}\right)$$

$$= \frac{p}{2}\sqrt{1 + \frac{4q^2}{p^2}} = \sqrt{\frac{p^2}{4} + q^2} \qquad Q.E.D.$$

In the example taken, p is a tensile stress : the same result holds for p as a compressive stress.

162. **Two Perpendicular Stresses.** CASE I. *Like Forces.* Let a block of material *ABCD* (Fig. 171) of unit thickness be subjected to Two Pulls, P_x and P_y at right angles to one another, and let p_x, p_y be the stresses per unit area across sections normal to these forces.

Let p_x be $> p_y$.

NOTE.—p_x and p_y are two principal stresses.

Let *EF* be any section making an angle θ with the direction of the force P_x.

The normal component of P_x across $EF = P_x \,.\, \sin\theta$

Normal stress on EF due to $P_x = \dfrac{P_x}{EF} \,.\, \sin\theta$

$$= p_x \cdot \frac{C_1F}{EF} \,.\, \sin\theta = p_x \sin^2\theta \quad . \quad . \quad . \quad (12)$$

Similarly the tangential stress on EF due to P_x

$$= \frac{P_x \cos\theta}{EF} = p_x \frac{C_1F}{EF} \,.\, \cos\theta$$

$$= p_x \sin\theta \,.\, \cos\theta \quad . \quad . \quad . \quad . \quad (13)$$

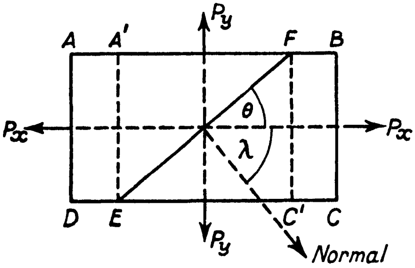


FIG. 171

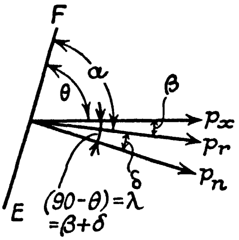


FIG. 172

Normal component of stress on EF due to P_y

$$= \frac{P_y}{EF} \cdot \cos\theta = p_y \cdot \frac{A'F}{EF} \cdot \cos\theta$$

$$= p_y \cos^2\theta \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Similarly the tangential stress on EF due to P_y

$$= - p_y \sin\theta \,.\, \cos\theta \quad . \quad . \quad . \quad (15)$$

It is negative, since this tangential force is of the opposite sign of that due to P_x.

The total normal component of the stress on EF

$$= p_n = p_x \sin^2\theta + p_y \cos^2\theta \quad . \quad . \quad . \quad (16)$$

The total tangential component of the stress on EF

$$= p_t = (p_x - p_y) \sin\theta \,.\, \cos\theta \quad . \quad . \quad . \quad (17)$$

The resultant of these components is

$$p_r = \sqrt{p_n^2 + p_t^2}$$

$$= \sqrt{(p_x \sin^2\theta + p_y \cos^2\theta)^2 + (p_x - p_y)^2 \sin^2\theta \,.\, \cos^2\theta}$$

$$= \sqrt{p_x^2 \sin^2\theta + p_y^2 \cos^2\theta} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If p_r makes an angle a with EF (Fig. 172),

$$\text{then}\quad \tan a = \frac{p_n}{p_t}$$

$$= \frac{p_x \sin^2 \theta + p_y \cos^2 \theta}{(p_x - p_y) \sin \theta \,.\, \cos \theta} = \frac{p_x \tan^2 \theta + p_y}{(p_x - p_y) \tan \theta} \quad . \quad . \quad (19)$$

If δ is the angle p_r makes with p_n, then

$$\tan a = \cot \delta$$

$$\text{or}\quad \cot a = \tan \delta$$

If β is the angle between p_r and P_x, then

$$\beta = a - \theta$$

$$\tan \beta = \tan(a - \theta) = \frac{\tan a - \tan \theta}{1 + \tan a \,.\, \tan \theta} \quad . \quad . \quad (20)$$

Substituting the value of $\tan a$ from (19) and working out,

$$\tan \beta = \frac{p_y}{p_x} \cot \theta \quad . \quad . \quad . \quad (21)$$

$$p_t = (p_x - p_y) \sin \theta \,.\, \cos \theta$$

$$= \frac{p_x - p_y}{2} \,.\, \sin 2\theta \quad . \quad . \quad . \quad (22)$$

and is a maximum when $2\theta = 90°$,

$$\text{thus}\quad p_{t\,max} = \frac{p_x - p_y}{2} \quad . \quad . \quad (23)$$

and acts on the plane making an angle of 45° with p_x.

163. **In the Case of Two Unlike Forces P_x and P_y**, that is, say, p_x tensile and p_y compressive, the above results hold good with a change in the sign of p_y.

In this case the maximum shear or value of p_t will be

$$p_{t\,max} = \frac{p_x + p_y}{2} \quad . \quad . \quad . \quad (24)$$

If $p_x = p_y$: the maximum p_t is on the plane at 45° to p_x and the corresponding $p_n = 0$, ... (25) these results corresponding exactly with the case of pure shear.

163*a*. (i) From equation (16), if $p_y = p_x$ and of the same sign, then $p_n = p_x$.

And from equation (18), $p_r = p_x \;\therefore\; p_r = p_n = p_x$.

Also $a = 90°$ and $\delta = 0°$.

From equation (17), if $p_y = p_x$, then $p_t = 0$.

Therefore, if the two principal stresses at a point are of like sign and of equal magnitude, then the stress on a third plane through the point is of the same intensity and is normal to the plane. These stresses are called "fluid" stresses. If the third plane makes an angle θ with the direction of p_x, then the normal and resultant stress on the third plane makes the angle θ with the plane (principal) on which p_x acts. If p_x is horizontal, then the angle θ is made with reference to the vertical through the point considered. (*See* (*A*), Fig. 172A.)

(ii) Also if $p_y = p_x$ but of unlike sign.

Then $p_r = p_x = p_y$ in magnitude.

From equation (19)

$$\tan a = \frac{p_x \tan^2 \theta - p_x}{2p_x \tan \theta} = \frac{\tan^2 \theta - 1}{2 \tan \theta} = -\cot (2\theta)$$

But $\tan a = \cot \delta = -\cot 2\theta$

$\therefore\ \delta = (-2\theta)$

Therefore, p_r will make an angle (-2θ) with the normal to its plane, which makes an angle θ with the direction of p_x.

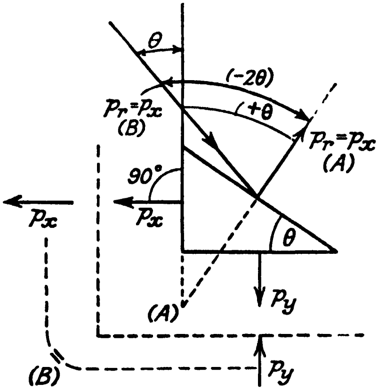


FIG. 172A

The interpretation of the above is as follows—

If a pair of principal stresses at a point be unlike (one tension and one compression) and be of equal intensity, then the resultant on any plane through the point is of the same intensity, and is inclined to the normal to the direction of p_x at an angle θ, but on the opposite side to the resultant obtained in (i) when $p_x = p_y$ of same sign. (*See* (*B*), Fig. 172A.)

164. EXAMPLE. (Refer to para. 162.)

Find the plane across which the resultant stress is most inclined to the normal, when $p_x > p_y$ but of the same sign. Here the angle a the resultant makes with the plane will be a minimum.

Let $\delta_{max} = \phi$ = maximum inclination of the resultant stress to the normal.

From equation (19), page 295, $\tan\delta = \cot\alpha$

$$= \frac{(p_x - p_y)\cos\theta \,.\, \sin\theta}{p_x \sin^2\theta + p_y \cos^2\theta} \quad (26)$$

when δ is a maximum, then $\tan\delta$ is a maximum ; so that $\dfrac{d(\tan\delta)}{d\theta}$ has to be equal to zero.

Differentiating and simplifying

$$(p_x \sin^2\theta + p_y \cos^2\theta)\cos 2\theta - (p_x - p_y)\cos\theta \,.\, \sin\theta \,.\, \sin 2\theta = 0$$

$$\text{Then } p_n \cos 2\theta - p_t \sin 2\theta = 0$$

$$\text{therefore } \tan 2\theta = \frac{p_n}{p_t} = \tan\alpha = \cot\delta_{max} = \cot\phi$$

$$\text{Now } \cot\phi = \tan\left(\frac{\pi}{2} - \phi\right)$$

$$\therefore\ 2\theta = \frac{\pi}{2} - \phi$$

$$\theta = \frac{\pi}{4} - \frac{\phi}{2} \text{ or } \frac{\phi}{2} = \frac{\pi}{4} - \theta \quad . \quad . \quad (27)$$

a relation between θ and ϕ when the maximum conditions hold.

Substituting the value of θ from (27) in (26),

$$\tan\phi = \frac{(p_x - p_y)\cos\phi}{p_x(1 - \sin\phi) + p_y(1 + \sin\phi)}$$

which, on simplifying, gives

$$\frac{p_x}{p_y} = \frac{1 + \sin\phi}{1 - \sin\phi} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$\text{or } \sin\phi = \frac{p_x - p_y}{p_x + p_y} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

From (29), ϕ can be found : Substituting in (27), θ can be found, thus giving the plane required.

The results obtained are used in the theory of retaining walls.

Equation (29) gives the maximum inclination to the normal, and Equation (27) gives the inclination of the plane to the direct stress p_x.

Let λ = inclination of the normal to the direct stress p_x.

$$\text{Then } \lambda = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \frac{\pi}{4} + \frac{\phi}{2}$$

$$\lambda = \frac{\pi}{4} + \frac{\phi}{2} \quad . \quad . \quad . \quad . \quad (30)$$

Equation (29) leads to a graphical method for finding the value of the resultant when $\sin \phi$ is known, and also for finding the principal stresses given the angles two resultant stresses make with their respective normals. (*See* Problem 41.)

165. **Ellipse of Stress.** (1) LIKE STRESSES. Draw two concentric circles as in Fig. 173, whose radii are respectively equal to p_x and p_y, (p_x and p_y are principal stresses) and $p_x > p_y$

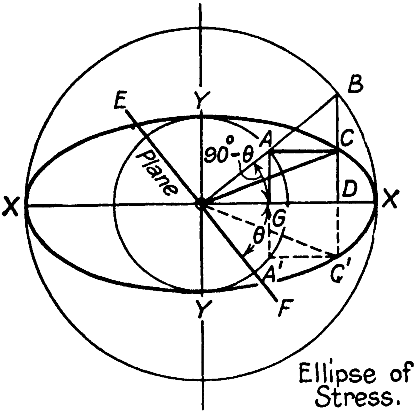


FIG. 173

$$OX = p_x \qquad OY = p_y$$

and let *EOF* be the plane inclined at an angle θ to *OX*; and *OAB*, the normal inclined at $(90 - \theta)$ to *OX*.

Draw *BD* and *AG* perpendicular to *OX*, *AC* perpendicular to *BD*, and join *OC*.

It can be shown that

$$OC = \sqrt{OD^2 + AG^2} = \sqrt{p_x{}^2 \sin^2\theta + p_y{}^2 \cos^2\theta} = p_r$$

$$\text{and that angle} = DOC = \beta$$

$$\text{and angle } COF = \alpha$$

It follows that C is on an ellipse whose major axis is $2p_x$ and whose minor axis is $2p_y$, because

$$\frac{OD^2}{p_x^{\,2}} + \frac{AG^2}{p_y^{\,2}} = \sin^2\theta + \cos^2\theta = 1$$

This ellipse is called the Ellipse of Stress. From it can be obtained the resultant stress on any plane, for by drawing OB at right angles to the given direction and BD parallel YY, to meet the ellipse in C, then OC is the resultant stress on the required plane in magnitude and direction, and the angle COF is the angle between this resultant stress and the given plane.

(2) UNLIKE STRESSES. Let p_x be negative, p_y positive. The construction is similar as before to drawing OAB. From A, drop a perpendicular to cut the circle p_y on the opposite side in A'. Drop a perpendicular from B, and draw $A'C'$ perpendicular to it to meet in C'. Join OC'. Then OC' is the resultant stress on the plane OF, and C' lies on an ellipse which is the Ellipse of Stress, for here $\tan\beta$ is negative.

CIRCLE OF STRESS. (*See* (*a*), Fig. 173A.) In the cases when the principal stress intensities p_x and p_y are of equal magnitude, the ellipse of stress becomes a circle. If they are of the same sign, the resultant stress OP on any and every oblique plane EF perpendicular to the figure is normal to that plane and equal in magnitude and sign to the stresses $p_x = p_y$ (*see* para. 163*a* (i)).

If the stress p_y is of opposite sign to p_x, then the resultant stress (OP_1) on any oblique plane EF perpendicular to the figure is of magnitude $p_x = p_y$ but makes the angle (-2θ) with the normal to the plane and where θ is the angle the plane makes with the direction of p_x.

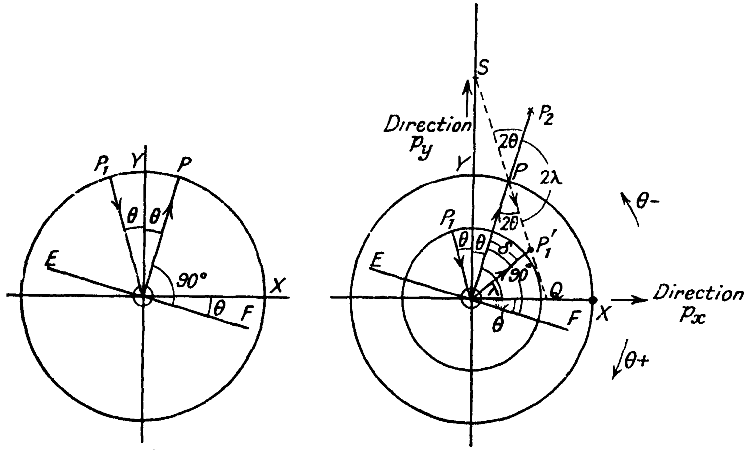
The case of unequal principal stresses of the same sign may be treated by the circle of stress by writing (when $p_x > p_y$)

$$p_x = \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2}$$

$$p_y = \frac{p_x + p_y}{2} - \frac{p_x - p_y}{2}$$

Every unit area of the plane EF is then subject to equal like normal stresses $\frac{p_x + p_y}{2}$ and to equal and opposite stresses $\frac{p_x - p_y}{2}$, but of opposite sign to $\frac{p_x + p_y}{2}$ (*See* para 163*a* (ii))

To find the resultant stress p_r in the plane EF it is necessary



(a) (b)

$OX = p_x = OY = p_y$.

$OP = p_x$ making angle θ with OY and normal to EF when $p_x = p_y$ and of the same sign. $OP_1 = p_x$ in magnitude, but makes angle (-2θ) with normal to plane EF when $p_x = -p_y$.

Let $OP = \frac{p_x + p_y}{2}$ Case (*a*)

$OP_1 = \frac{p_x - p_y}{2}$ Case (*b*)

where $p_x > p_y$ but of the same sign. From P draw PP_1' equal and parallel to OP_1, then OP_1' is the resultant stress p_r on EF in magnitude and direction. PP_1' makes the angle 2θ with OP. Produce OP to P_2.

The bisector of OPP_1' is parallel to OY and the bisector of $P_1'PP_2$ is parallel to OX

FIG. 173A

to find the resultant of the two stresses $\frac{p_x + p_y}{2}$ and $\frac{p_x - p_y}{2}$ which can be geometrically added together.

Referring to (*b*), Fig. 173A, $OP_1' = p_r$ in magnitude and direction.

If $P_1'P$ is produced to cut OY in S and PP_1' produced to cut OX in Q, it can be shown that

$$OP = SP = PQ = \frac{p_x + p_y}{2}$$

and therefore SQ is the diameter of a semicircle having P as centre.

Also, $SP_1' = p_x$ and $P_1'Q = p_y$.

Now as the plane EOF moves through all angles, the point P will describe a circle with centre O and radius $\frac{p_x + p_y}{2}$ and P_1' will describe a circle about P with a radius $\frac{p_x - p_y}{2}$, OP and PP_1' keeping equally inclined to the vertical OY but on opposite sides of it. The locus of the point P_1' is an ellipse which is the ellipse of stress for the point O, within the body or material at the point O.

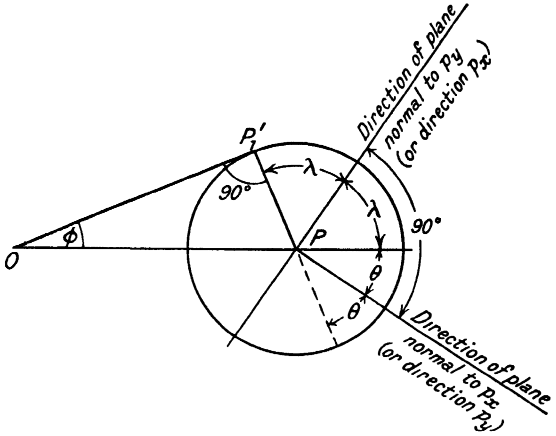


FIG. 173B

When $\theta = 0°$, $p_r = p_y$: when $\theta = 90°$, $p_r = p_x$.

The maximum value of $\delta = \phi$, the angle the resultant makes with the normal to the plane EOF, will be when p_r is tangential to the circle traced out by the point P_1' about P. (*See* Fig. 173B.)

$$PP_1' = \frac{p_x - p_y}{2}: \quad OP = \frac{p_x + p_y}{2}$$

$OP_1' = p_r$ making the angle $\delta = \phi$ with the normal to the plane. OP_1' is tangential to the circle of radius $\frac{p_x - p_y}{2}$.

From the right-angled triangle $OP_1'P$,

$$\frac{PP_1'}{OP} = \sin\phi = \frac{p_x - p_y}{p_x + p_y} \quad . \quad . \quad \text{Equation (29)}$$

$$\text{or } \frac{p_x}{p_y} = \frac{1 + \sin\phi}{1 - \sin\phi} \text{ or } \frac{p_y}{p_x} = \frac{1 - \sin\phi}{1 + \sin\phi}$$

$$\text{Also } 2\lambda = \frac{\pi}{2} + \phi$$

$$\text{or } \lambda = \frac{\pi}{4} + \frac{\phi}{2} \quad . \quad . \quad . \quad . \quad . \quad \text{Equation (30)}$$

$$\text{also } 2\theta = \pi - \frac{\pi - \phi}{2}$$

$$\therefore \theta = \frac{\pi}{4} - \frac{\phi}{2} \quad . \quad . \quad . \quad . \quad . \quad \text{Equation (27)}$$

Conjugate Stresses. (*See* Fig. 173C.)

Take an elementary prism represented by the face *ABCD*, of unit thickness normal to the plane *ABCD*. Let this be acted upon by the equal and opposite stresses p_r on the faces *AB*, *CD* and the equal and opposite stresses p_r' on the faces *AC*, *BD*. The direction of p_r is parallel to the faces *AC*, *BD* and that of p_r' parallel to the faces *AB* and *CD*. That is, the stresses p_r

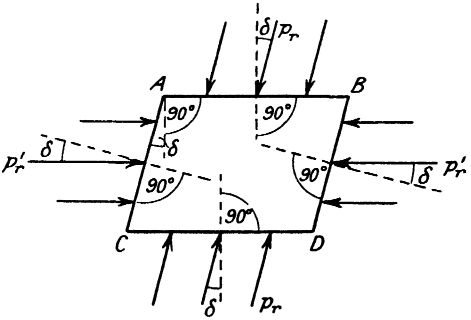


FIG. 173C

and p_r' all make the same angle, δ with the normal to their planes. The prism will be in equilibrium under the system of forces, and the stresses p_r and p_r' are known as conjugate stresses. Principal stresses as considered are a particular case of conjugate stresses.

Problem.

Let there be within a mass of material at a point O a pair of conjugate stresses p_r and p_r' acting on their respective planes. Let the angle δ be their obliquity with respect to the normals to their planes. Find the ratio of p_r and p_r'. Assume they are of the same sign.

Refer to Fig. 173D, draw any line ON; draw OP_1 making an angle δ with ON; make $OP_1 = p_r$ and OQ on $OP_1 = p_r'$.

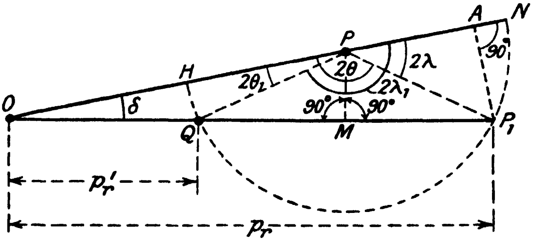


FIG. 173D

From M the middle point of QP_1 erect the perpendicular MP to cut ON in P. Draw QP, PP_1 and the semicircular arc HQP_1N.

$$OP \text{ represents } \frac{p_x + p_y}{2} \text{ and } PP_1, \frac{p_x - p_y}{2}$$

where p_x and p_y are the principal stresses of the system.

It can be shown that

$$OP = \frac{p_r + p_r'}{2\cos\delta} = \frac{p_x + p_y}{2}$$

$$\text{and } \frac{p_x - p_y}{2} = \sqrt{\left(\frac{p_r + p_r'}{2\cos\delta}\right)^2 - p_r p_r'}$$

$$\text{and } \left(\frac{p_x - p_y}{p_x + p_y}\right)^2 = 1 - \frac{4p_r p_r' \cos^2\delta}{(p_r + p_r')^2} \quad . \quad . \quad . \quad (30a)$$

$OA/OP_1 = \cos\delta$ and $\cos 2\lambda = (2p_r \cos\delta - p_x - p_y)/p_x - p_y$.

When the resultant makes the maximum angle $\delta = \phi$ with the normal

$$\text{then } \frac{p_x}{p_y} = \frac{1 + \sin\phi}{1 - \sin\phi} \text{ or } \frac{p_x - p_y}{p_x + p_y} = \sin\phi \quad . \quad . \quad (30b)$$

From equations (30*a*) and (30*b*) it can be shown that

$$\frac{p_r - p_r'}{p_r + p_r'} = \pm\sqrt{\frac{\cos^2\delta - \cos^2\phi}{\cos^2\delta}}$$

$$\text{Then } \frac{p_r}{p_r'} = \frac{\cos\delta \pm \sqrt{\cos^2\delta - \cos^2\phi}}{\cos\delta \mp \sqrt{\cos^2\delta - \cos^2\phi}} \quad . \quad . \quad . \quad (30c)$$

$$\text{also } OP_1 = OA \sec\delta = p_r \quad . \quad . \quad . \quad . \quad . \quad . \quad (30d)$$

An interpretation of this equation is given in para. 172A in the chapter on Retaining Walls.

Illustrative Problem 41.

At a point in a material subjected to two direct stresses on planes at right angles, the resultant stress on a plane *A* is 4 tons per square inch, inclined at 30° to the normal and on a plane *B* is 1 ton per square inch, inclined at 45° to the normal. Find the principal stresses, and show the position of the two planes *A* and *B* relative to the two principal stresses.

Let the stresses be of the same kind. Referring to Fig. 174. Draw $AB = 1$ ton/sq. in. at an angle of 45° to the line AD. Draw $AC = 4$ ton/sq. in. at an angle of 30° to the line AD.

Construct a circle to pass through B and C, and to have its centre O on the line AD.

$$\text{Then } AO = \frac{\text{sum of the principal stresses}}{2} = \frac{R_{max} + R_{min}}{2}$$

$$\text{and } OB = OC = \frac{\text{difference of the principal stresses}}{2}$$

$$= \frac{R_{max} - R_{min}}{2}$$

Scaling off from the figure,

$$\frac{R_{max} + R_{min}}{2} = 2{\cdot}72$$

$$\frac{R_{max} - R_{min}}{2} = 2{\cdot}13$$

from which $R_{max} =$ 4·85 tons per sq. in.

,, $R_{min} =$ ·59 ton per sq. in.

Let AF be the tangent to the circle of radius

$$\frac{R_{max} - R_{min}}{2}$$

Then AF is inclined to AO at an angle ϕ, which is the plane

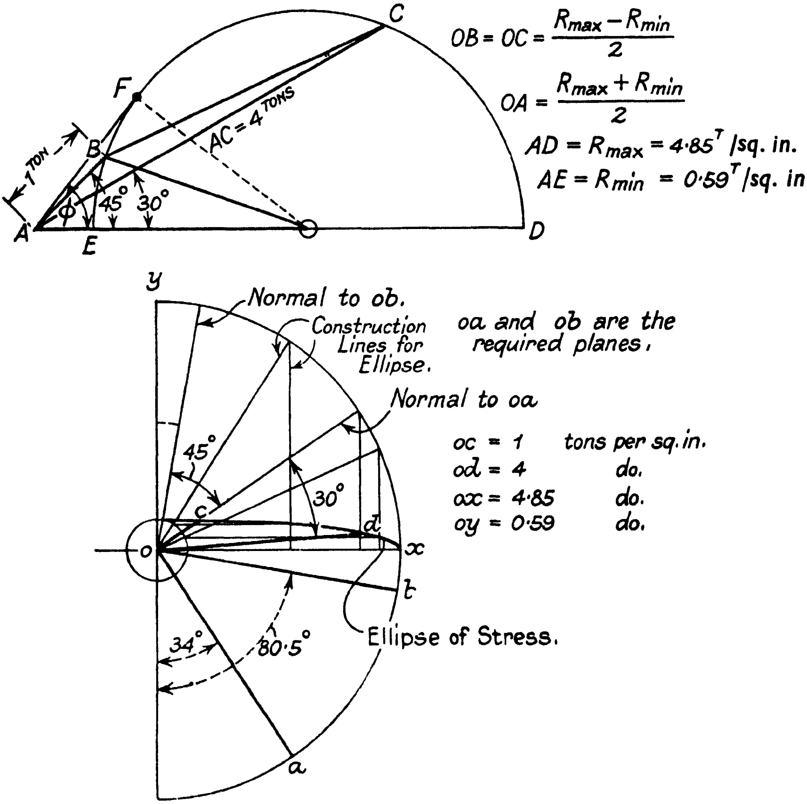


FIG. 174

on which the resultant is inclined at the possible maximum angle to the normal,

$$\text{for } \sin\phi = \frac{R_{max} - R_{min}}{R_{max} + R_{min}} \quad \text{(verifying Equation (29))}$$

The positions of the two planes required may be found by constructing the ellipse of stress. From the origin of the two axes, find where forces of 1 ton/sq. in. and 4 tons/sq. in. cut the ellipse; then the direction of the normal to the plane has been established, the plane being at right angles to the normal. (*See* Fig. 174.) Or the directions of the planes with reference to the principal maximum stress may be found by solving the necessary equations.

Using the Equation (17),

$$p_t = (R_{max} - R_{min}) \sin\theta \, . \cos\theta = p_r \sin\phi$$

p_r and the principal stresses are known, as well as ϕ; therefore θ can be found.

From which the plane of the

1 ton resultant is at 9·5° or (90 − 9·5°) = 80·5° to the maximum principal plane

and of the 4 ton resultant at 35° to the max. principal plane.

From the graphical method (Fig. 174), the angles are 80·5 and 34° respectively.

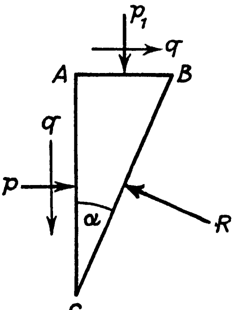


FIG. 175

166. EXAMPLE.

On two mutually perpendicular planes, normal stresses—one of intensity p and one of intensity p_1—act, in addition to two equal shear stresses of intensity q. Find the direction of the principal planes and the intensity of the principal (normal) stresses upon them.

As before, let *AB* and *AC* be the two planes at right angles. Let *BC* be a principal plane and *R* the maximum principal stress. Let the wedge *ABC* be of unit thickness. (*See* Fig. 175.)

Let *BC* be at an angle a to *AC*.

Let p and p_1 be two compressive stresses and q the intensity of the shear stresses.

Resolving along *AB* and *AC*,

$$p \, . AC + q \, . AB = R \, . BC \cos a \quad . \quad . \quad . \quad (A)$$

$$p_1 \, . AB + q \, . AC = R \, . BC \sin a \quad . \quad . \quad . \quad (B)$$

From (A), $p \cdot \frac{AC}{BC} + q \cdot \frac{AB}{BC} = R \cos a$

$$p \, . \cos a + q \, . \sin a = R \cos a \quad . \quad . \quad (31)$$

From (B), $p_1 \frac{AB}{BC} + q \cdot \frac{AC}{BC} = R \sin a$

$$p_1 \sin a + q \cos a = R \sin a \, . \quad . \quad . \quad (32)$$

From (31), $q \sin a = (R - p) \cos a$

$$q \tan a = R - p \qquad (33)$$

From (32), $q \cos a = (R - p_1) \sin a$

$$q \cot a = R - p_1 \qquad (34)$$

From (33) and (34),

$$q \tan a + p = q \cot a + p_1$$

$$q (\tan a - \cot a) = p_1 - p,$$

$$\text{or } q (\cot a - \tan a) = p - p_1$$

$$\text{or } \frac{2q}{\tan 2a} = p - p_1$$

$$\tan 2a = \frac{2q}{p - p_1} \qquad (35)$$

Solving Equation (35) for a will give the direction of the two principal planes, as $2a$ will have two values differing by 180°, and which will consequently give the inclination to AC of the two principal planes which are mutually perpendicular.

From (33) and (34),

$$(R - p)(R - p_1) = q \tan a \, . \, q \cot a$$

$$(R - p)(R - p_1) = q^2 \qquad (36)$$

$$R^2 - R(p + p_1) - (q^2 - pp_1) = O$$

$$R = \frac{p + p_1}{2} \pm \sqrt{\frac{(p - p_1)^2}{4} + q^2} \qquad (37)$$

Therefore $R_{max} = \frac{p + p_1}{2} + \sqrt{\frac{(p - p_1)^2}{4} + q^2} \qquad (38)$

and it is of the same sign as p and p_1.

$$R_{min} = \frac{p + p_1}{2} - \sqrt{\frac{(p - p_1)^2}{4} + q^2} \qquad (39)$$

and is of the opposite sign to p and p_1, if $q^2 > pp_1$.

These results are used in the analysis of the stresses in a dam given in Chapter XIII.

The planes of maximum shear stress are inclined at angles of $\frac{\pi}{4}$ to the principal planes found, and the maximum shear stress is

$$q_{max} = \frac{R_{max} - R_{min}}{2} = \sqrt{\frac{(p - p_1)^2}{4} + q^2} \qquad (40)$$

Note. If p is of the opposite to sign p_1, the modifications necessary are easily made by a substitution of the necessary signs in the preceding formula.

If $p_1 = O$,

$$R_{max} = \frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2}$$

$$\text{and } q_{max} = \sqrt{\frac{p^2}{4} + q^2}$$

both of which agree with the results obtained in Equations (9) and (11).

167. Having found the principal planes and stresses, and their directions, the ellipse of stress can be drawn, from which can be found the resultant stress on any plane. The maximum principal plane will be drawn at an angle a to the vertical; this fixes the axis YY of principal stress. The axis XX of principal stress for which $p_x = R_{max}$ measured is at right angles YY ; thus the position of the ellipse of stress will be fixed in space. Obviously the minimum principal plane makes an angle of $(90° + a)$ with the vertical.

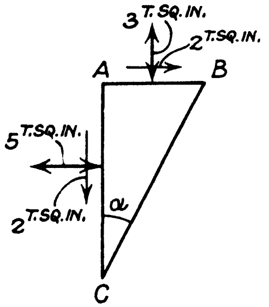


FIG. 176

Illustrative Problem 42.

At a point in a structural member there are two tensile stresses of 5 and 3 tons per square inch on two planes at right angles to each other, accompanied by a shear stress of 2 tons per square inch. Find the direction and magnitude of the principal stresses. (Fig. 176.)

Let a = angle the maximum principal plane makes with the 5-ton stress.

$$\tan 2a = \frac{2 \times 2}{5 - 3} = 2 \quad (\textit{See}\ \text{Equation (35)})$$

$$2a = 63° \, 26' \text{ or } 180 + 63° \, 26'$$

$$\therefore \ a = 31° . 43' \text{ and } a_1 = (90° + 31° \, 43') = 121°43'.$$

The maximum principal plane is at an angle of 31°43′ with the 5-ton stress plane and the minimum principal plane is at an angle of (a_1) 121°43′ with the 5-ton stress plane working from the vertical.

$$R = \tfrac{1}{2}(5 + 3) \pm \sqrt{\frac{(5 - 3)^2}{4} + (2)^2} \text{ (\textit{See} Equation (37))}$$

$$= 4 \pm \sqrt{5}$$

$$= 6{\cdot}236 \text{ or } 1{\cdot}764 \text{ tons/sq. in.}$$

$$R_{max} = 6{\cdot}236 \text{ tons/sq. in. (tension).}$$

$$R_{min} = 1{\cdot}764 \quad ,, \quad \text{(tension).}$$

REFERENCES

For further data regarding *The Theory of Stresses,* the student is requested to refer to works on *Strengths of Materials* and on *The Theory of Elasticity.*

Walls, Bins and Grain Elevators, by M. S. Ketchum. (McGraw-Hill.)

Strength of Materials, Vol. I. S. Timoshenko. Mohr's Circle.

Theory of Elasticity, Southwell.

EXAMPLES

1. Define principal planes and principal axes. If a specimen (cross-sectional area, A sq. in.) carries a tensile load of P tons, show that a maximum shear stress of $\frac{P}{2A}$ tons per square inch exists on a plane at 45° in the direction of the load. (U. of B.)

2. Define principal stresses and principal planes. At a point in a specimen there is a normal tensile stress of 5 tons per square inch on a certain plane, accompanied by a shear stress of 2 tons per square inch. Find the maximum principal stress and the angle the direction of this makes with the direction of the 5-ton tensile stress. (U. of B.)

3. In a bar subjected to pure tension, show graphically (e.g. "polar" diagrams) the magnitude of the normal and shear stresses on any plane inclined at an angle θ to a cross-section at right angles to the axis of pull, when θ varies from 0 to 2π. A bar, 1 in. diameter, is loaded with 5 tons. Determine the normal and shear stress in a plane inclined at 60° to the axis of the bar. (U. of B.)

4. The normal tensile stresses on two planes at right angles in a solid are 3 tons and 2 tons per square inch respectively, and the shear stress is 1 ton per square inch. Determine the principal stresses in direction and magnitude. (U. of B.)

5. Show that the two principal stresses at a point in a member are equal to half the sum of the normal stresses on any two planes at right angles through the point plus or minus the maximum intensity of shearing stress at that point. (I.C.E.)

6. A boiler is 6 ft. 6 in. diameter, ¾ in. thick, and is subjected to an internal pressure of 150 lb. per square inch, the ends being unstayed. Find the intensities of longitudinal and circumferential stress in the material. and of the normal and shearing stresses on a plane at 45° to the length of the cylinder. (I.C.E.)

7. At a point in a piece of steel there is a shear stress of 1 ton per square inch, and tensile stresses of 3 tons per square inch and 2 tons per square

inch respectively acting on planes at right angles. Determine graphically, or otherwise, the maximum principal stress, its direction, and the maximum shear stress. Draw the ellipse of stress. (U. of B.)

8. Prove that the sum of the normal components of the stresses at a point in a member on any two planes at right angles is a constant quantity for that point.

9. A system of loads is applied to a body and produces principal stresses at a certain point as follows: Tensile stress of 4 tons per square inch, acting on a horizontal section; compressive stress of 3 tons per square inch, acting on a vertical section. The system of loads is removed, and a second system is applied which produces at the same point principal stresses as follows: Tensile stress of 3 tons per square inch, acting on a section at 30° to the horizontal; compressive stress of 4 tons per square inch, acting on a section at 120° to the horizontal. All these sections may be taken at 90° to the plane of the paper. Find the principal stresses and the sections on which they act (showing them clearly in a diagram) when both systems of loads are applied simultaneously. (U. of L.)

10. At a point in a material under stress, the intensity of the resultant stress on a certain plane is 3 tons per square inch (tensile) inclined at an angle of 30° to the normal. The stress on a plane at right angles to this has a normal tensile component of intensity of 2 tons per square inch. Find (*a*) the resultant stress on the second plane; (*b*) the principal planes and stresses.

CHAPTER XIII

RETAINING WALLS AND GRAVITY DAMS

168. **A Retaining Wall** is one for sustaining the pressure of earth, or other filling or backing which possesses some frictional stability. The backing may be level with the top of the wall, or it may be sloped upwards from the wall when the backing is higher than the wall; in this case the wall is positively surcharged.

If the earth surface slopes downwards from the top of the wall, then the surcharge is a negative one. The pressure of the supported material will depend upon the material, the method of placing, moisture content, and other factors. It will be assumed that the materials are semi-fluids, possessing no cohesion, of indefinite extent, the particles being held in place by friction on each other. Loose earth will remain in equilibrium with its faces at slopes whose inclinations are less than an angle ϕ, which is called the *angle of repose*, or more properly the *angle of internal friction*. The coefficient of friction will be $\mu = \tan \phi$. Now, if a homogeneous, unlimited granular mass is in equilibrium, and if p_x and p_y are the two principal stresses at a point within the mass of the material, then the greatest angle which the resultant on a plane at the same point can make with the normal to the plane is ϕ, the angle of repose. The greatest ratio between p_x and p_y will be

$$\frac{p_y}{p_x} = \frac{1 - \sin \phi}{1 + \sin \phi}$$

To determine fully the pressure of the filling on a retaining wall it is necessary that the resultant pressure be known (*a*) in magnitude, (*b*) in line of action, and (*c*) in point of application. Theories for the design of retaining walls come into two classes—

(1) The Theory of Conjugate Pressures, due to Rankine, and known as *Rankine's Theory*; and

(2) The Theory of the Maximum Wedge, commonly known as *Coulomb's Theory*.

Rankine's theory completely determines the thrust in magnitude, direction and point of application. In Coulomb's theory, the magnitude of the thrust is ascertained, but the direction of it and its point of application must be assumed, thus leading to numerous solutions of more or less merit. Experimental work has been carried out within recent years, notably in Great Britain, by Professor Jenkin. The results of this work must be referred to in technical publications. The solution of the thrust of the filling for the simplest cases of retaining walls only are given in this chapter. For the graphical solutions using the ellipse of stress and the earth pressure triangle (*wedge theory*), reference should be made to more advanced works. (*See* list of references at the end of this chapter.)

169. **Theory of Rankine.** In this theory the filling is assumed to consist of an incompressible, homogeneous, granular mass, not possessing the property of cohesion or resistance to shear, the particles being held in position by friction on each other. The mass is of indefinite extent, having a plane surface, resting on a homogeneous foundation and being subjected to its own weight. These assumptions lead to the ellipse of stress and the development and use of formulae already found in the previous chapter. If a wall is vertical, then the pressure or thrust of the earth on the wall will be parallel to the top surface. The pressure on other than vertical walls can be determined from the construction of the particular ellipse of stress, although this method gives indeterminate values for some walls when they lean towards the filling. The earth face (i.e. the face of the wall in contact with the filling) of the wall is looked upon as a plane passing through points within the filling itself. The work given will determine the pressure actively exerted by the filling upon the wall which is less than the passive resistance which may be developed by pushing the wall against the earth.*

Conditions at the moment of failure when the retaining wall begins to slide. The space between the back of the wall and the earth filling as soon as the wall begins to slide is presumably filled up by a vertical fall of earth, which exerts a tangential effect μP, where μ is the coefficient of friction for the earth and wall, and P is the earth pressure normal to the wall. In Rankine's theory μP is not considered, and the retaining wall is made thus automatically a little more stable than is required.

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* See Ketchum, *Walls, Bins and Grain Elevators* (McGraw-Hill).

SURFACE OF THE EARTH BACKING HORIZONTAL WITH THE TOP OF THE WALL, BACK OF WALL VERTICAL. In Fig. 177, take a column of earth height h and having unit area. Neglecting friction on the sides of the column, and assuming the filling is subjected to its own weight, the pressure per unit horizontal area at depth h will be $w_e h$ where w_e = weight of 1 cu. ft. of earth. This pressure will be the maximum principal pressure, and consequently on a vertical surface, which will be a principal plane, there will be a minimum principal horizontal pressure. The problem is to find the intensity of this horizontal stress.

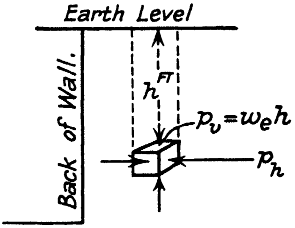

FIG. 177

Consider any plane intermediate between the two principal planes, then the condition that sliding shall just not take place is that a resultant pressure p_r on this plane shall just not make an angle with the normal = ϕ = angle of friction.

p_v is the maximum principal stress or pressure on a small cube of earth; p_h is then the minimum principal stress at right angles to p_v.

Now sliding will take place along some intermediate plane, on which the resulting stress is at an angle ϕ (the angle of repose) with the normal to that plane.

Equation (28), Chapter XII, shows that the relation between two principal stresses p_y and p_x for this condition is

$$p_x = p_y \frac{1 + \sin\phi}{1 - \sin\phi} \text{ where } p_x > p_y$$

Now $p_v > p_h$, so that changing over and substituting v for x and h for y,

$$p_v = p_h \frac{1 + \sin\phi}{1 - \sin\phi}$$

$$\text{and } p_h = w_e h \left(\frac{1 - \sin\phi}{1 + \sin\phi}\right) \quad . \quad . \quad . \quad (1)$$

$$\text{or } p_h = w_e h \,.\, \tan^2\left(45^\circ - \frac{\phi}{2}\right) \quad . \quad . \quad . \quad (2)$$

This is the horizontal intensity of pressure due to the earth on the back of the wall at a depth of h ft.

The total horizontal force on a wall of height h per unit length or run of wall is because p_h is proportional to the depth of a point,

$$P_h = \tfrac{1}{2}hw_e h\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)$$

$$= \tfrac{1}{2}w_e h^2\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right). \quad . \quad . \quad . \quad . \qquad (3)$$

and it acts at a depth of $\frac{2}{3}h$ from the top of the wall.

170. **Graphical Method of Finding p_h.** (Fig. 178.) O is the centre of the semicircle $CDBOC$. Produce BOC to some point A, from which the tangent AD to the semicircle makes an angle ϕ with ACB.

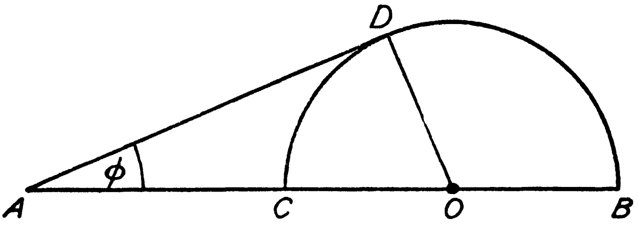

FIG. 178

Now AB will represent $p_v = w_e h$ and AC will represent $p_h = w_e h\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)$* to the same scale $p_v = AB$.

PROOF. $\dfrac{OD}{OA} = \dfrac{OD}{AB - OD} = \sin\phi$

then $OD = AB\sin\phi - OD\sin\phi$

$OD(1 + \sin\phi) = AB\sin\phi$

$$OD = AB \,.\, \frac{\sin\phi}{1 + \sin\phi}$$

$$AC = AB - 2OD$$

$$= AB - \frac{2AB\sin\phi}{1 + \sin\phi} = AB\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)$$

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* $OA = \dfrac{p_v + p_h}{2}$; $OD = \dfrac{p_v - p_h}{2}$

$AB = OA + OD = p_v$

$AC = OA - OD = p_h$

See Chap. XII.

If AB represents $w_e h$,

$$\text{then } AC = w_e h\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right) = p_h$$

171. **Sloping Back of Wall and Horizontal Earth Surface** (Fig. 179). Let θ be the slope of the back wall face with the vertical, the wall sloping away from the filling. Then the intensity of pressure across the face at a depth h is given by

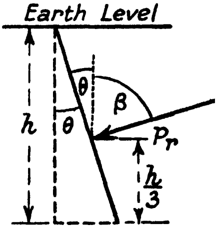

FIG. 179

$$p_r = \sqrt{p_x{}^2\sin^2\theta + p_y{}^2\cos^2\theta}$$

(Eqn. 18, Chap. XII)

where $p_x > p_y$.

Therefore, in this case of the retaining wall,

$$\text{where } w_e h \,.\, \frac{(1 - \sin\phi)}{1 + \sin\phi} = p_h = p_v$$

$$\text{and} \quad p_x = w_e h = p_v$$

$$p_r = \sqrt{p_v{}^2\sin^2\theta + p_h{}^2\cos^2\theta}$$

$$\therefore\ p_r = \sqrt{w_e{}^2h^2\sin^2\theta + w_e{}^2h^2\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)^2\cos^2\theta}$$

$$= w_e h\sqrt{\sin^2\theta + \cos^2\theta \,.\, \tan^4\left(45^\circ - \frac{\phi}{2}\right)} \quad . \quad (4)$$

Total P_r per foot-run of wall

$$= P_r = \tfrac{1}{2}w_e h \times \frac{h}{\cos\theta} \times \sqrt{\sin^2\theta + \cos^2\theta\,\tan^4\left(45^\circ - \frac{\phi}{2}\right)} \quad . \quad . \quad . \quad (5)$$

$$P_r = \tfrac{1}{2}w_e h^2\sqrt{\tan^2\theta + \tan^4\left(45 - \frac{\phi}{2}\right)} \quad . \quad . \quad (6)$$

and it acts at a point on the back of the wall at a depth of two-thirds the height of the wall.

The angle β at which it will act with the direction of the maximum principal stress is given by

$$\tan\beta = \frac{R_{min}}{R_{max}} \,.\, \cot\theta \qquad \text{(Eqn. 21, Chap. XII)}$$

where R_{min} and R_{max} are the two principal stresses equal to p_v and p_h

$$\therefore \tan\beta = \frac{w_e h}{w_e h}\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right) . \cot\theta$$

$$= \frac{\tan^2\left(45^\circ - \frac{\phi}{2}\right)}{\tan\theta} \qquad (7)$$

172. **Graphical Method of Finding** p_r (of para. 171) (Fig. 180). p_h and p_v will be found either by calculation or p_h by the graphical method given in paragraph 170.

Now p_h and p_v are principal stresses

$$p_v > p_h$$

To find the resultant force p_r acting on a plane making an angle θ with the direction of the maximum principal stress.

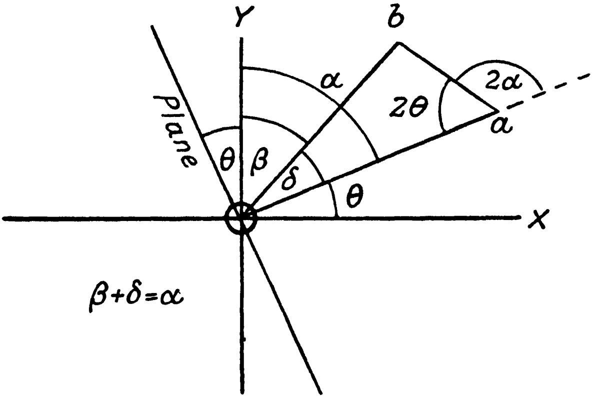

Fig. 180

Referring to Fig. 180, OY and OX are the directions of the maximum and minimum principal stresses respectively. Draw $Oa = \frac{p_v + p_h}{2}$ to scale and making an angle θ with OX, i.e. Oa is normal to the plane.

From a, draw $ab = \frac{p_v - p_h}{2}$ to scale and making an angle 2θ with Oa.

Join b to O, when Ob will represent p_r.

PROOF. $(Ob)^2 = (Oa)^2 + (ab)^2 - 2 \,.\, Oa \,.\, ab \,.\, \cos 2\theta$

$$= \frac{(p_v + p_h)^2}{4} + \frac{(p_v - p_h)^2}{4} - \tfrac{1}{2}(p_v + p_h)(p_v - p_h)\cos 2\theta$$

$$= \frac{p_v^2}{2}(1 - \cos 2\theta) + \frac{p_h^2}{2}(1 + \cos 2\theta)$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\therefore \quad (Ob)^2 = p_v^2 \sin^2\theta + p_h^2 \cos^2\theta = p_r^2$$

If $\theta =$ zero,

$$p_r = \left(\frac{p_v + p_h}{2}\right) - \left(\frac{p_v - p_h}{2}\right) = p_h$$

If θ could be equal to 90°,

$$p_r = \left(\frac{p_v + p_h}{2}\right) + \left(\frac{p_v - p_h}{2}\right) = p_v$$

172A. **Vertical Earth Face. Retaining Wall with Positive Surcharge** δ. In Fig. 181A take a small parallelopipedon of earth at a point at a depth h below the surface. It is held in equilibrium by the forces, p_r vertical, z normal, and p_r' whose direction is not yet known. The stresses on every part or any imaginary plane in a granular mass will be parallel. The stresses on a vertical plane will be parallel to the plane of surcharge where the surcharge is positive. The unit pressure $p_r = \dfrac{wh}{\sec \delta} = wh \cos \delta$ is uniform over the surface inclined at the positive angle δ to the horizontal, and it is vertical in direction. p_r' acts on a vertical plane, and will therefore have the direction of the inclined plane on which p_r acts. Therefore p_r and p_r' are conjugate stresses. The resultant pressure P_r' on the back of the wall per foot-run will therefore be parallel to the plane of surcharge.

NOTE. The resultant pressure on a wall not vertical will not be parallel to the top surface.

To find the intensity of p_r'. It was shown in para. 165, Chapter XII, that

$$\frac{p_r}{p_r'} = \frac{\cos \delta \pm \sqrt{\cos^2\delta - \cos^2\phi}}{\cos \delta \mp \sqrt{\cos^2\delta - \cos^2\phi}} \quad . \quad . \quad . \quad (8)$$

where, in this case, ϕ is the angle of repose of the filling.

Equation (8) represents both the active and passive thrust at the point, the two stresses being equal in amount but opposite in direction.

Since p_r' is less than p_r for active forces, equilibrium of the wall will take place with the upper signs. Reversing the fractions, solving for p_r', and putting

$$p_r = w_e h \cos \delta$$

we obtain

$$p_r' = w_e h \,.\, \cos \,.\, \delta \,.\, \frac{\cos \delta - \sqrt{\cos^2\delta - \cos^2\phi}}{\cos \delta + \sqrt{\cos^2\delta - \cos^2\phi}} \quad . \quad . \quad (8a)$$

Therefore for a wall of height h, as p_r' is proportional to the depth of a point, then per foot-run of wall, the resultant thrust will be

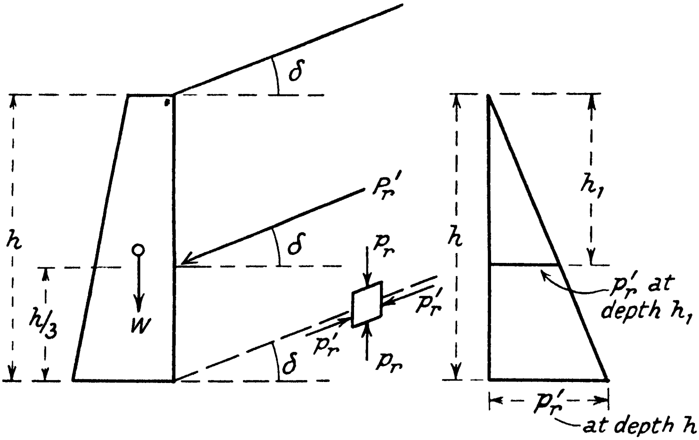

FIG. 181A

$$P_r' = \frac{w_e h^2}{2} \,.\, \cos \delta \,.\, \frac{\cos \delta - \sqrt{\cos^2\delta - \cos^2\phi}}{\cos \delta + \sqrt{\cos^2\delta - \cos^2\phi}}$$

If $\delta = \phi$ $\qquad P_r' = \frac{w_e h^2}{2} \,.\, \cos \phi$

If $\delta = O$ $\qquad P_r' = \frac{w_e h^2}{2} \,.\, \frac{1 - \sin \phi}{1 + \sin \phi}$ *See* para. 169.

INCLINED RETAINING WALL. The earth face leans away from the filling which is positively surcharged to the angle δ.

The ellipse of stress can be used to determine the resultant pressure on such an inclined retaining wall. This solution determines the amount and direction of the resultant. For the method and proof, recourse should be made to more advanced works on retaining walls.* The same results may be obtained directly from the discussion of the pressure on vertical walls.

AB represents the earth face of the wall inclined at an angle θ to the vertical. In Fig. 181B, let $P_r{}'$ = pressure on a vertical wall *BC* per foot-run as given by equation (8*a*). $P_r{}'$ acts parallel to the top slope and at a point *BC*/3 above *B*. Let *W* represent the weight of the triangle of earth *ABC* and of unit thickness which acts through the centroid of the triangle. It intersects $P_r{}'$ at the point *D* on the face of the wall *AB*. Then *P*, the resultant of $P_r{}'$ and *W*, will be the resultant pressure per foot-run of wall at *D*. The angle it makes with the normal to the wall, and with the horizontal, can easily be found from the force polygon constructed. The algebraic equation for *P* is complicated, but is given in textbooks on retaining walls.

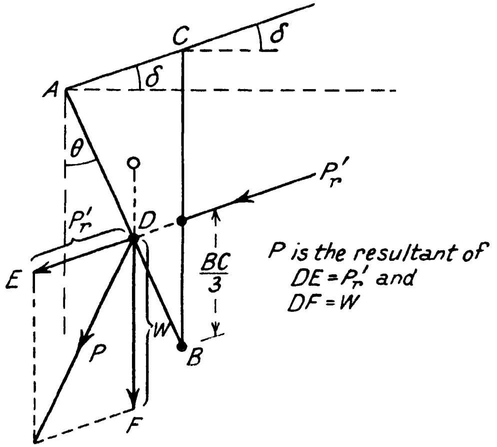

FIG. 181B

173. **Wedge Theories.** In these theories, it is assumed there is a wedge of the filling, having the earth face of the wall as one side, and a plane called the plane of rupture as the other side, which exerts a maximum thrust on the wall. The plane of rupture lies between the earth face of the wall and a line,

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* e.g. *Walls, Bins and Grain Elevators*, Ketchum.

drawn from the bottom of the wall, making the angle of repose of the filling with the horizontal. The theories do not determine the direction of the thrust or its point of application. There are many assumptions as to the direction of the thrust, but its point of application is generally assumed to be one-third up from the base, when the filling starts from the top of the wall.

In Fig. 181c, let AB be the back of the wall making an angle ρ with the horizontal: let the filling be positively surcharged to the angle δ, which cannot be greater than ϕ, the angle of repose of the filling. Let AF be a trace of the plane of rupture, which will lie between AB and the line of repose AG, of the earth drawn from the base of the wall. It is assumed that the triangular prism of earth above AF will produce the maximum pressure and that in turn the prism will be supported by the reaction of the wall and the earth. When the prism is just on the point of moving, P_r', the thrust of the earth on the wall, will make an angle β with the normal to the earth face. It can be shown* that, per foot-run of wall,

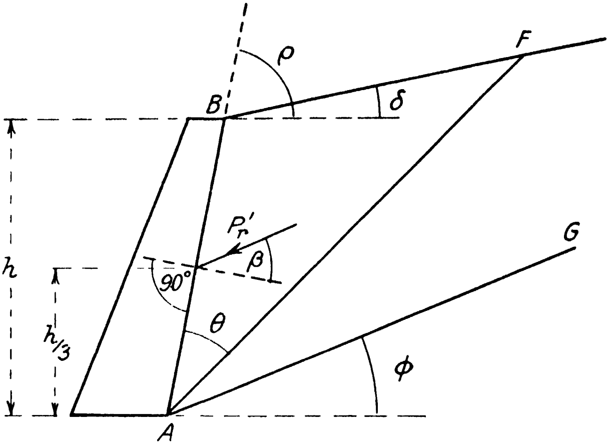

FIG. 181c

$$P_r' = \tfrac{1}{2}w_e h^2 \cdot \frac{\sin^2(\rho-\phi)}{\sin^2\rho \cdot \sin(\rho+\beta)\left[1+\sqrt{\dfrac{\sin(\beta+\phi)\cdot\sin(\phi-\delta)}{\sin(\rho+\beta)\cdot\sin(\rho-\delta)}}\right]^2} \quad . \quad . \quad . \quad . \quad (9)$$

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* *See* Ketchum, *Walls, Bins and Grain Elevators* (McGraw-Hill).

where ϕ = angle of repose of the earth and h = height of the wall. P_r' is assumed to act at a height $h/3$ above the base of the wall.

The value of P_r' will therefore depend upon the angle β, and various experimenters have given various values to this angle. One assumption is that $\beta = \phi$, another that $\beta = \phi/2$, and that β is equal to the angle of friction of the filling on the back of the wall.

Using equation (9), P_r' can be calculated from the various values of h, ρ, δ, β, and ϕ, or the formula may be simplified for known conditions.

e.g. if $\rho = 90°$, $\delta = O$, and $\beta = O$,

$$\text{then } P_r' = \frac{w_e h^2}{2} \tan^2\left(45 - \frac{\phi}{2}\right)$$

$$= \frac{w_e h^2}{2} \cdot \frac{1 - \sin\phi}{1 + \sin\phi} \quad . \quad . \quad . \quad (10)$$

and acts normal to the wall. If $\rho = 90°$, $\delta = O$, $\beta = \phi$

$$\text{then } P_r' = \frac{w_e h^2}{2} \cdot \frac{\cos\phi}{(1 + \sqrt{2}\sin\phi)^2} \quad . \quad . \quad (11)$$

and makes the angle ϕ with the normal to the wall.

If the wall is vertical and the surcharge and β are zero, it can be shown that the plane of rupture makes the angle $45 - \frac{\phi}{2}$ with the vertical, i.e. it bisects the angle between the back of the wall and the line of repose. If the wall is vertical and the positive surcharge is $\delta = \phi$, the plane of rupture coincides with the plane of repose.

174. **Resistance of Masonry Retaining Walls.*** DISTRIBUTION OF NORMAL STRESSES ON A HORIZONTAL SECTION. Take unit length of wall. (Fig. 182.)

Let R = resultant of the earth pressure and the weight of the wall on the rectangular area $AB \times 1$.

Its point of action on AB is at D, which is distant x from C, the centre point of AB. It can be resolved into V, vertical, and H, horizontal. H will cause shear.

V can be replaced by a couple Vx and a force V acting at the point C. The normal stresses at points on AB will therefore be the algebraic sum of a bending stress and a direct stress.

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* Only masonry walls are considered in this book. The student is referred to textbooks on Concrete for the design of reinforced concrete walls.

Thus the resultant stress at A will be the sum of two compressive stresses, and where D is between A and C,

$$p_A = \frac{V}{d \times 1} + Vx \times \frac{d}{2 \times d^3} \times 12. \quad \left(M = f\frac{I}{y}\right)$$

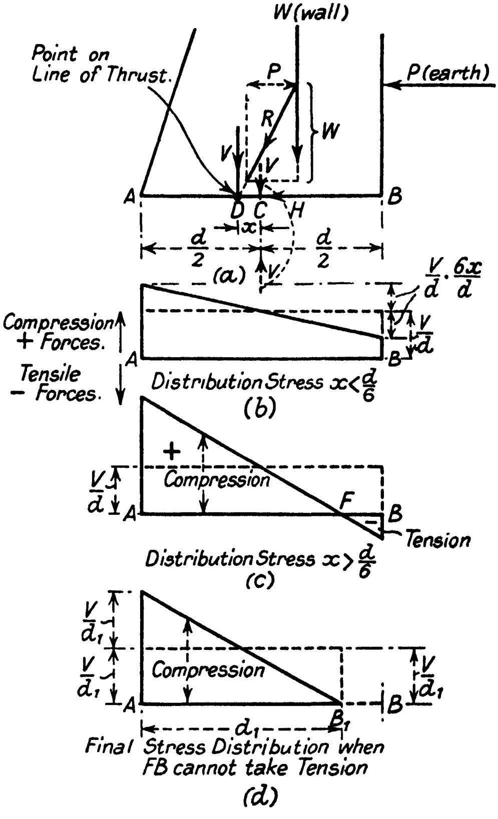

FIG. 182

$$= \frac{V}{d}\left(1 + \frac{6x}{d}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$p_B = \frac{V}{d}\left(1 - \frac{6x}{d}\right) \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If $x = \frac{d}{6}$, $\quad p_B = O.$

If $x > \frac{d}{6}$, p_B becomes a tensile force.

Mortar joints cannot resist a tensile stress, so the limit of the point of action (D) of the reaction is $\frac{AB}{3}$.

If R falls within the middle third (i.e. when $x < d/6$), no tensile stresses are possible; but the wall is heavy in material.

Let R fall outside the middle third at a distance from $A = y$, such that $y < \frac{d}{3}$

then a tensile stress will be developed at B; as the mortar is assumed to take no tension, then a crack will be developed until the tensile stresses disappear. This will occur at some point B', where

$$AB' = d_1, \qquad x_1 = \frac{d_1}{6} \qquad \text{and } AD = \frac{d_1}{3}$$

The effective width of the base is now d_1; and for no stress at B', y must be equal to $\frac{d_1}{3}$.

Thus from A to B', compressive stress; and from B' to B, no stress.

$$B'B = \text{length of crack.}$$

The maximum stress at A_1 will now be $\frac{2V}{d_1}$: and if this compressive stress is within the safe limits of the material, the wall will be safe, unless water gets into the crack at B, when it will exert an upward pressure on the wall, thus throwing R further towards A, and increasing the compressive stress at A until such a time when the material fails.

175. **To Find the Line of Thrust for a Wall.** Take a number of horizontal sections within the wall; to ascertain the point of action of the reaction, find the resultant of the weight of the material above the wall (acting through the centroid of this piece of the wall), and the total earth pressure above the section acting at a depth of two-thirds the height of the wall above the section. By the parallelogram of forces, the resultant can be obtained in magnitude and direction; and

the point where its line of action cuts the section gives the point D required. Join up all the points D for different sections and the line of thrust is found.

Note. In working out the loads, it will be found convenient to work in cubic feet of wall, and equivalent cubic feet of wall for the earth per unit length of wall. The total earth pressure divided by the weight of a cubic foot of wall will give the equivalent cubic feet of wall. To convert into force units of lbs. or tons, multiply the force in cubic feet of wall by the weight of 1 cu. ft. of wall.

For the method of working, see the example for a dam given in paragraph 185, and Figs. 184 and 185.

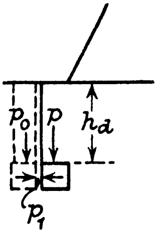

FIG. 183

176. **Foundations.** (Fig. 183). When the ground is sufficiently firm to support a structure without any reinforcing, such as piles, the average safe or normal bearing pressure

$$= \frac{\text{total weight borne}}{\text{area of the foundation}}$$

$$\text{or area of foundation} = \frac{\text{total weight borne}}{\text{safe unit-bearing pressure}}$$

At the front edge of a foundation, let the normal bearing pressure be p, assumed uniform. In order to resist any squeezing out of the earth, there must be a horizontal pressure p_1 to resist this. This in its turn is supported by a virtual pressure p_0 at the outside of the base, and p_0 must be equal to $w_e h_a$ where h_d is the depth of the footing.

$$\therefore\ p_0 = p_1\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right) = p\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)^2 \quad . \quad (14)$$

$$\therefore\ w_e h_d = p\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)^2$$

$$h_d = \frac{p}{w_e}\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)^2 \quad . \quad . \quad (15)$$

If p = average unit-bearing pressure,

$$h_d \geq \frac{W}{w_e A}\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)^2 \quad . \quad . \quad . \quad (16)$$

Illustrative Problem 43.

A concrete foundation for a wall has to carry 6 tons per linear foot at 1·5 tons per square foot bearing pressure. Estimate the necessary depth of the foundations if the angle of repose of the earth is 35°, and its weight 110 lb. per cubic foot.

$$p = 1{\cdot}5 \text{ tons/square foot.}$$

The pressure at right angles to p is

$$p_1 = 1{\cdot}5\left(\frac{1 - \sin 35^\circ}{1 + \sin 35^\circ}\right) = 1{\cdot}5 \times {\cdot}27$$

$$= {\cdot}405 \text{ ton/square foot.}$$

Let h_d = depth of foundation in feet,

$$\text{then } w_e h_d = 110\, h_d = {\cdot}405 \times {\cdot}27 \times 2240$$

$$\text{therefore, } h_d = 2{\cdot}24 \text{ ft.}$$

If the weight of the concrete foundation is not included in the weight of the wall, it must be allowed for in designing the depth of the foundation. The depth of concrete will be h_d ft.*

177. **Dams.** Dams, which are walls of masonry or concrete, are used for impounding or holding up large depths of water; they can be put into two main classes—gravity, and arched dams. This section will only deal with gravity dams.

178. **Notes on Gravity Dams.** (1) The resultant thrust, whether the reservoir be full or empty, must be within the base, or the dam will overturn; and no normal tensile stresses are developed if the resultant thrust fall within the middle third. (The water face of a dam must be nearly vertical.)

(2) The maximum compressive stress on any section must not be greater than a safe working stress.

(3) The resistance to sliding on any horizontal plane must be greater than the horizontal pressure H, i.e. $\mu W > H$, where μ is a coefficient of friction and W the weight of the dam above the plane. The base of the dam is not generally the critical plane as regards slipping.

(4) The shear stress must not exceed a specified amount.

(5) The maximum principal stress developed must not exceed the safe compressive working stress.

(6) On the water face there shall be no tensile principal stress.

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* If the foundation pressure varies uniformly from a maximum p_1 to a minimum p_2, then

$$\frac{p_1}{p_2} \lessdot \left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)^2 \text{where } p_1 \lessdot w_e h_d\left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)^2$$

179. **The Analysis of a Gravity Dam.** Assume the form, then find the stresses. For loads, work in cubic feet of water and consider unit length of dam. The equivalent water load on a cross-section due to the dam above it is equal to the area of the cross-section of the dam above it (× unit length) multiplied by the density of the masonry $= \rho$

$$\rho = \frac{\text{weight of 1 cu. foot of masonry}}{\text{weight of 1 cu. foot of water}}$$

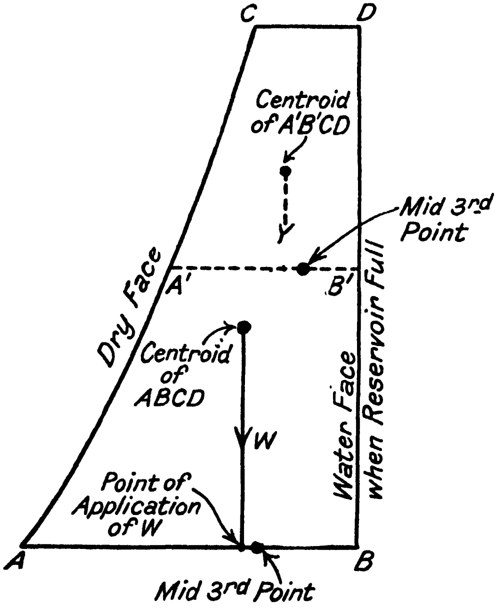


FIG. 184

180. **First Case.** Consider the stability of the dam with no water in the reservoir. (*See* Fig. 184 and Plate I.) The force on any horizontal cross-section AB = weight of masonry above it, acting through the centroid of the mass of masonry.

Find its point of application on the cross-section. Take similar planes at distances, say, 10 to 20 ft. apart, and find corresponding points of application.

Connect all these points, and the line of thrust is obtained equal to the locus of the points of action of the thrusts on the horizontal planes.

For no tensile stresses on any plane or section, the resultant load W on such a plane AB must be within the middle third. Thus the line of thrust has to be within the middle third, and as near as possible to the water face boundary line of the middle thirds of all sections.

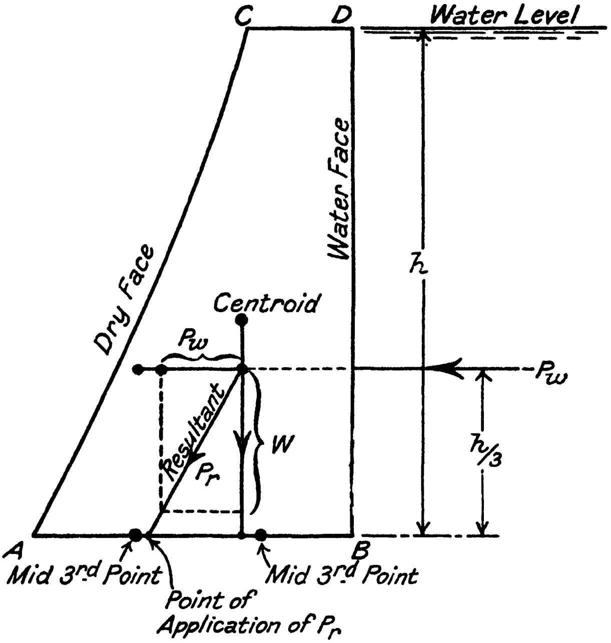


FIG. 185

181. **Second Case.** CONSIDER THE DAM WHEN THE WATER IN THE RESERVOIR IS AT ITS MAXIMUM DEPTH AT THE DAM. (*See* Fig. 185 and Plate II, page 334.) The total water load on the dam above the section taken (AB, say) will be equal to $P_w = \frac{wh}{2} \times h = wh^2$ per unit length, and where h = height of water above the section; P_w will act normal to the water face at a height $\frac{h}{3}$ above AB (w = weight of 1 cu. foot of water = 62·4 lb.)

The resultant force acting on the section will be found by compounding the water load, and the load due to the masonry above the section. This will throw the point of action of the resultant away from the water face towards the downstream face; and, for no tensile stress in the dam, the point of action of the resultant must fall within the middle third. For no stress at the water face, the point will be the further middle third point from the water face. Therefore, the line of thrust with the reservoir full will be towards the middle third boundary line farthest from the water face.

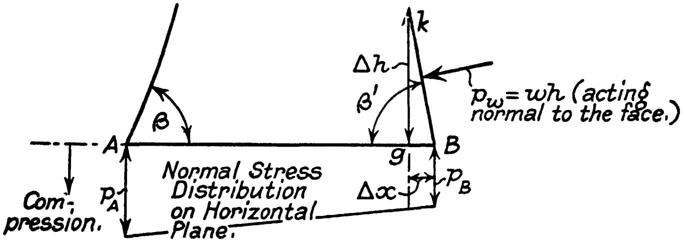

k
Δh
β'
$p_w = wh$ (acting normal to the face.)
β
A
B
g
Δx
p_B
p_A
Com-pression.
Normal Stress Distribution on Horizontal Plane.

FIG. 186

If the water face is battered, then the water load normal to the face will tend to make the resultant force steeper, and, therefore, its points of action on the horizontal planes will lie nearer the centre and tend towards increased stability.

The normal stresses on horizontal planes can be found as for retaining walls; the maximum compressive stresses must be within the safe limits of the masonry.

182. General Case of Analysis of a Dam. SHEAR STRESSES ON HORIZONTAL PLANES. (Figs. 186 and 187.)

$AB =$ any horizontal plane at depth h

$p_B =$ normal stress at B

$p_A =$,, ,, A

At B take a small length along $AB = \Delta x = Bg$.

At g erect a vertical to cut the face of the dam in k; let $gk = \Delta h$. kB is very small.

Let the mean intensity of water pressure on kB be $p_w = wh$. Angle $kBg = \beta'$.

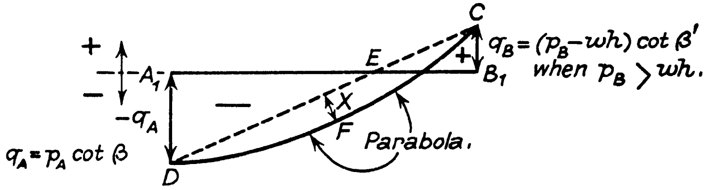
Consider the equilibrium of the triangle gkB.

The total upward load on $gB = p_B \cdot \Delta x$

$$\text{The vertical component of } p_w = p_w \sin(90 - \beta')$$
$$= p_w \cos \beta'$$

$$\text{Total load on } Bk = wh \,.\, Bk$$
$$\text{whose vertical component } = wh \,.\, Bk \cos \beta'$$
$$= wh \,.\, Bk \,.\, \frac{\Delta x}{Bk}$$
$$= wh \,.\, \Delta x$$

Neglecting the weight of masonry gkB, the shear force on $gk = p_B \Delta x - wh \,.\, \Delta x$.



Horizontal Shear Stress Diagram.

FIG. 187

$$\text{Shear stress on } gk = p_B \,.\, \frac{\Delta x}{\Delta h} - wh \,.\, \frac{\Delta x}{\Delta h}$$
$$= p_B \cot \beta' - wh \cot \beta'$$

The intensity of the complementary horizontal shear stress on $gB = \Delta x$ is equal to the intensity of the vertical shear stress at B;

$$\text{then } q_B = (p_B - wh) \cot \beta'$$

If $\beta' = 90°$, $q_B = O$, i.e. there is no horizontal shear stress at B.

If $p_B > wh$, then the vertical shear stress on gk acts downwards, and the complementary shear stress on Δx acts from right to left; that is, it acts in conjunction with the horizontal water pressure.

Note. Δh is to the left of B.

Similarly at A, the vertical shear stress on a vertical section to the right of A will be

$$p_A \cot \beta$$

and it will act downwards.

The complementary shear stress on a small length Δx at A will be $p_A \cot \beta$, and will act from left to right, resisting the horizontal water pressure and the tendency to slide.

If $p_B > wh$, q_B is positive, q_A is negative.

Or if p_B and p_A are of the same sign, then always q_B is of the opposite sign from q_A.

This curve of horizontal shear is a parabola.*

To construct the curve.

q_B and q_A can be found. Erect perpendiculars B_1C and A_1D to scale $= q_B$ and q_A at B_1 and A_1 (Fig. 187).

Now the total shear force along AB is equal to the total horizontal water load on the dam above the section AB; this is easily ascertained. Let it be H.

The total shear force on AB = area of shear stress diagram A_1DFCB_1 where DFC is a parabola. Exception *see* paragraph 186. Join C to D to cut A_1B_1 in E. Find the area of A_1DECB_1 and to scale = total shear load = S.

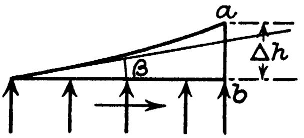


FIG. 188

Making allowances for the different kinds of shear stress (positive or negative), $H - S$ or $S - H$ will be nearly equal to the area $DECF$ to scale, and will be so when $q_B = O$ or is of the same sign as q_A. Measure DC and at the centre point F of DC erect a perpendicular X in the necessary direction, such that $\left(CD \times \frac{2X}{3}\right)$ to scale $= H - S$ or $S - H$.

If q_B of the opposite sign from q_A, then re-check, to ascertain if $A_1DFCB_1 = H$. If not, by further trial the correct curve can be ascertained.

As q_B must be small, β' must not be much less than 90°. If β is small, ab, Fig. 188, becomes subjected to a bending moment which may be fairly big. Concrete is weak in tension, and it is supposed with concrete dams that small cracks occur at points b which upset the values of the stresses found theoretically.

183. **Normal Stresses on Vertical Planes.** (*See* Fig. 189.) Assume that the horizontal shear stresses on two planes $A'B'$ and AB, Δh apart, are known.

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* *See* paper, *Proceedings Institution of Civil Engineers*, Vol. clxxii, p. 89: "Stresses in Masonry Dams," by E. P. Hill.

Shear stresses opposing the water pressure are negative.

The difference in shear force between the planes acts as a normal force on *ab*.

Let X = area of shear stress diagram for $A'B'$ to the left of ab
and Y = ,, ,, ,, AB ,,

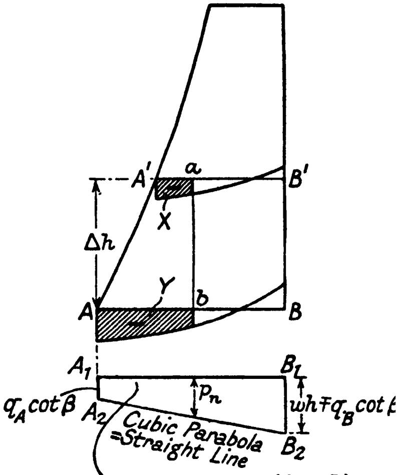
Then $Y - X = P_N$ on ab = total normal force on ab,

then the normal stress $p_n = \dfrac{Y - X}{\Delta h}$

At the point A, p_n is $q_A \cot \beta$; and at B it is equal to the mean intensity of water pressure on Δh due to the height of water above the mean point of Δh, less or plus $q_B \cot \beta'$. Δh is very small, therefore p_n at B

$$= wh \mp q_B \cot \beta'$$

The curve of p_n is a cubic parabola, which is nearly a straight line; thus set down an ordinate at $B = B_1B_2 = wh \mp q_B \cot B'$ and join to A_1A_2 where $A_1A_2 = q_A \cot \beta$, and the diagram for the normal vertical pressures is found.



Normal Stresses (for AB) on Vertical Planes.

FIG. 189

184. **Theory of Stress.** The stresses acting on a small element at a section of the dam are shown in Fig. 190.

p and p_n = normal stresses on the horizontal and vertical planes respectively.

It has been shown that the stresses can be compounded so that on two planes the stresses are normal. These stresses are the principal stresses R_{max} and R_{min}.

The connecting formula is

$$(R - p_n)(R - p) = q^2. \quad (\textit{See } \text{Eqn. 36, Chap. XII})$$

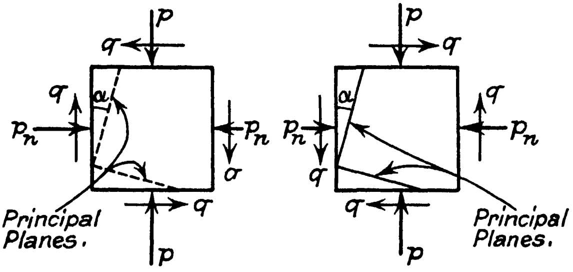
The maximum principal plane is inclined at an angle a to the vertical,

$$\text{where } \tan 2a = \frac{2q}{p_n - p} \quad (\textit{See } \text{Eqn. 35, Chap. XII})$$

For any section, the normal, horizontal, and shear stresses are known; by the use of the formula, the two principal stresses and the direction of the principal plane can be found. Thus the ellipse of stress can be drawn. (*See* Plate II.)

Both principal stresses must be positive, i.e. both compressive.

If the water face of the dam is vertical and there are no tensile stresses, then no tensile forces at all will occur. It is, therefore, important that neither of the principal stresses is tensile.



q negative on the section. *q positive on the section.*

FIG. 190

CONSIDER THE WATER FACE. It is essential no tensile stresses should occur on any plane normal to the water face.

Referring to Fig. 191,

One principal stress $= R_1 = wh$

$$q_B = (p_B - wh) \cot \beta'$$

$$\text{and } (R - p_n)(R - p) = q_B^2 \qquad (C)$$

p_n, p, and q_B are known and also R_1.

Solving Equation (C), two values of R are found, one of which is $R_1 = wh$. The other is on a plane at right angles to the water face, and can be found from (C). It must be positive, i.e. compressive.

An analysis of a dam is given in Plates I and II.

185. **Notes on Plates I and II.** Take unit foot-run of the dam. Work in weight units of 62·5 lb. (the weight of 1 cu. ft. of water); then the density of the masonry will be

$$\rho = \frac{\text{weight of 1 cu. ft. of masonry in lb.}}{62{\cdot}5 \text{ lb.}}$$

Divide the dam section by horizontal planes equidistant apart (if possible), such as *aa*, *bb*, *cc*, . . . *ll*. Find the weight in the required units of each section of masonry, *aabb* . . . *jjkk*, *llmm*, which will be equal to the area in square feet $\times \rho$. These weights will act through the centroids of their sections. By known means, find the centroids and join together by a curve to obtain the centroid locus.

C_k is the centroid for block *jjkk*

C_l is that for *kkjj*.

From C_l drop a perpendicular to cut *ll* in *l′* ; from C_k drop a perpendicular to cut *kk* in *k′*. Similarly for the other sections.

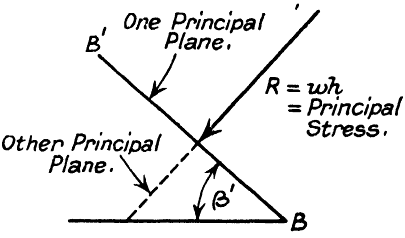


FIG. 191

Construct the polar diagram for the masonry loads, and from this the link or funicular polygon for these loads. With the reservoir empty, the total load acting on the base *ll* is the total masonry load per foot-run. To find its point of application on the base *ll*, produce the outer lines of the masonry link polygon to meet in a point *L*. Draw a vertical through *L*. This vertical gives the position of the resultant total load with respect to *l′* on *ll*. The resultant acts at the distance l^2l^1 from *l′* on the base. Similarly for the other planes; join up the points of application and the line of thrust, dam empty is found.

RESERVOIR FULL. Calculate the water pressure acting at each section point *a*, *b*, *c*, . . *l* per foot-run of dam.

$(p = wh: \ w = \text{unit weight of } 62{\cdot}5 \text{ lbs.} \ \therefore p = h)$.

Set out to scale these pressures at right angles to the water face at their section points, and join up the ordinates to obtain a water pressure line al_w. To find the water loads acting on each section of masonry, such as *ab*, *bc*, etc., find the areas of each section of water pressure diagram to scale, such as $kk_w ll_w$, and these give in water units the water loads on the similar faces kl; these act through the centroids of their particular sections and at right angles to the wall. These loads are represented by P_b, P_c . . . P_l. The total water load is the sum of these. On the area *aacc*, the water load will be $P_b + P_c$, and similarly for the other sections. Construct the water load polar diagram, attaching it to the masonry load polar diagram.

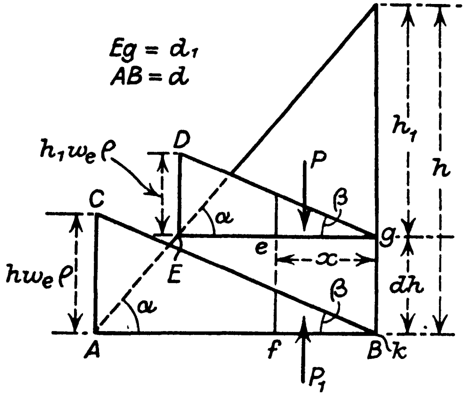


FIG. 192

Construct the funicular or link polygon for the water loads from the polar diagram, and from this link polygon can be found the direction of application of the resultants of the loads required. The resultant water load above the section *cc* is $P_b + P_c = p_c$ and acting as shown.

Produce p_c to meet the resultant weight of the masonry *aacc* in C_3. From the joined polar diagrams, the resultants of the corresponding total water and masonry loads are found in magnitude and direction; these are R_b . R_c . R_d . . . .

Through C_3 draw a line parallel to R_c to cut the base *cc*, and the cutting point is the point at which R_c acts. Similarly for the other section lines; join up the points of application and the line of thrust, reservoir full is obtained.

By the methods indicated, the direct normal horizontal and vertical, also shear stresses, are found for the different sections and the stress distribution diagrams drawn.

In Plate II are given the distribution of stress diagrams, and the ellipses of stress for various horizontal sections and points in these sections.

TABLE FOR PLATE II

Unit stress = 62·5 lb./square foot.

Base.	Wet Face Stress Units.			Dry Face Stress Units.		
	Horizontal.	Vertical.	Shear.	Horizontal.	Vertical.	Shear.
CC	15·0	21·02	·145	3·34	41·5	12·32
FF	37·47	1·14	·996	26·44	105·5	52·78
II	59·98	14·12	2·84	90·75	119·1	104·4
LL	82·46	61·4	2·32	220·0	87·3	138·5

Ellipse.	Principal Stresses.	
	Minor.	Major.
A	15·02	21·02
B	5·98	34·75
C	·35	37·5
D	9·7	74·3
E	11·95	60·15
F	25·0	137·5
G	33·8	84·5
H	61·14	84·9
I	65·85	118·95
J	61·7	164·3
K	40·35	226·45

186. **Notes on Question 4 in Examples** (page 337, Chap. XIII). (Fig. 192, page 334).

h = height of dam

d = required width of the base

w = weight of 1 cu. ft. of water

ρw = weight of 1 cu. ft. of masonry

$$\frac{\text{Total water load}}{\text{Total masonry load}} = \frac{\frac{d}{3}}{\frac{h}{3}} \text{ for resultant to hit middle third point}$$

$$\therefore \quad \frac{h}{3} \times \frac{1}{2} wh^2 = \frac{d}{3} \cdot \frac{1}{2} dhw\rho \qquad \therefore \quad d = \frac{h}{\sqrt{\rho}}$$

Max. stress $= p = \frac{2W}{d} = \frac{2}{d}(\frac{1}{2}dhw\rho) = hw\rho$ and is a function of h

Triangles DEg and CAB are the normal horizontal stress distribution diagrams.

Shear stress in $\triangle^r$ dam for the conditions given.

Take two section depths h and h_1, and dh apart.

Max. stress on base depth $h = hw\rho$

" " $h_1 = h_1 w\rho$

Slope of horizontal normal stress diagrams,

$$\tan\beta = \frac{w\rho h}{d}; \qquad \tan\beta_1 = \frac{w\rho h_1}{d_1}$$

$$\frac{h}{d} = \frac{h_1}{d_1} \qquad \therefore\ \beta = \beta_1$$

$$\begin{aligned}\text{Shear on } ef \text{ section} &= p - p + \text{weight of } efgk\\ &= \text{weight of } efgk\\ &= w\rho \,.\, x \,.\, dh\end{aligned}$$

$$\text{Shear stress on } ef = \frac{w\rho \,.\, x \,.\, dh}{dh} = w\rho x$$

∴ Shear stress diagram is a straight line when there are just no tensile stresses on the base.

From the general case considered in paragraph 182

$$p_A = w\rho \,.\, h_1$$

$$q_A = w\rho \,.\, h_1 \cot a$$

$$\cot a = \frac{d_1}{h_1}$$

$$\text{and } q_A = w\rho \,.\, d$$

∴ shear stress is a function of the length of the base.

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EXAMPLES

1. A wall of rectangular section 14 ft. high is to retain an embankment of dry earth having an angle of repose which is 30°. Find the necessary thickness of the wall, and the maximum vertical stress on the base if the earth is horizontal and level with the top of the wall.
Weight of earth, 100 lb. per cubic foot.
Weight of wall, 150 lb. per cubic foot. (U. of B.)

2. A retaining wall is 10 ft. high and 5 ft. wide at its base, and 2 ft. wide at its top, which is level with the ground surface. The back of the wall is vertical. It carries a super-load equal to 10 cwt. per square foot at 3 ft. below the ground level. Determine the position of the resultant pressure at the base, using Rankine's formula for the lateral pressure of the earth, if the specific gravity of the masonry is 2¼ and that of the earth 1¾, and the angle of friction of the earth 45°. (I.C.E.)

3. A piece of level ground is to have a portion of the surface excavated to a depth of 14 ft., and it is necessary to support the earth at the boundaries of this excavation by concrete retaining walls. The earth face is vertical and the width at the top of the wall is 3 ft. Determine a suitable trapezoidal cross-section for the retaining walls, if the earth weighs 125 lb. per cubic foot and has a natural slope of 2 to 1. The concrete may be assumed to weigh 140 lb. per cubic foot. Discuss the validity of any formula used in connection with the calculation.

4. A masonry dam 50 ft. high has a vertical water face. Assuming the dam has a triangular section, determine the width of the base so that there shall just be no tensile stresses on the base. Show that the shear stress diagram on the base is a triangle. Construct the ellipses of stress for various points on the base. w for masonry = 144 lb./cu. ft. (U. of B.)

5. A wall 15 ft. high of rectangular section has to retain earth, the surface of which is horizontal. The angle of repose of the earth is 30°. Determine the dimensions of the wall so that the line of thrust shall be in the middle third of the base. Weight of earth, 100 lb. per cubic foot; weight of wall, 150 lb. per cubic foot. (U. of B.)

6. A masonry dam of trapezoidal section 100 ft. high has a base 60 ft. wide. The water face is vertical and the width at the top is 10 ft. Find the normal stress diagram for the base, and deduce approximately the shear-stress diagram. Weight of masonry, 140 lb. per cubic foot. (U. of B.)

7. A trapezoidal masonry dam has a height of 42 ft. and the water face is vertical. The base is 25 ft. and the thickness at the top 8 ft. Weight of a cubic foot of masonry, 150 lb. Determine—
(1) The resultant thrust on the base per foot length of dam ;
(2) The distribution of normal stress on the base. (U. of B.)

8. A concrete retaining wall with a vertical face is 12 ft. high and 5 ft. wide at the base, and 2 ft. 6 in. wide at the top. If the concrete weighs

140 lb. per cubic foot, find the horizontal force per foot run of the wall necessary to overturn the wall, the force being applied on the vertical face 4 ft. from the base.

9. Prove the Rule of the Middle Third, as stated for solid masonry structures with rectangular bases, subjected to overturning forces. A parallel brick chimney of hollow square section is 2 ft. by 2 ft. inside, and the thickness is also 2 ft. The chimney is 40 ft. high. Find the greatest allowable intensity of wind pressure perpendicular to one face, so as not to cause tension at one edge. The brickwork weighs 120 lb. per cubic foot. (U. of B.)

10. Determine the width and depth of a concrete foundation which supports a wall having a load on the base of 6 tons per foot run, if the earth has a bearing pressure of 1½ tons per square foot and an angle of repose of 30°. The weights of concrete and earth are 140 and 100 lb. per cubic foot respectively. (U. of L.)

11. Prove that the intensity of the horizontal pressure per unit area on the vertical back of a retaining wall at a depth h is

$$p = w_e h \frac{(1 - \sin\phi)}{(1 + \sin\phi)}$$

and hence deduce a formula for the safe depth of a foundation on which the maximum pressure is 2 tons per square foot. (U. of B.)

12. The earth face of a retaining wall 20 ft. high is vertical. The angle of friction both for earth on earth and for earth on masonry is 40°; the earth weighs 110 lb. per cubic foot. Take account of friction between the earth and wall, and find the resultant earth pressure on the wall. Also find the pressure by Rankine's theory. The earth surface is horizontal.

13. Taking the dam in Question 7, determine also (*a*) the distribution of shear stress on the base; (*b*) the normal stresses on vertical planes at the base. Construct the ellipses of stress for sections on the base distant 0, 5, 12·5, 20, and 25 ft. from the water face.

14. The base of a retaining wall is 9 ft. wide; the vertical component of the resultant thrust is 12 tons, and it acts (*a*) at the centre, (*b*) at 6 ft. from the earth face; (*c*) at 7 ft. from the earth face. For each case, draw the normal stress diagram for the base. Assuming a mortar joint along the base which cannot resist tension, for case (*c*) what is the maximum compressive stress on the base? and draw the diagram of normal stress distribution.

15. Work Questions 1, 3, and 5 by both the Rankine and Wedge theories; the coefficient of friction of earth on wall being the same as for earth and earth; i.e. = tan (angle of repose of the earth).

CHAPTER XIV

REINFORCED BEAMS

187. Flitch Beams. A flitch beam is one consisting of timber to which are bolted steel plates. The aim is to obtain a beam which is stronger than the timber, yet economical in cost. The reinforcing material must therefore have an elastic modulus (E) greater than that of the reinforced material. In some flitch beams the plates are bolted to the outsides of the timber; in others, a single plate runs down the centre of the timber beam. The plates may or may not be of the same depth as the timber beam.

The beam will be built up symmetrically with respect to the neutral axis, as shown in Fig. 193.

188. **Consider the Case where there is a Centre Plate,** whose depth is less than that of the timber; the following theory will apply also to two or more plates, as all the plates can be put together to make an equivalent single plate.

$E_s =$ Young's modulus for the steel plates.
$E_t =$,, ,, ,, timber.
$d_t =$ depth of the timber.
$d_s =$,, plates $=$ or $< d_t$
$b_t =$ total breadth of the timber.
$b_s =$,, ,, steel plates.
$I_t =$ moment of inertia for the timber portion only.
$I_s =$,, ,, ,, steel plates only.
$I_B =$ effective moment of inertia of the whole beam, working, say, on a timber basis.
$f_s =$ skin stress in the plates.
$f_t =$,, ,, timber.

$$\frac{d_s}{d_t} = k; \quad \frac{E_s}{E_t} = m = \text{modular ratio.}$$

For any beam, the external moment at a section $=$ internal moment of resistance at that section.

External moment $=$ moment of resistance.

$$M = f_s \frac{I_s}{\frac{d_s}{2}} + f_t \cdot \frac{I_t}{\frac{d_t}{2}}$$

$$= 2f_s \frac{I_s}{d_s} + 2f_t \cdot \frac{I_t}{d_t} \quad . \quad . \quad . \quad (1)$$

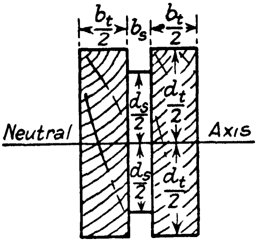

FIG. 193

Now $\frac{2f_s}{d_s} = \frac{E_s}{R}$; and $\frac{2f_t}{d_t} = \frac{E_t}{R}$

The radius of curvature being the same,

$$\frac{2f_s}{d_s E_s} = \frac{2f_t}{d_t E_t}$$

$$f_s = f_t \cdot \frac{d_s}{d_t} \cdot \frac{E_s}{E_t} \quad . \quad . \quad (2)$$

$$\therefore \; M = 2f_t \cdot \frac{d_s}{d_t} \cdot \frac{E_s}{E_t} \cdot \frac{I_s}{d_s} + 2f_t \frac{I_t}{d_t} \quad . \quad . \quad . \quad (3)$$

$$\frac{E_s}{E_t} = m$$

Then $M = 2\frac{f_t}{d_t}(m \, . \, I_s + I_t)$. . . . . (4)

$$= \frac{2f_t}{d_t}(I_E) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $I_E = mI_s + I_t$

= effective moment of inertia of the beam on a timber basis . . . (6)

Let I_{tw} = moment of inertia of a whole rectangular cross-section of the beam, given that it is only made of timber

$$= \frac{(b_t + b_s)d_t^3}{12}$$

Then the ratio of the loads which can be carried for the given beams to develop the same working stress in the timber is

$$\frac{I_E}{I_{tw}}$$

Illustrative Problem 44.

A timber beam, 4 in. × 2 in. cross-section, 100 in. long, is simply supported. Find the size of two steel plates 4 in. deep to be fixed to the timber beam, so that it may carry a central load of 2500 lb. with a maximum stress in the timber of 2000 lb./sq. inch.

$$E_{timber} = 1{\cdot}5 \times 10^6 \text{ lb./sq. in.}; \quad E_{steel} = 30 \times 10^6 \text{ lb./sq. in.}$$

$$m = \frac{E_{steel}}{E_{timber}} = 20$$

Then the maximum external moment on the compound beam has to be equal to the internal moment of resistance.

$$\therefore \frac{2500 \times 100}{4} = 2000 \times \frac{\dfrac{2 \times 4^3}{12}}{2} + 20 \times 2000 \times \frac{b \times \dfrac{4^3}{12}}{2}$$

where b in inches is the total width of the plates required.

$$62{,}500 = 1000 \times \frac{64}{6} + 20{,}000 \times b \times \frac{64}{12}$$

$$b = \frac{30}{64} \simeq \tfrac{1}{2} \text{ in.}$$

Two plates are required, to be fastened one on each side of the beam and to be of cross-section 4 in. × ¼ in.

189. **Reinforced Concrete Beams.*** Concrete is a heterogeneous material (consisting of cement, sand, and stone) having a fairly good compressive strength, but a very low tensile strength, which is usually assumed negligible. It is cheap, economical, and easy to make, and can be adapted to many purposes. For use in beams and structures where tensile stresses may be developed, some material is required in the concrete to take the tensile stresses. Mild steel in round and other shaped rods is used, and the two materials together give a reinforced concrete beam, column, and other structures. This section will only deal with steel rods placed in the tension side of a beam; in many cases, steel rods are also placed on the compression side to assist the concrete in taking the compressive strains.

In designing beams of this reinforced character, and knowing the safe load to be carried, first find the position of the neutral axis of the cross-section, and then—

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* For a further development in the design and fabrication of reinforced concrete refer to Chapter XVII, which deals in an elementary way with "Pre-stressed Concrete."

External moment at a section = internal moment of resistance, which is a couple; that is, the total tensile force on one side of the neutral axis = total compressive force on the other.

For steel only on the tension side, it is assumed that the concrete will only take compressive forces, leaving the steel to take the tensile forces; then the total internal compressive force in the concrete equals the total tensile force in the steel.

If steel rods are also put in on the compression side, then the total internal compressive force in the concrete plus the compressive force in the steel on the compression side is equal to the total tensile force in the steel on the tension side.

The notation given is that stated in the L.C.C. regulations for reinforced concrete work.

190. Notation for Beams and Slabs.

A = area of tensile reinforcement in sq. in.

A_c = „ compressive „ „

a = arm of internal moment of resistance in inches.

B = bending moment due to external loads or forces.

b = breadth of a rectangular beam in inches or the breadth of the flange of a Tee beam.

*c = permissible compressive working stress in lb./sq. in. of the extreme edge of the concrete in compression. It depends upon the mix and grade used: varying, for instance, from 750 lb./sq. in. for a 1 : 2 : 4 mix to 975 lb./sq. in. for a 1 : 1 : 2 mix, both of ordinary grade concrete. These values are increased to 950 and 1,250 for high grade mixed concrete.

d_s = total depth of slab or beam in inches.

d = effective depth of beam in inches; i.e., the distance from the compression edge of the concrete to the centre of gravity of the steel reinforcement in tension.

d_c = depth of the centre of gravity of the compression reinforcement (when used) from the compression edge of the concrete.

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* See Table 4, *Code of Practice, C.P. 114* (1948). The stresses given above are now increased—

for a 1 : 1 : 2 mix, compression due to bending, 1500 lb./sq. in.

for a 1 : 2 : 4 mix, compression due to bending, 1000 lb./sq. in.

E_c = modulus of elasticity of concrete in compression in lb./sq. in.

E_s = modulus of elasticity of steel in tension, lb./sq. in.

l = length of effective span of a beam in inches.

*$m = \frac{E_s}{E_c}$ = modular ratio. A suggested value is $\frac{40{,}000}{3c}$ where $3c$ = minimum cube strength at 28 days (works tests).

n = distance of the neutral axis from the compression edge of the concrete in inches.

$n_1 = \frac{n}{d}$ = neutral axis ratio, so that $n = n_1 d$.

p_c = percentage of tensile reinforcement = $100r$ where $r = \frac{A}{bd}$.

R_c = internal moment of resistance in terms of the compressive force.

R_s = internal moment of resistance in terms of the tensile force.

$r = \frac{A}{bd}$ and $r_c = \frac{A_c}{bd}$

s_1 = slab or beam depth ratio = $\frac{d_s}{d}$

t = tensile working stress in the steel in lb./sq. in. (18,000 to 20,000 lb./sq. in. suggested values for mild steel.)

t_1 = ratio of the tensile stress in the steel to the skin compressive stress in the concrete.

W = working load in lb.

191. **Assumptions Involved in the Theory of Reinforced Concrete Beams.**

(1) A plane section before bending remains plane after bending.

(2) Tension in the concrete is neglected.

(3) The stress in the concrete is proportional to the strain. (*See* note in assumption 4.)

(4) The modulus of elasticity for the concrete is assumed to be constant. The stress-strain diagram for most concrete in

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* The *Code of Practice C.P. 114* (1948) now says that "the Modular Ratio is equal to 15." The student is, however, advised after reading the theory developed later to give himself problems for solution in which m has values other than 15.

compression is a smooth curve right from the start; therefore, the slope of this curve varies for all stresses, and, consequently, E_c, which is the slope of the stress-strain curve. If working the concrete at, say, 600 lb./sq. in., then for the particular kind of concrete, E_c, corresponding to this stress, should be used.*

(5) Adhesion between the steel and the concrete is perfect within the limit of proportionality of the steel.

192. **Rectangular Beam.** ANALYSIS WHEN THE LIMITING STRESSES ARE KNOWN. Reinforced on the tension side only. Assume one row of rods area A. (Fig. 194.) The beam will bend about the neutral axis (N.A.), which is at a depth n from the compression skin. Here b, d, m, t and c are known.

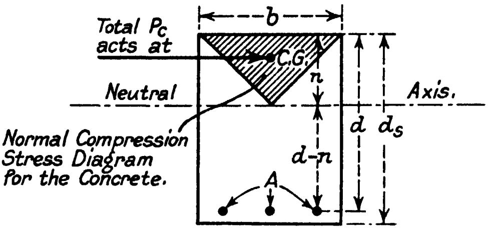

FIG. 194

The problems are to find (*a*) the position of the neutral axis, (*b*) the maximum permissible bending moment for the beam with given limiting stresses, and (*c*) the steel area A.

$$\text{Now strain } e = \frac{\text{stress}}{E}$$

$$\frac{\text{Maximum strain in the steel}}{\text{Maximum strain in the concrete}} = \frac{d-n}{n}$$

$$\therefore \qquad \frac{t/E_s}{c/E_c} = \frac{d-n}{n}$$

$$\frac{t}{c} = \frac{E_s}{E_c}\left(\frac{d-n}{n}\right) = m\left(\frac{d-n}{n}\right) \quad . \quad . \quad . \quad (7)$$

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* See *Modulus of Elasticity of Concrete*, by Professor Lea, Institution of Concrete.

$$n = n_1 d$$

$$\therefore \frac{t}{c} = \frac{m(d - n_1 d)}{n_1 d}$$

$$= \frac{m(1 - n_1)}{n_1} \quad . \quad . \quad . \quad (8)$$

As t, c, and m are known, n_1 can be found and therefore n.

Assuming working stresses of $t = 16{,}000$ lb./sq. in.

$c = 600$ „

and $m = 15$

$$\frac{16{,}000}{600} = 15\left(\frac{1 - n_1}{n_1}\right)$$

$$n_1 = \cdot 36$$

$$\text{or } n = \cdot 36d \quad . \quad . \quad . \quad . \quad (9)$$

The total tensile force in the steel = the compressive force in the concrete.

$$\text{Then } tA = \left(\frac{0 + c}{2}\right)bn = \frac{bnc}{2}$$

$\left(\frac{0 + c}{2}\right.$ = the average compressive stress in the concrete.$\left.\right)$

The internal moment of resistance is $R_c = R_s$

$$R_c = \frac{bnc}{2} \times \left(d - \frac{n}{3}\right) = tA\left(d - \frac{n}{3}\right) = R_s \quad . \quad . \quad (10)$$

$$\text{Now } a = d - \frac{n}{3}$$

$$= d\left(1 - \frac{n_1}{3}\right) = d\left(1 - \frac{\cdot 36}{3}\right)$$

i.e. when $t = 16{,}000$ lb./sq. in., $c = 600$ lb./sq. in., $m = 15$,

$n_1 = \cdot 36$, $a = \cdot 88d$ . . (11)

$$\therefore R_c = \frac{bnc}{2} \times a \qquad \text{(From (10))}$$

$$= \frac{b \times \cdot 36d \times 600 \times \cdot 88d}{2} = 95\ bd^2$$

therefore the external moment = $B = 95\,bd^2$ . . (12)

In general $R_c = R \,.\, bd^2$ where $R = \frac{c}{2} \,.\, n_1 \,.\, \left(1 - \frac{n_1}{3}\right)$

also $R_s = R \,.\, bd^2$ where $R = t \,.\, r \,.\, \left(1 - \frac{n_1}{3}\right)$

The quantity R is called the coefficient of resistance.

Relation between A and bd,

$$A = rbd \text{ (by definition)}$$

$$\text{then } trbd = \frac{bnc}{2} \quad \text{and} \quad \frac{t}{c} = \frac{n}{2rd}$$

$$\text{therefore } \frac{n}{2rd} = \frac{m(d-n)}{n} \quad \text{from Equation (8)}$$

$$\text{and } n^2 = 2mrd(d-n)$$

Solving $n = \{-mr \pm \sqrt{m^2r^2 + 2mr}\}d$

n cannot be minus.

$$\therefore\ n = n_1d = (\sqrt{m^2r^2 + 2mr} - mr)d \quad . \quad . \quad (13)$$

$$\text{or } n_1 = \sqrt{m^2r^2 + 2mr} - mr. \quad . \quad . \quad (14)$$

If $n_1 = \cdot 36$ and $m = 15$ (for the conditions taken),

$$\cdot 36 = \sqrt{225r^2 + 30r} - 15r \quad . \quad . \quad (15)$$

Solving Equation (15),

$$r = \cdot 0068 \quad . \quad . \quad . \quad (16)$$

$$\text{and } \therefore\ A = \cdot 0068bd \quad . \quad . \quad . \quad (17)$$

Equations which give the relation between the area of the steel and the effective area of the beam when

$t = 16{,}000$ lb./sq. in., $c = 600$ lb./sq. in., and $m = 15$

(The economical percentage of steel $= p_c = 100\,r = \cdot 68$.)

Rework examples given, taking $c = 800$ lb. per sq. in. $t = 18{,}000$ lb. per sq. in., and $m = \frac{40{,}000}{2400}$

$\simeq 16$, say.

$$\text{From Equation (7)} \quad \left(\frac{t}{m}\right)\left(\frac{1}{c}\right) = \frac{d-n}{n} = \frac{c_t}{c}.^{*}$$

$$\text{Also } tA = \frac{bnc}{2}.$$

---

[*] $C_t = t/m$ is the stress at the depth d of the section assuming a straight line distribution of stress on the concrete basis. *See* page 347.

$$\therefore\ mc\,.\,\frac{d-n}{n}\,.\,A = \frac{bnc}{2}.$$

$$\therefore\ (mA)(d-n) = bn \times \frac{n}{2}\ .\quad .\quad .\quad .\quad . \quad (18)$$

$$\text{and } c_t\,.\,(mA) = \frac{bnc}{2}\quad .\quad .\quad .\quad .\quad . \quad (19)$$

ANALYSIS WHEN THE STEEL AREA A IS KNOWN.

Interpretation of Equations (18) *and* (19). c_t is the equivalent tensile stress in an equivalent concrete area (mA) on the tension side. The equivalent elastic modulus of the equivalent concrete area (mA) or transformed concrete area is E_c. That is, the reinforced section has been transformed into an equivalent concrete section having a transformed concrete area on the tension side equal to mA. (*See* Fig. 194A.)

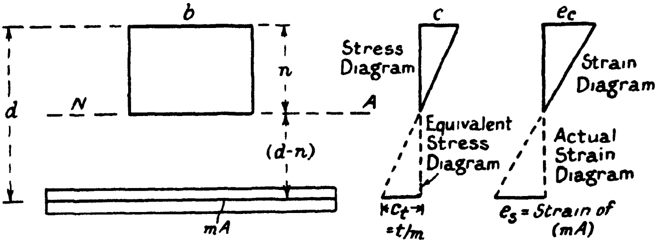

FIG. 194A

The transformed concrete area is distributed parallel to the neutral axis and its axis is parallel to the neutral axis. The area is concentrated along the axis. From equation (18), to find the neutral axis of the transformed section, take the moment of the transformed section about the neutral axis and equate it to the moment of the compression area about the neutral axis.

We have $\frac{bn^2}{2} + mAn = mAd$ from equation (18).

$$\therefore\ n^2 + \frac{2mA}{b}\,.\,n = \frac{2mAd}{b}.$$

$$\therefore\ n^2 + \frac{2mA}{b}\,.\,n + \frac{m^2A^2}{b^2} = \frac{m^2A^2}{b^2} + \frac{2mAd}{b}$$

$$\therefore\ n = -\frac{mA}{b} \pm \sqrt{\frac{m^2A^2}{b^2} + \frac{2mAd}{b}}.$$

Thus, in any given beam when b, d, A and m are known, the neutral axis has a fixed position (through the centroid of the transformed section) and therefore the ratio of c and t is constant. Also from equation (18), it can be seen that any increase in A for a given m will increase the value of n, because $(d - n)$ decreases; in other words, the neutral axis is lowered. For a given A, a lower value of m raises the neutral axis, that is, decreases the value of n. A higher value of m lowers the neutral axis for a given value of A, i.e. the value of n is increased.

EXAMPLE.

A rectangular reinforced concrete beam has $b = 10$ in., $d = 20$ in., $A = 2$ sq. in., $m = 15$. The beam carries a bending moment of 480,000 in.-lb. Calculate c and t.

To find the neutral axis position.

$$mA(d - n) = \frac{bn^2}{2}$$

$$30(20 - n) = 5n^2.$$

$$\therefore\ n^2 + 6n = 120.$$

$$n^2 + 6n + 9 = 129.$$

$$n = 11{\cdot}36 - 3 = 8{\cdot}4 \text{ in.}$$

$$d - \frac{n}{3} = 20 - 2{\cdot}8 = 17{\cdot}2 \text{ in.}$$

$$\therefore\ \frac{bnc}{2} = tA = \frac{\text{Bending Moment}}{\left(d - \frac{n}{3}\right)} = \frac{480{,}000}{17{\cdot}2}.$$

$$\therefore\quad c = \frac{480{,}000}{17{\cdot}2} \times \frac{1}{42} = 665 \text{ lb. per sq. in.}$$

$$t = \frac{480{,}000}{17{\cdot}2 \times 2} = 13{,}950 \text{ lb. per sq. in.}$$

What is the maximum moment which the beam of the foregoing problem can carry, assuming that the limiting stresses for c and t are 750 lb. per sq. in. and 18,000 lb. per sq. in. respectively?

If $m = 15$, $n = 8{\cdot}4$ in. again, and the arm of the resisting couple is therefore 17·2 in. If c has its maximum value of 750 lb. per sq. in., then the corresponding resisting couple is

$$10 \times \frac{750}{2} \times 8{\cdot}4 \times 17{\cdot}2 = 541{,}800 \text{ lb.-in.}$$

If t has its maximum value of 18,000 lb. per sq. in., then the corresponding resisting couple is

$$2 \times 18{,}000 \times 17{\cdot}2 = 619{,}200 \text{ lb.-in.}$$

The maximum bending moment which can be carried is 541,800 in.-lb. when $c = 750$ lb. per sq. in. and $t = 15{,}267$ lb. per sq. in. Thus the limiting steel stress is not realized.

192*a*. **Design a Rectangular Beam with Tension Reinforcement to Carry a Given Bending Moment at Given Stresses.** The most economical beam results when both materials are stressed to the limit and the problem is to determine b, d, and A for a given value of m, such that the permissible stresses will be realized simultaneously when the internal moment of resistance is equal to the stated bending moment. If A is the steel area and t the given tensile stress, then mA will be the transformed concrete area, and the equivalent tensile stress in it equal to t/m. If c is the limiting compression stress, then n is found from the equation

$$\frac{c}{n} = \frac{t}{m(d-n)} \text{ or } \frac{t}{m \, . \, c} = \frac{d-n}{n} \text{ from equation (7)}$$

$$\therefore \; n = d\left(\frac{c}{\frac{t}{m} + c}\right) \text{ or } n_1 = \frac{c}{\frac{t}{m} + c}$$

$$B = R \, . \, bd^2 \text{ where } R = \frac{c}{2} \, . \, n_1\left(1 - \frac{n_1}{3}\right)$$

We have, therefore, a single equation containing the two unknowns b and d. It is thus necessary to make an assumption regarding b in relation to d, i.e. $b = xd$. For small rectangular beams $x = \frac{1}{2}$ to $\frac{3}{4}$ and for large beams $x = \frac{1}{4}$ to $\frac{1}{3}$.

Now d can be calculated. In practice b and d would commonly be made an integral number of inches, and if d as calculated consisted of a whole number and a fraction of inches, then it would be necessary to re-design the beam to meet the condition that d would be an integral number of inches. Knowing B, b, d, t, C, and n, A can be found.

Reference should be made to advanced works on design for the principles underlying the method of changing from the theoretical dimensions to an exact integer.

Therefore, to find n or n_1, when the fibre or skin stresses are known, the formulae

$$n = d\left(\frac{c}{\frac{t}{m} + c}\right) \text{ or } n_1 = \frac{c}{\frac{t}{m} + c} \text{ will be used.}$$

193. **Tee-Beams.** In practice tee-beams usually form part of a floor system and act integrally with the slab on either side, which forms a flange giving added strength in the compressive part. If the beams are widely spaced, the compressive stresses are not distributed uniformly across the whole width of the slab. In order to investigate or design the usual tee-beam, it is necessary to make some assumption regarding the width of the slab which will be considered to act reasonably with the stem or rib and be uniformly stressed over the whole width. In British practice, for the breadth of the flange, the least of the following is taken—

(*a*) one-fourth of the effective span of the tee-beam,

(*b*) the distance between the centres of the ribs of the tee-beams, or

(*c*) twelve times the thickness of the slabs.

The minimum breadth of the rib should not be less than one-third of the depth of the rib below the slab. There are, however, two methods of designing a tee-beam with the flange provided by a floor slab. The first method assumes that the full breadth of flange is available for use. The compressive stress in the flange is usually found to be low by this method. The other method assumes that the limiting stresses are realized, and that the breadth of slab called into play is only that necessary. This breadth is usually less than the limit set by the various codes. The position of the neutral axis, and the arm of the resisting couple, will have different values by these two methods. Both assumptions are no more than conveniences which give satisfactory results. The design of tee-beams consists in proportioning the stem or rib, and determining the tension steel area. Proportioning the stem requires a consideration of the shearing stresses.

STEEL RODS IN THE TENSION SIDE ONLY. (1) *When the neutral axis falls within the slab*, the analysis is similar to that of the rectangular beam with reinforcement in the tension side only, remembering that b applies to the slab breadth and not the rib breadth.

(2) When the neutral axis falls below the slab. (Fig. 195.) The compressive stress in the small portion of the rib will be neglected. Assume that the breadth b is such that the limiting stress of c is realized in the slab, and t in the steel.

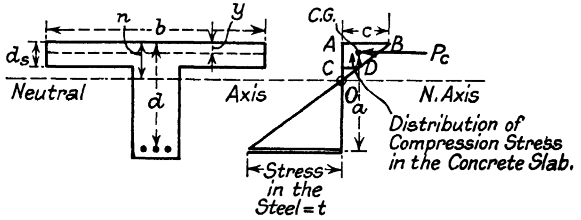

FIG. 195

b, d_s, d, t, c and n are known

$$\frac{\text{Maximum strain in the concrete}}{\text{Maximum strain in the steel}} = \frac{\frac{c}{E_c}}{\frac{t}{E_s}}$$

$$= \frac{c}{t} \cdot \frac{E_s}{E_c} = \frac{n}{d-n}$$

$$\frac{c}{t} = \frac{n}{m(d-n)} \quad . \quad . \quad . \quad (20)$$

knowing c and t, n can be found.

The compressive stress in the concrete at the base of the slab

$$= c \times \frac{n - d_s}{n}; \qquad (d_s = \text{depth of slab}) \qquad (21)$$

The total compressive force in the concrete will act at a depth y from the maximum compression edge.

(From Fig. 195.)

$$\text{Area } ABCD \times y = AOB \times \frac{AO}{3} - \frac{OCD}{3}\left(\frac{CO}{3} + AC\right)$$

from the stress diagram.

Now $n = n_1 d$ and $d_s = s_1 d$

$$\text{Then } y = \frac{s_1 d(3n_1 - 2s_1)}{3(2n_1 - s_1)} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$\text{and } d - y = a = d\left\{1 - \frac{s_1}{3}\left(\frac{3n_1 - 2s_1}{2n_1 - s_1}\right)\right\} \quad . \quad . \quad . \quad (23)$$

Total tensile force = total compressive force

$$tA = c\left(\frac{2n - d_s}{2n}\right)bd_s \quad . \quad . \quad . \quad (24)$$

where $\frac{c(2n - d_s)}{2n}$ = average stress in the slab

From the equation, the steel area A can be calculated. Substituting for n and d_s as before, it can be shown

$$n_1 = \frac{2rm + s_1^2}{2rm + 2s_1} \quad . \quad . \quad . \quad . \quad (25)$$

$$\text{And consequently } a = d\left\{\frac{s_1^3 + 4mrs_1^2 - 12mrs_1 + 12mr}{6mr(2 - s_1)}\right\} \quad . \quad . \quad . \quad (26)$$

$$\text{Approximately,} \quad a = d - \frac{d_s}{2} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$\text{The internal moment of resistance} = R_t = tA \, . \, a \quad . \quad (28)$$

$$= R_c = cbd_s\left(\frac{2n - d_s}{2n}\right)a \quad . \quad (29)$$

Substituting the value for a, the moment of resistance in terms of c or t can be found. The problem can again be easily solved by the transformed area method.

194. **The Investigation of the Maximum Stresses at a Given Section of a Tee-beam for a Stated Moment, and of the Maximum Permissible Moment for a given Tee-beam with Certain Limiting Stresses Given.** Here b, d, d_s, A and m will be known. The analysis follows the same lines as for the similar case of the rectangular beam. It will be best considered from the transformed area method, and the compression in the stem will again be neglected. The compression area will therefore

be considered as acting at a depth of $d_s/2$ from the compression skin. A will again be transformed to mA, and from the equation

$$bd_s\left(n - \frac{d_s}{2}\right) = mA \,.\, (d - n)$$

the value of n and hence the position of the neutral axis is found.

Draw the equivalent stress diagram, where the maximum compressive stress is c, the stress at the under-edge of the slab is equal to $\frac{c}{n}(n - d_s)$, and the tensile stress in the transformed steel area is t/m.

If the moment B is given, it is a simple matter to compute the total compressive stress C in terms of c and hence the lever arm a.

$Ca = B$, an equation from which C and hence c can be found. Also $C = \frac{t}{m} \,.\, mA = tA$ from which t is calculated.

If the limiting stresses and A are given but not b the breadth of the flange, the investigation may be concerned with the maximum moment that can be carried by the beam and also the breadth of the flange. As in the preceding discussion, locate the position of the neutral axis from the stress diagram drawn with the extreme stresses taken as equal to the limiting stresses, and calculate the lever arm a. As in the case of the rectangular beam, calculate the moment assuming the limiting stress of t is realized. $B = tAa$.

The total compressive stress will be $C = T$. Calculate the average C_a per unit breadth of the flange: then $b = \frac{C}{C_a}$. Compare this calculated b with the value of b allowed from one of the limiting formulae.

$b = \frac{C}{C_a}$ will in general be less than the maximum allowable value of b.

Illustrative Problem 45.

A reinforced concrete beam is 3·5 in. wide and 4·25 in. deep to the centre of the tensile steel reinforcement. The working stresses are 600 and 16,000 lb. per square inch for the concrete and steel respectively,

$$\frac{E_s}{E_c} = 15$$

Find the depth of the neutral axis from the compression flange, the area of the steel in tension, the percentage reinforcement, and the moment of resistance.

From paragraph 192.

$$\frac{16{,}000}{600} = 15\left(\frac{4{\cdot}25 - n}{n}\right)$$

Solving $n = 1{\cdot}53$ in.

Total force in the steel

$$= \frac{600 \times 3{\cdot}5 \times 1{\cdot}53}{2} \text{ lb.} = 1605 \text{ lb.}$$

$$\text{Area of steel} = \frac{1605}{16{,}000} = {\cdot}1 \text{ sq. in. nearly}$$

$$\text{Percentage reinforcement} = \frac{{\cdot}1 \times 100}{4{\cdot}25 \times 3{\cdot}5} = {\cdot}617 \text{ per cent}$$

$$\text{Moment of resistance} = 1605 \times \left(4{\cdot}25 - \frac{1{\cdot}53}{3}\right)$$

$$= 6000 \text{ lb.-in.}$$

195. **Distribution of Shear Stress in a Reinforced Concrete Beam of Rectangular Cross-section and with Only Reinforcements on the Tension Side.** (Fig. 196.) Consider the forces acting on a length of beam δx between two vertical planes AD and BC. The concrete again carries no load in tension.

Let the compressive stress in the outside fibre at A be c and at B, $c + \delta c$.

Let q be the shear stress on a horizontal plane of area $b \,.\, \delta x$ situated at a height y above the neutral axis; then $q \,.\, b \,.\, \delta x$

$$= -\frac{1}{2}\left(c + \frac{c \,.\, y}{n}\right)(n - y)b \,. + \frac{1}{2}\left(c + \delta c + \frac{(c + \delta c)y}{n}\right)(n - y)b$$

$$= \frac{\delta c}{2}\left(1 + \frac{y}{n}\right)(n - y)b$$

$$\therefore\ q = \frac{1}{2} \cdot \frac{\delta c(n^2 - y^2)}{n \,.\, \delta x} \qquad (30)$$

This is zero when $y = n$, and a maximum when $y = 0$

$$q_{max} \text{ at the neutral axis} = \frac{\delta c \,.\, n}{\delta x \,.\, 2} \qquad (31)$$

$= q_{max}$ on the vertical plane through a point on the neutral axis.

The curve of vertical shear distribution in the compression side of the beam is a parabola, being zero at the skin and a maximum at the neutral axis.

Below the neutral axis, assuming the tensile forces in the concrete negligible, the shear stress must be constant between the neutral plane and the plane of the tensile reinforcement, where the whole of it is balanced by the difference in the tensile forces in the steel at K and L.

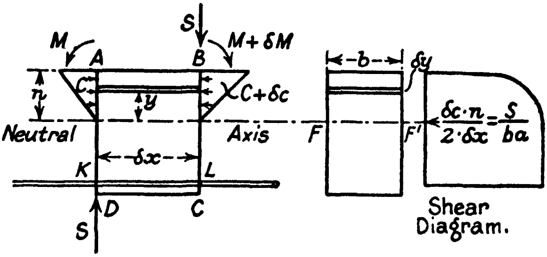

FIG. 196

Let M be the bending moment at AD and $M + \delta M$ at BC. Let a be the arm of the internal moment of resistance. The internal moment of resistance at AD is $Ca = P_t a = M$

where C = total compressive force in the concrete

and P_t = „ tensile „ steel.

$$\text{Now } -\frac{M}{a} + \frac{M + \delta M}{a} = -C + (C + \delta C)$$

$$= +\delta C \quad . \quad . \quad . \quad (32)$$

$$\text{Now } \delta C = (c + \delta c)\frac{bn}{2} - c\frac{bn}{2} = \delta c \, . \frac{bn}{2}$$

$$\therefore \ \delta c \, . \frac{bn}{2} = \frac{\delta M}{a} = \frac{S \, . \, dx}{a} \quad . \quad . \quad . \quad . \quad (33)$$

where S is the total shearing force at the section.

But $\frac{1}{2} \, . \, \delta c \, . \, nb = q_{max} \, . \, b \, . \, \delta x$ (From (31))

where q_{max} is the shear stress at the N.A.

$$\therefore \ q_{max} = \frac{S}{b \, . \, a} = \text{vertical shear at the N.A.} \quad . \quad (34)$$

For rectangular beams and for tee-beams where b is replaced by b_1 the breadth of the stem, an average value of a is usually taken as $\frac{7}{8}d$. Actually $a = d - \frac{n}{3}$.

196. Diagonal Tension in Reinforced Concrete Beams. In the chapter on principal stresses, it was shown that a tensile force could be compounded with complementary shear forces acting at the same point within the material to produce principal stresses

$$R = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q^2}$$

where $+ p$ is the tensile stress and q the shear stresses. If it is assumed that the concrete cannot carry tensile forces then

$$R = \pm q$$

that is, there will be two principal stresses, one a tensile stress equal to $+ q$, and the other a compressive stress equal to $- q$. These planes will be at angles of 45° and 135° to the plane of the initial tensile stress. The tensile stress $R = + q$ acting at an angle of 45° to the plane of the initial tensile stress is known as the *diagonal tension* stress. The concrete in a reinforced beam is no stronger in itself than when unreinforced, and it cracks in any loaded beam when the tensile stress limit is exceeded. In all loaded reinforced beams, therefore, on the assumption that the concrete cannot take tensile load below the N.A., lines of cracking (cracks inclined at 45° to the line of the beam) will develop in the beam when the tensile limit stress of the concrete is exceeded. The direction of the cracks will be away from the supports. Now if reinforcement known as *diagonal tension* or *web* (or more incorrectly, *shear*) *reinforcement* is placed in the beam, either vertically to the line of the tensionreinforcement or normal to the probable line of diagonal tension crack, then this web reinforcement will function to keep any one crack from opening up widely and compel the formation of many minute cracks in place of the single large one which would cause failure. In order to proportion the web reinforcement, knowledge must be had of the diagonal tension.

The web reinforcement in practice consists of stirrups, generally vertical, looped about the main steel, and of main longitudinal rods bent up at an angle across the region of diagonal tension stress in those portions of the beam where they are no longer needed to resist the normal tension. Tests indicate that the concrete is effective in resisting small amounts of diagonal tension, and may be counted upon with safety to perform this duty when the shearing stress is less than about 40 lb. per sq. in. for most ordinary mixes. When q exceeds this limit, the concrete is ordinarily still counted upon as carrying a portion of the diagonal tension. The principal compressive stress $= -q$ can, of course, be carried effectively by the concrete itself. The working maximum allowable value of q is usually 120 lb. per sq. in., but the suggested code recommends values dependent upon the proportion and grade of concrete to be used.*

STRESSES IN DIAGONAL TENSION REINFORCEMENT. The methods given are purely approximate, producing empirical rules that have been found to give safe and economical results.

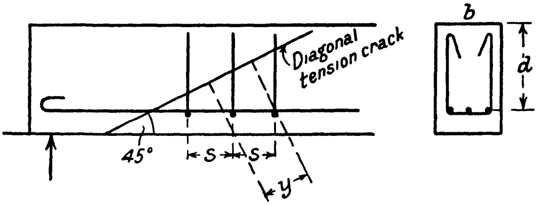

FIG. 197A

(*a*) *Vertical Stirrups.* The centre stirrup of the three shown in Fig. 197A is assumed to carry all or part of the vertical component of the diagonal tension acting over the distance y along the 45 degree line of potential cracking. The horizontal opening of the crack is prevented by the longitudinal steel, which may be considered to carry the horizontal component of the diagonal tension. The concrete, too, is credited with carrying a certain portion of the vertical component of the diagonal tension. The total amount of the diagonal tension within the distance y is $q \,.\, b \,.\, y$ and the vertical component is $q \,.\, b \,.\, s$.

The stress (T_s) in the stirrup then is $q_1 bs$ where q_1 is the amount of shear denoting the share of diagonal tension carried by the stirrup. q_1 is often assumed to be $\frac{2}{3}q$ or equal to $(q - 40)$ lb. per sq. in.

NOTE. The vertical component of diagonal tension in any distance along a R.C. beam is taken as equal to the total horizontal shear in that distance.

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* Suggested Code of Practice, *see* References at end of chapter.

If t_s lb. per sq. in. is the tensile stress in the vertical stirrup and A_s is the total cross-sectional area of the legs of the stirrup

$$\text{then } t_sA_s = q_1 . b . s = \frac{S_1}{ba} . b . s = \frac{S_1 s}{a} \quad . \quad . \quad . \quad . \quad (35)$$

where S_1 is the proportion of the average shear force over the distance s and a is the lever arm of the internal moment of resistance.

(*b*) *Inclined Rods.* Assuming that the vertical component of diagonal tension in a horizontal spacing of s is the vertical component of the tensile stress in the 45° inclined rod, then

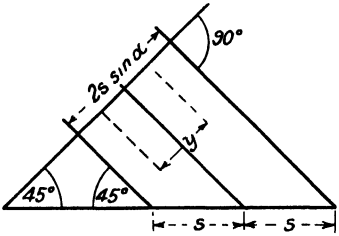

FIG. 197B

$$\begin{aligned} t_sA_s &= q_1by \\ &= q_1bs \sin a \\ &= 0{\cdot}707q_1bs = 0{\cdot}707S_1s/a \quad . \quad (36) \end{aligned}$$

Comparing the spacing of the vertical and inclined rods, if t_s, A_s q_1 and b are the same for both cases, then s for the vertical stirrups is equal to 0·707 of the spacing (s) for the inclined rods. Therefore inclined rods are more effective than vertical stirrups for taking up the diagonal tension. The limiting spacing for vertical stirrups is one-half the depth to the main steel centre, and for inclined rods, three-quarters the depth. Of course, the vertical stirrups can act in conjunction with inclined rods.

SPACING OF THE DIAGONAL TENSION REINFORCEMENT. *Analytical Method.* Let the distance between two sections A and B of a reinforced concrete beam be ($\triangle l$) equal to s. Let the moments at A and B be M_A and M_B respectively (say $M_B > M_A$).

Then the average rate of change of moment over the length $(\triangle l)$

$$= \frac{M_B - M_A}{(\triangle l)} = S_A$$

$$\text{or } S_A\,(\triangle l) = M_B - M_A.$$

Therefore equations (35) and (36) can read

$$t_s\,.\,A_s\,.\,a = k(M_B - M_A) \text{ or } 0{\cdot}707k(M_B - M_A)$$

If A_s is assumed and constant, then $t_s\,.\,A_s\,.\,a$, will be constant for a beam of uniform cross-section, if a is assumed constant.

Then $M_B - M_A$ is a constant, for k and $0{\cdot}707k$ are constant.

$$\therefore\ M_B - M_A = \frac{t_s A_s a}{k} \text{ or } = \frac{t_s A_s a}{0{\cdot}707k} = \text{constant.}$$

To find $M_B - M_A$; assume a value of A_s; t_s, a and k will be known, and the change of moment can be found.

Construct the moment diagram for the beam on the length of beam as base : on a vertical axis and from the horizontal axis mark off intervals corresponding to $M_B - M_A$. Draw lines parallel to the base of the moment diagram from these interval points, to cut the moment diagram (generally) in two points. From these points of intersection drop perpendiculars on to the base (length of beam to scale) of the moment diagram. Where these perpendiculars touch, the base will give the position of the stirrups at the tensile reinforcement.*

If A_s is not assumed, the first stirrup is usually placed at a distance not less than $\frac{a}{2}$ from the support. $\left(\frac{a}{2}\right.$ or other distance will correspond to $\left.\triangle l.\right)$ The change of moment over this distance can be measured from the moment diagram, or it can be calculated. A_s can then be found using the required equations (35 – 36). The spacing can be found as before. It will be found that for more or less uniform loading the stirrups are spaced closer together near the supports than at the centre of the beam.

If stirrups are too far apart there will be opportunity for inclined cracks to open between them, and the limiting spacing is often taken as $d/2$ for vertical stirrups, and as $3d/4$ for inclined rods. A further point to be watched is the anchorage at the ends of the stirrups and bars. They may be very highly stressed at the neutral axis and, therefore, the question of the "bond" between the bars and the concrete is called into play. If the ends are anchored by a hook, a satisfactory anchorage is obtained. The discussion of bond and anchorages must be left to books on reinforced concrete design.

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* For practical rules of spacing, *see* Problem 45*b*, page 360.

Illustrative Problem 45*a*.

A rectangular reinforced concrete cantilever beam 12 ft. long carries an end load of 20,000 lb. The beam section is $b = 10$ in. $d = 20$ in. If one vertical stirrup can carry a load of 4500 lb., how many stirrups will be required? Assume the concrete can carry one-third of the diagonal tension.

We have $q = \dfrac{S}{b\,.\,a} = \dfrac{20{,}000}{10 \times \frac{7}{8} \times 20}$ lb. per sq. in.

Load to be carried by vertical stirrups

$$= \frac{2}{3} q\,.\,l\,.\,b \text{ lb.}$$

$$= \frac{2}{3}\,\frac{20{,}000}{10 \times \frac{7}{8} \times 20} \times 12 \times 12 \times 10 \text{ lb.}$$

$$= 109{,}500 \text{ lb.}$$

No. of stirrups $= \dfrac{109{,}500}{4{,}500} = 24.$

Spacing of the stirrups $= \dfrac{144}{24} = 6$ in.

Illustrative Problem 45*b*.

A simple reinforced rectangular concrete beam is 24 in. deep to steel centre and 12 in. wide. It supports a total uniformly-distributed load of 3000 lb. per ft. run on a clear span of 20 ft. Use single loop stirrups of $\frac{3}{8}$ in. round material, which can be assumed to be stressed to a maximum of 18,000 lb. per sq. in. Design the diagonal tension reinforcement, using vertical stirrups, and assume no stirrups are required where the maximum shear stress is less than 40 lb. per sq. in.

The total end shear is 30,000 lb. decreasing uniformly to zero at the middle of the beam. The intensity of maximum shear stress at the end of the beam is

$$q = \frac{30{,}000}{12 \times \frac{7}{8} \times 24} = 119 \text{ lb. per sq. in, say } 120.$$

Let y in. be the distance from the middle of the beam along which no stirrups are required.

$$\text{Then } \frac{120}{120} = \frac{40}{y} \qquad \therefore\ y = 40 \text{ in.}$$

The vertical stirrups will be required over 80 in. at each end.

A single loop stirrup of $\frac{3}{8}$ in. round material can carry

$$2 \times 0{\cdot}11 \times 18{,}000 \text{ lb.} = 3960 \text{ lb.}$$

Assuming that the concrete can carry diagonal tension represented by $q = 40$ lb. per sq. in., then the total diagonal tension load to be carried by the stirrups in 80 in. length from the support is

$$\tfrac{1}{2}(120 - 40) \times 12 \times 80 = 38{,}400 \text{ lb.}$$

The number of stirrups required is

$$\frac{38{,}400}{3960} \simeq 10.$$

The average spacing will be

$$\frac{80}{10} = 8 \text{ in.}$$

Note that as the spacing varies inversely as the shear, then the end spacing will be

$$\frac{\frac{1}{2}(120 - 40)}{80} \times 8 = 4 \text{ in.}$$

whilst at a distance of 80 in. from the support it will be

$$\frac{\frac{1}{2}(120 - 40)}{0} \times 8 = \text{infinity}$$

which is greater than $\frac{24}{2}$ the maximum spacing allowable.

In choosing stirrups for a beam, a convenient practice is to calculate the number required in each end and space them approximately, the spacing being given in multiples of 2 or 3 in. Some designers prefer to calculate the spacing at several points, and place the stirrups accordingly; others prefer to employ more stirrups than required, in order to keep down the spacing to the maximum allowable.

For the stirrups required in the present problem, a suggested spacing is

2–4–6–6–6–9–9–9–12–12–12

the first dimension being from the support. In practice the stirruping would be carried through the whole length of beam, a number of stirrups at about maximum spacing being placed across the central portion of the beam where q is less than 40 lb. per sq. in.

Illustrative Problem 45*c*.

Two $\frac{7}{8}$ in. steel bars are bent up at an angle of 45°, 15 in. from the support of a reinforced concrete beam of effective depth 20 in. and breadth 10 in. The average shear over the 15 in. is 110 lb. per sq. in. Are vertical stirrups required to assist the bent-up bars in taking diagonal tension? Assume limiting stress in bent-up bars to be 18,000 lb. per sq. in.

Assume concrete can take diagonal tension to the amount represented by 40 lb. per sq. in.

Then using equation

$$t_sA_s = 0{\cdot}707q_1bs$$

$$\text{where } A_s = 2 \times 0{\cdot}785 \times \frac{49}{64} = 1{\cdot}2 \text{ sq. in.}$$

$$q_1 = (120 - 40) = 80 \text{ lb. per sq. in.}$$

$$b = 10 \text{ in.} \quad s = 15 \text{ in.}$$

$$\text{then } t_s = \frac{0{\cdot}707 \times 80 \times 10 \times 15}{1{\cdot}2} = 7070 \text{ lb. per sq. in.}$$

which is less than 18,000 allowable.

Therefore vertical stirrups are not required. As the shear, say, at the point of bend is 100 lb. per sq. in., vertical stirrups will be required in the length of beam from this point to the point where $q = 40$ lb. per sq. in.

196*a*. **Bond Stress and Anchorage or Embedment.** The question of proper length of embedment of steel in concrete arises whenever there is stressed steel in concrete. Whether the stress is tension or compression, a rod must extend beyond any point of stress a distance sufficient to develop in bond the total stress there existing. For instance, suppose a number of tension bars of a cantilever beam to be anchored in a supporting mass of concrete. What is the length the bars must extend into the mass concrete, such that the resistance to pulling out developed with the allowable bond stress is equal to, or greater than, the total stress in the bars at the face of the mass concrete?

Let u be the allowable bond stress between the steel and the concrete in the mass concrete. Let Σo represent the total perimeters of the bars, l the length of embedment or anchorage, f_s the stress in the bars at the face of the mass concrete, and ΣA_s the total sectional area of the bars. There results as a general expression

$$u \,.\, l \,.\, \Sigma o = f_s \Sigma A_s$$

For a single square bar of side D and for a single round bar of diameter D it can be shown that

$$l = \frac{f_s}{4u} \,.\, D.$$

There are other ways to secure the necessary anchorage such as hooks and mechanical devices. Specifications lay down acceptance rules and dimensions for such devices.

Bond also plays its part when considering the rate at which stress passes from the concrete to the rod, in the case of reinforcement for tension or compression in beams.

Consider the rate of transfer of stress from concrete to the tension steel in beams. Referring to para. 195, it will be seen that the bond stress, the tendency of the rods to slip, equals the horizontal shear; and therefore if we consider unit length of beam of breadth b and a horizontal plane below the neutral axis, and if q be the average horizontal shear stress over the plane, and u the average bond stress between the steel and the concrete, then

$$u \,.\, \Sigma o \,.\, 1 = q \,.\, b \,.\, 1$$

where Σo is the sum of the perimeters of the rods.

$$\text{Then } u = \frac{q \,.\, b}{\Sigma o}$$

u in general is kept below 100 lb. per sq. in.: this value is, however, sometimes exceeded where effective end anchorage is provided.*

However, tests show that this theoretical relation between u and q does not quite cover all the facts. Nevertheless, it forms a useful basis for comparison in beams in which the dimensions and general make up are similar.

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* *See* also suggested values for u in the recommended Code of Practice.

If S is the shear force at a section

$$\text{Then } u = \frac{qb}{\Sigma o} = \frac{S \cdot b}{b \cdot a \cdot \Sigma o} = \frac{S}{a \cdot \Sigma o}$$

where a is the arm of the resisting moment.

$$\text{Now } a = \frac{d(3 - n_1)}{3}$$

$$\therefore\ u = \frac{3S}{(3 - n_1)d\Sigma o}$$

Comparing the values of u and q for the same steel area,

$$u = \frac{S}{a\Sigma o} \text{ and } q = \frac{S}{a \cdot b}$$

If $\Sigma o < b$, then $u > q$.

Therefore to make $u =$ or $< q$, for the same steel area, it is only necessary to increase the number of bars, and at the same time decrease the diameter, so that $\Sigma o =$ or $> b$.

E.g. assume one bar, $d = 1$, perimeter $= \pi$, area $= \frac{\pi}{4}$.

Assume $d = \frac{1}{2}$, number of bars required for area $\frac{\pi}{4}$ is 4, giving a perimeter of 2π.

$\therefore$ 4 bars of dia. $= \frac{1}{2}$ have twice the perimeter of 1 bar of dia. $= 1$.

Illustrative Problem 45*d*.

The overall dimensions of a rectangular reinforced concrete beam are 26 in. by 12 in. The centre of the steel area of 3 sq. in. for 3-1 in. square bars is 2 in. from the nearer edge. The shear load at a section is 24,000 lb. Calculate the intensity of the bond and shear at this section.

Assuming $a = \frac{7}{8}d$, then for the section given

$$a = \tfrac{7}{8}(26 - 2) = 21 \text{ in.}$$

$$\text{Then } q_{max} = \frac{S}{ba} = \frac{24{,}000}{12 \times 21} = 96 \text{ lb. per sq. in.}$$

$$u = \frac{S}{a \cdot \Sigma o} = \frac{24{,}000}{21 \times 12} = 96 \text{ lb. per sq. in.}$$

Assuming $m = 15$, to find $a = d - \frac{n}{3}$

$$\text{We have } mA_s(d - n) = \frac{bn^2}{2}$$

$$\therefore\ 15 \times 3\,(24 - n) = \frac{12}{2} \times n^2$$

$$\therefore\ 180 - 7{\cdot}5n = n^2$$

$$n^2 + 7{\cdot}5n = 180$$

$$n^2 + 7{\cdot}5n + \frac{7{\cdot}5^2}{4} = 180 + \frac{7{\cdot}5^2}{4}$$

$$n + \frac{7{\cdot}5}{2} = \pm\sqrt{194}$$

$$n = 13{\cdot}9 - 3{\cdot}75 = 10{\cdot}15 \text{ in.}$$

$$\frac{n}{3} = 3{\cdot}38 \text{ in.}$$

$$\therefore\ a = (24 - 3{\cdot}38) = 20{\cdot}62 \text{ in.}$$

$$\text{cf. } a = \tfrac{7}{8} \times 24 = 21 \text{ in.}$$

Assuming there are 3 – 1 in. round bars instead of 3 – 1 in. square bars, and $a = 21$ in.

$$q = \frac{24{,}000}{12 \times 21} = 96 \text{ lb. per sq. in.}$$

$$u = \frac{24{,}000}{21 \times 3 \times \pi} = 121 \text{ lb. per sq. in.}$$

197. **Reinforced Concrete Columns with Axial Loads.** The usual type of reinforced concrete compression member has a circular, octagonal, or rectangular concrete section with a series of rods, parallel to the longitudinal axis of the member, set about 2 in. back from the surface all around the perimeter. The steel reinforcement is from 0·8 to 8·0 per cent of the cross-sectional area of the concrete. The main reinforcing bars are held in place either by being wired to an encircling series of hoops or ties, or to a closely spaced steel wire spiral. The vertical or longitudinal reinforcement deforms the same as the surrounding concrete, as the column shortens under load. The action of the ties is to bind the rods together and into the mass of concrete in such a way that they themselves will not

buckle and cause failure. As heavy initial stresses are induced in the longitudinal reinforcement during the shrinkage of the concrete whilst hardening, the function of the ties is important. The task is more efficiently performed by the spiral reinforcement, as it serves to restrain lateral movement of the concrete during shortening. As the spiral only comes into play after the column passes its elastic limit, certain authorities consider that no allowance should be made for the increased strength which it affords, whilst others make this allowance. Reference should be made to the work of Considére in this respect. Most reinforced columns are so short that their ultimate strength is not limited by any tendency towards buckling or bending; their length is ordinarily less than 15 to 18 times their least dimension. For slenderer columns, the working stresses must be reduced below those allowable on short columns of the same section. In columns exposed to fire, the steel must be adequately protected by a covering of concrete at least 2 in. thick. Sometimes this additional concrete is credited with being part of the column; in other cases it is only the concrete within the spiral or longitudinal bars which is effective in carrying load, i.e. the core is the effective concrete. The value of m is taken the same for columns as for beams.

(1) SHORT COLUMNS. (*a*) *With lateral ties or hoops.*

Let $A_c =$ area of concrete only, in sq. ins.; not including any finishing material applied after column is cast.

„ $A_s =$ area of steel only, in sq. ins.

„ $A = $ total area $= A_c + A_s$

„ $W =$ axial load in lb.

„ $c_c =$ compressive stress in concrete lb. per sq. in.

„ $c_s =$ „ „ steel „ „ „ „

„ $m = \dfrac{E_s}{E_c}$

Then $W = c_cA_c + c_sA_s$ * . . . . . . (37)

Now, as the strain in the concrete and the steel is the same,

$$\text{then } \frac{c_c}{E_c} = \frac{c_s}{E_s}$$

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* The code states that W shall not be greater than given by this equation: therefore c_s and c_c shall not exceed their maximum permissible values.

$$c_s = \frac{E_s}{E_c} \cdot c_c = mc_c$$

From (37) $$W = c_cA_c + mc_cA_s$$

$$= c_c(A_c + mA_s) \quad . \quad . \quad . \quad . \quad (38)$$

$$= c_c \text{ (effective area on a concrete basis)}$$

$$= c_c(A_{Ec}) \quad . \quad . \quad . \quad . \quad . \quad (39)$$

$$A_c = A - A_s$$

then $$W = c_c\{A + A_s(m - 1)\} \quad . \quad . \quad . \quad (40)$$

(*b*) *With spiral reinforcement.* The Code of Practice for reinforced concrete suggests the following. Where spiral reinforcement is used, the axial load W on the column shall not exceed the value given by equations (40), or (41), below, whichever is the greater.

$$W = c_cA_k + c_sA_s + 2{\cdot}0\, t_bA_b \quad . \quad . \quad (41)$$

where A_k = cross-sectional area of concrete in the core.

t_b = permissible stress in tension in spiral reinforcement.

and A_b = equivalent area of spiral reinforcement (volume of spiral per unit length of the column).

In no case shall the sum of the loads contributed by the concrete in the core, and by the spiral, exceed $0{\cdot}50f_cA_c$ where f_c is the crushing strength of the concrete required from the works tests, and A_c is gross sectional area of the concrete.

The value of c_c is dependent on the kind of concrete used, and varies from 600 to 1250 lb. per sq. in., whilst the maximum c_s and t_b depend on the kind of mild steel used, but vary from 13,500 to 15,000 lb. per sq. in.

(2) LONG COLUMNS. The Code suggests the following. The permissible working loads of axially-loaded long columns shall not exceed the values given by the equations for short columns multiplied by a buckling coefficient which depends upon the ratio of effective length to least lateral dimension of the column, or ratio of effective length to least radius of gyration. For further information reference should be made to the Code.

Illustrative Problem 46.

A reinforced concrete column is 14 in. square. It is reinforced longitudinally with 4 2-in. diameter steel rods placed near the corners; it carries 60 tons. Find the load carried by the concrete and steel respectively.

$$E_s \text{ (steel)} = 29 \times 10^6 \text{ lb./sq. in.}$$

$$E_c \text{ (concrete)} = 3 \times 10^6$$

$$m = \frac{E_s}{E_c} = \frac{9\frac{2}{3}}{1}$$

Total area of steel = 4π sq. in.

Area of concrete = $196 - 4\pi = 183{\cdot}44$ sq. in

W_c = load taken by concrete.

W_s = „ steel.

60 tons = $W_c + W_s$

c_s = stress in steel. c_c = stress in concrete

$$\frac{c_s}{E_s} = \frac{c_c}{E_c} \qquad \therefore c_s = 9{\cdot}66c_c$$

$$60 \text{ tons} = 183{\cdot}44c_c + (9{\cdot}66c_c \times 4\pi)$$

$$c_c = {\cdot}197 \text{ tons/sq. in.} = 440 \text{ lb. per sq. in.}$$

Load taken by the concrete = $183{\cdot}44 \times {\cdot}197 = 36{\cdot}7$ tons

„ „ steel = $60 - 36{\cdot}7 = 23{\cdot}3$ „

Stress in the steel = $\dfrac{23{\cdot}3}{4\pi} = 1{\cdot}90$ tons/sq. in.

or $c_s = {\cdot}197 \times 9\frac{2}{3} = 1{\cdot}90$ tons/sq. in. = 4,260 lb. per sq. in.

Illustrative Problem 47.

Design a column to carry an axial load of 100,000 lb. Take c_c = 600 lb. per sq. in. and m = 18.

(*a*) c_s will be equal to $18 \times 600 = 10{,}800$ lb. per sq. in. Total area of concrete required for the transformed section is

$$\frac{100{,}000}{600} = 167 \text{ sq. in.}$$

A 12 × 12 section furnishes 144 sq. in., leaving 23 as the transformed concrete area of the steel.

$$\therefore (18 - 1)A_s = 23$$

$$\text{and } A_s = \frac{23}{17} = 1{\cdot}35 \text{ sq. in.}$$

$$\frac{1{\cdot}35}{144} \times 100 = 0{\cdot}94 \text{ per cent.}$$

If the cross-section and steel first chosen are unsatisfactory for any reason, further trials must be made. The design is facilitated by the use of tables and diagrams.

In the above example stress has controlled the design.

(*b*) If the effective length of the column is, say, 18 ft., then the dimension of a square section would be

$$\frac{18}{15} = 1{\cdot}2 \text{ ft.} = \text{say } 15 \text{ in.}$$

The total load on the column is 100,000 lb.

$$\text{Then } c_c = \frac{100{,}000}{225 + (18 - 1)A_s}$$

The steel area should be between 0·8 and 8 per cent of the section, or between 1·8 sq. in. and 18 sq. in. As stiffness is controlling the design of the section, the steel area need not be too high—say four, 1 in. round bars (area 3·14 sq. in.).

$$\begin{aligned} \text{Then } c_c &= \frac{100{,}000}{225 + (17 \times 3{\cdot}14)} \\ &= \frac{100{,}000}{278{\cdot}5} \simeq 360 \text{ lb. per sq. in.} \\ c_s &= 360 \times 18 \simeq 6500 \text{ lb. per sq. in.} \end{aligned}$$

Therefore, as A_s diminishes, c_c and c_s will both increase.

Also as the column dimensions decrease, c_c and c_s will increase.*

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* For examples on Columns with spiral reinforcement consult "The New Code applied to Design" by C. E. Reynolds, *Concrete and Constructional Engineering*, p. 489, 1934. Also Mr. Probst's book (ref. p. 370).

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EXAMPLES

1. Design the central section of a reinforced concrete beam of 20 ft. span to carry a load of 600 lb. per foot run. Tensile stress allowable in the steel, 16,000 lb. per square inch. Compressive stress in the concrete, 600 lb. per square inch. Ratio of modulus of elasticity of steel to that of concrete, 12. Ratio of breadth to depth, 1 to 3. Show how you would make provision for resisting the shear stress in the beam. (U. of L.)

2. Define resilience. A concrete pillar is 12 in. square in section and has four 1 in. diameter rods as vertical reinforcement. A load of 20 tons is placed on the column. Find—

(*a*) The shortening of the column ;
(*b*) Resilience of the column.

$$E_s = 30 \times 10^6 \text{ lb./sq. in.} \quad E_c = 2 \times 10^6 \text{ lb./sq. in.}$$

(U. of B.)

3. A load of 2 tons has to be supported midway between two walls 10 ft. apart. Design a suitable reinforced concrete beam for the purpose. The outside dimensions are to be 18 in. deep and 10 in. wide. (U. of B.)

4. The vertical sides of a circular reinforced concrete tank, 60 ft. in diameter, are reinforced with circular steel rods embedded in the concrete. If the tensile stress in the steel is limited to 5 tons per square inch, calculate the area of steel required per foot in depth between 8 ft. and 9 ft. depth of water, and between 9 ft. and 10 ft. depth of water. (I.C.E.)

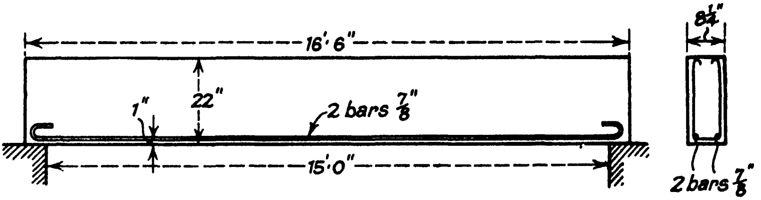


FIG. 198

5. The reinforced concrete beam shown in Fig. 198 carries two equal concentrated loads symmetrically placed on the span and 5 ft. apart. If the modular ratio is 12, and *c* and *t* are not to exceed 600 lb. per square inch and 16,000 lb. per square inch respectively, find the maximum values of the loads, taking into account the weight of the beam itself. If vertical stirrups made from ⅜-in. round steel take the whole shear, find the necessary spacing. (U. of L.)

6. A reinforced concrete slab, 7½ in. thick, has an effective span of 10 ft. The reinforcement consists of ½ in. diameter bars at 6-in. centres placed 1½ in. above the bottom of the slab. Determine what uniform load per square foot the slab will carry in addition to its own weight, if the allowable

maximum stresses are 18,000 lb. per square inch for steel and 650 lb. per square inch for concrete, and if the ratio of the modulus of elasticity of steel to that of concrete is 12. Weight of slab, 150 lb. per cubic foot. (U. of L.)

7. Write down the expressions for, and explain what you understand by (*a*) the equivalent area of a reinforced concrete column; (*b*) the equivalent moment of inertia of a reinforced concrete beam 12 in. deep, 6 in. wide, reinforced with two steel bars $\frac{1}{2}$ in. diameter at 1 in. from the underside of the beam. (U. of B.)

8. A reinforced concrete beam 6 ft. long, 5 in. wide, and 7 in. effective depth, carries a load of 2 tons applied at two points symmetrically placed relative to the centre of the beam. The area of the reinforcement is 0·6 sq. in. The distance between the supports is 5 ft., and the distance between the points of application of the load is 3 ft. Find the maximum stress in the concrete and in the steel. $m = 15$. (U. of B.)

9. A flitched timber beam consists of two timber joists, each 4 in. wide by 12 in. deep, with a steel plate $\frac{3}{4}$ in. thick and 8 in. deep, placed symmetrically between them and firmly fixed in place. If the span is 20 ft. and the ends are simply supported, calculate the maximum uniformly-distributed load the beam can carry if the stress intensity in the timber is not to exceed 1000 lb. per square inch. What will then be the maximum stress in the steel?

E for steel = 30,000,000 lb. per square inch.
E for timber = 1,500,000 lb. ,, (U. of L.)

10. What do you understand by the "equivalent area" and "equivalent moment of inertia" of a reinforced concrete column? State what assumptions are made in deducing mathematical expressions for these.

A concrete column is 15 in. × 15 in. square in section; it is reinforced with four 2 in. diameter mild steel bars. If the maximum stresses allowable are 600 lb. square inch in the concrete and 16,000 lb. square inch in the steel, what load can the column safely carry? Take the modular ratio as 15. (U. of B.)

11. What is meant by the "economical percentage of steel" in a reinforced concrete beam with tension reinforcement only? Deduce expressions for the depth of the neutral axis, the percentage reinforcement, and the moment of resistance of an "economical beam," assuming maximum stresses of 600 lb./sq. in. (compression) in concrete, and 16,000 lb./sq. in. (tension) in steel. (Take $m = 16$.) (U. of B.)

12. If kd is the depth of the neutral axis of a rectangular concrete beam reinforced on the tension side only, p the steel ratio, and m the modular ratio, prove that

$$k = \sqrt{(2pm + p^2m^2)} - pm$$

A rectangular reinforced concrete beam is 15 in. wide and 30 in. deep to the centre of the steel bars, which have an area of 4 sq. in. If $m = 15$ and the limiting stresses f_s and f_c are respectively 18,000 and 800 lb. per sq. in., determine (*a*) the position of the neutral axis, and (*b*) the maximum moment which can be carried. (U. of B.)

13. A rectangular concrete beam reinforced in the tension side only is to be designed to resist a moment of 600,000 inch-pounds. Design the economical section for an ordinary grade concrete having a minimum 28 days crushing strength based on works tests of 2250 lb. per sq. in. and a mild steel having an ultimate strength of 30 tons per sq. in. Take $m = 18$. (U. of B.)

14. What are the functions of bent-up bars and vertical stirrups in reinforced concrete beams?

The maximum compressive stress in a reinforced concrete beam is 800 lb. per sq. in. and the stress in the steel is 18,000 lb. per sq. in. The effective depth of the beam is 25 in. and m is 15. Calculate the stress in ⅜ in. U vertical stirrups at 4 in. spacing where the shearing force is 8 tons. Assume that the concrete can carry one-third of the diagonal tension. (U. of B.)

15. A double reinforced rectangular concrete beam is 15 in. wide and 30 in. deep from the compression edge to the centre of the tension steel. The areas of the compression and tension steel are both 4 sq. in., and the centre of the compression steel is one-third of the neutral axis depth from the compression edge. If m is 15 and the bending moment at the section is 120,000 pound-feet, calculate the maximum stress in the concrete and the stresses in both the compression and tension steel. (U. of B.)

16. A reinforced concrete pit prop is 6 ft. long and 6 in. square in section. The hooped longitudinal reinforcement consists of four ¼ in. square mild steel bars. If m is 15 and f_c is 500 lb. per sq. in., calculate the working load of the prop. (U. of B.)

17. Design a square reinforced concrete column, 14 ft. high, reinforced with 1 per cent longitudinal bars and ordinary hoops to carry an axial load of 100 tons. The permissible stress in the concrete is 500 lb. per sq. in. and $m = 12$. (U. of B.)

18. The section *abcd* of a short reinforced concrete column is 15 in. by 15 in. The reinforcement consists of four 1 in. square bars whose centres are 1½ in. from the edges of the column. A load of 70,000 lb. is applied on one of the axes of symmetry and at a distance of 2 in. from the centre of the section. Calculate the maximum and minimum stresses developed in the steel and in the concrete. $m = 12$. (U. of B.)

(*Note.* Transform the whole section and solve as for a homogeneous short column.)

19. Discuss the design of short reinforced concrete compression members. (U. of B.)

20. Deduce an expression showing the relation between the bond and shearing stresses in a reinforced concrete beam.

Loads of 12 tons are applied at points 6 ft. and 12 ft. from one end of a simply supported reinforced concrete beam 18 ft. long. The tensile reinforcement consists of eight 1-in. round bars with their centre of gravity 18 in. from the outside compressive fibre of the beam. Calculate the maximum bond stress developed, if the maximum compressive stress in the concrete is 750 lb. per sq. in., the tensile stress in the steel is 18,000 lb. per sq. in., and m is 18.

How far into the supports of a cantilever beam should a 1-in. round bar be carried to develop its full working tensile strength, assuming a safe bond stress of 110 lb. per sq. in.? (U. of B.)

21. The flange of a Tee beam is 60 in. by 3½ in., and the stem is 10 in. wide. The area of the tension reinforcement is 2 sq. in., and its centre of gravity is 20 in. from the upper face of the flange. Determine the moment of resistance of the beam and the steel stress developed, if $f_c = 220$ lb. per sq. in. Take $m = 15$.

Over a length of this beam the shearing force is constant and equal to 14,000 lb. The spacing of ⅜-in. vertical U stirrups is 6 in. Assuming that the concrete carries diagonal tension represented by a shear stress of 40 lb. per sq. in., calculate the load carried by a stirrup. (U. of B.)

22. A simply supported reinforced concrete beam has a breadth of 9 in. and an effective depth of 18 in. If the limiting stresses, f_c = 750 lb. per sq. in. and f_t = 18,000 lb. per sq. in. are realized, calculate the moment of resistance and the area of the tensile steel. Assume $m = 12$.

If the moment of resistance is increased by 30 per cent, find the tensile and compressive steel areas required, assuming that the centre of the compressive steel is 2 in. below the compression skin, and that the limiting stresses are again realized. (U. of B.)

NOTE ON QUESTION 7 (*b*)—EXAMPLES XIV

Refer to Fig. 194

Moment of inertia of the concrete above the N.A. and about the N.A.

$$= \frac{bn^3}{3}$$

Moment of inertia of the steel about the N.A. on a concrete basis

$$= mA\,(d-n)^2$$

The equivalent moment of inertia of the beam on a concrete basis about the N.A.

$$I_{EC} = \frac{bn^3}{3} + mA\,(d-n)^2$$

$$\text{Now } tA = \frac{bnc}{2}$$

$$\text{and } t = \frac{mc\,(d-n)}{n} \quad . \quad . \quad . \text{ (equation 7)}$$

$$\therefore A = \frac{bn^2}{2m\,(d-n)}$$

$$\text{Then } I_{EC} = \frac{bn^3}{3} + \frac{bn^2\,(d-n)^2}{2\,(d-n)}$$

$$= \frac{bn^2}{2}\left(d-\frac{n}{3}\right)$$

$$\text{Equivalent compression modulus} = \frac{I_{EC}}{n} = \frac{bn}{2}\left(d-\frac{n}{3}\right)$$

$$\text{Moment of Resistance} = R_c$$

$$= \frac{bnc}{2}\left(d-\frac{n}{3}\right)$$

CHAPTER XV

THE SLOPE - DEFLECTION AND MOMENT - DISTRIBUTION METHODS OF THE SOLUTION OF RIGID OR CONTINUOUS FRAMED STRUCTURES, WHERE BENDING MOMENTS ARE THE DETERMINING FACTOR IN DESIGN

THE discussion of the methods will be based upon the assumption that the moment of inertia remains constant throughout the length of each member in the structures: i.e. members are prismatic members.

198. **Preliminary Discussion and Sign Convention.** A frame with rigid joints is a structure in which the member intersections are so constructed that the original angle between the members is maintained under any loading. For a steel frame joint with heavy gusset plates and ample riveting, and for welded connections, it is assumed that complete rigidity is justifiable. For monolithic reinforced concrete construction the assumption is made without question. By far the most important case of the rigid joint frame is to be found in the column and girder combination in building construction. However, there are other examples of rigid frame construction, e.g. culverts, trapezoidal frames, roofs with sloping members, the framed bridge span or open webbed girder (Vierendeel Frame). In the past, it was the custom to analyse a building frame as independent beams and columns without due regard to its essentially monolithic character. It is now recognized that such a method is inadequate, and that such a structure can be economically designed only by treating it as a multiple rigid frame. The present chapter will be devoted to two modern methods for the analysis of the rigid frame and to the study of a number of simple numerical problems. Special problems of a more complex character and of more complex structures can be found in the references given at the end of the chapter.

In a rigid frame all the members meet at joints, and because of the interaction of members and the applied loads, the joints will rotate. The amount of rotation will depend upon the restraint offered by the connecting members. Any or all of the members of the frame may be loaded transversely, and any joint may be deflected relative to its original position. Building frames, bents, culverts, lurch sideways or "sway" under the action of lateral forces or of unsymmetrically placed vertical loads. In general, when loads are applied to a rigid frame which is unsymmetrical in outline, in the cross-sectional dimensions of its members, the end or support conditions, or in the nature and position of the loads, or in any combination of these, there is a tendency for the joints to be displaced or translated, resulting in a lateral sway of the frame. (Obviously with a framed bridged span there will be a translation of the joints under the vertical applied loads.)

This effect causes joints to be translated as well as rotated under the action of the applied loads. However, in a great many cases, the relative linear displacements of the joints are negligible. Therefore a general form of equation giving the end moments in a member, must include terms stating the effect of joint restraint, the effect of applied loads, and the effect of joint movements.

Let a member AB tie into joints at A and B. Let it be subjected to transverse loads, so that there is rotation of the joints A and B. Let us consider the internal moments of resistance induced at the ends A and B of the member. If we cut the member at these ends then we must place at the cuts, (*a*) a pair of equal and opposite couples (moments of resistance), one acting on the joint and one on the end of the member, (*b*) an equal and opposite shearing force pair, and (*c*) an equal and opposite normal force pair. In general, the effects of the shearing and normal force strains on the moment distribution will be neglected. That is, the moment distribution is due to flexure only.

The end moment in a member will be designated as M with a subscript. The first letter of the subscript indicates the end of the member at which the moment exists: e.g. M_{AB} is the moment at A in member AB, and M_{BA} is the moment at B in the same member. If the ends of the members are fixed in direction (built-in or restrained against rotation), then the end moments will be designated as, e.g. M_{FAB} and M_{FBA} for the member AB. The determination of the magnitudes of these fixing moments is given in Chapter IV.

199. **Sign Convention.** Note carefully the convention for the algebraic sign of end moments, which will be used in considering statically indeterminate structures by the slope-deflection method and later by the moment-distribution method, the two

methods which are considered in this chapter. The student will find this convention different from that given in the previous work on simple, built-in, and continuous beams. While in general it is not desirable to use two such conventions, the advantages which result would seem to make the two conventions desirable. The student should thoroughly familiarize himself with these conventions and learn to recognize at a glance whether a moment is positive or negative.

The convention now given is as follows—

A moment which tends to rotate the end of a member in a 'counter-clockwise' direction is 'positive (+)' the moment which tends to rotate the end of a member in a 'clockwise' direction is 'negative (—).' (Conversely, moments acting on a joint in a 'clockwise' direction are 'positive,' and those in a 'counter-clockwise' direction are 'negative.')

The rotation of the tangent at the end of a member (or the rotation of a joint), is measured from the original direction of the axis of the member. A 'counter-clockwise' rotation of the tangent at the end of a member is 'positive' and a 'clockwise' direction is 'negative.' These rotations are denoted by the sign θ: e.g. θ_{AB}, θ_{BA}, etc., and they are in radian measure. If the ends *A* and *B* of a member *AB* are direction-fixed (or built-in), then $\theta_{AB} = \theta_{BA} = 0$.

The deflection or displacement (Δ) of one end of a member relative to the other end is measured perpendicular to the original direction of the axis of the member and it is termed 'positive' when the movement of the deflected end is 'clockwise' with respect to the other end. The opposite movement of the deflected end is 'negative.'

If l is the length of the member, then the angle ϕ (in radian measure $= \Delta/l$) through which the axis of the member rotates, is 'positive' if it is in a 'clockwise' direction, and 'negative' if it is in a 'counter-clockwise' direction.

200. Examples of the Sign Conventions are given in Figs. 199A to 205B. In these figures, *pi* stands for point of inflection, *TT* indicates tension on the top side of the beam and *TB* on the bottom side of the beam. These are given, as the bending-moment diagrams. (Figs. 199B to 205B) are drawn on the tension sides of the beams.

Fig. 199A denotes the conditions for a built-in or direction-fixed ended beam subjected to transverse loading. In Figs. 200A to 205B, the beam *AB* is not subjected to transverse loading.

In Figs. 200A and 201A the end A is caused to rotate by means of an external couple through an angle of $\pm\ \theta_{AB}$ radians, whilst the end B remains fixed in direction, i.e. $\theta_{BA} = 0$. Moments M_{AB} are induced in the member at the end A, and note that moments M_{BA} are induced in the beam at the end B: in the moment-distribution method of analysis, this moment

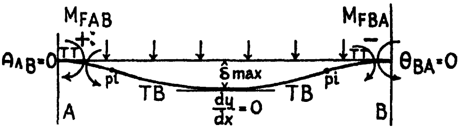


Fig. 199A

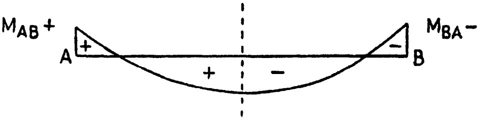


Fig. 199B

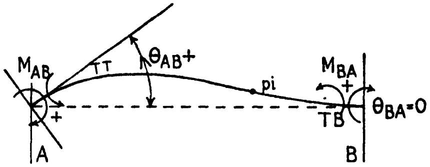


Fig. 200A

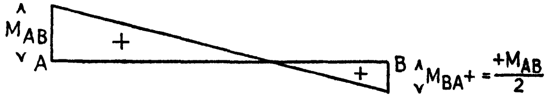


Fig. 200B

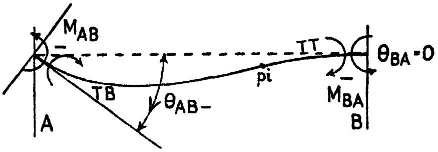
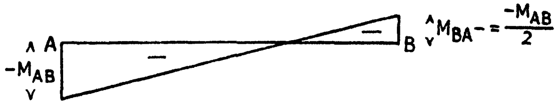


Fig. 201A



Fig, 201B

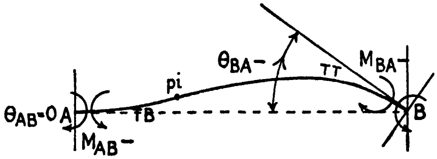


FIG. 202A

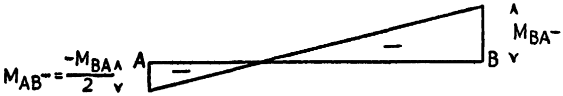


FIG. 202B

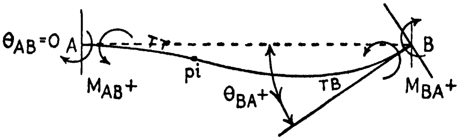


FIG. 203A

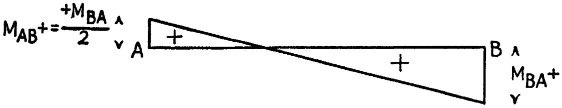


FIG. 203B

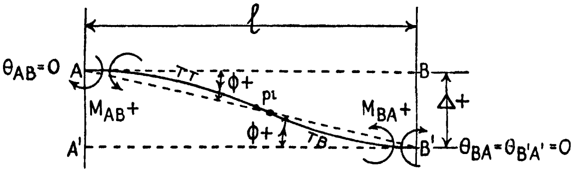


FIG. 204A

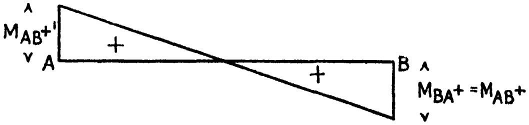


FIG. 204B

NOTE: A minus sign before a moment indicates that it is a negative moment.

M_{BA} is called a 'carry-over' moment. In Figs. 202A and 203A, the end *B* is rotated through an angle of $\pm\ \theta_{BA}$, whilst the end *A* remains direction-fixed.

In Fig. 204A the end *B* is displaced or translated by means of an external lateral force relative to *A* in a positive direction through a distance $+\ \Delta$, both the ends *A* and *B* remaining direction-fixed (i.e. $\theta_{AB} = \theta_{BA}$ remaining $= 0$). The student can deduce for himself the conditions for *AB* when *B* is displaced through a distance equal to $-\ \Delta$. In Fig. 205A the end *A* is translated relative to *B* through a distance $= -\ \Delta$, $\theta_{AB} = \theta_{BA}$ remaining $= 0$.

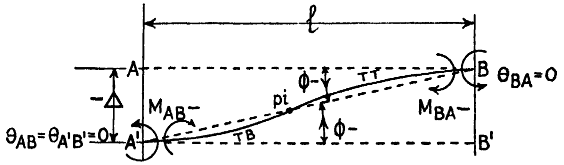


FIG. 205A

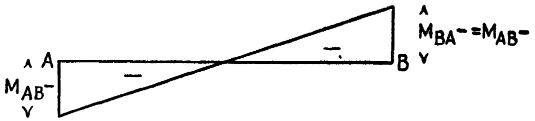


FIG. 205B

The bending moment diagrams corresponding to the loading conditions given in Figs. 199A to 205A are given in Figs. 199B to 205B. In the Figs. 200B and 201B, it is indicated that the induced moments in the beam at *B* are $M_{BA} = \pm\ M_{AB}/2$ for the member *AB* when it is of constant cross-section. When the end *B* is rotated, the moment in the beam at the end *A* is $M_{AB} = \pm\ M_{BA}/2$ for a beam of uniform cross-section. The proof of this is given in a subsequent paragraph. *For beams of non-uniform section, reference must be made to other works (references to which are given at the end of the chapter) for the value of the carry-over moments.* In the diagrams, Figs. 204B and 205B, it will be noted that the moments induced in the beam at the ends *A* and *B* due to the relative displacement of these ends are equal in magnitude; their values are determined in another paragraph.

In Fig. 206 there is indicated the signs of the induced moments in a length of a continuous beam.

Fig. 207 (*a*) shows the displacement of an unsymmetrical portal, direction-fixed at the bases *A* and *B*, due to a transverse load acting from left to right on the column *AB*: the moment

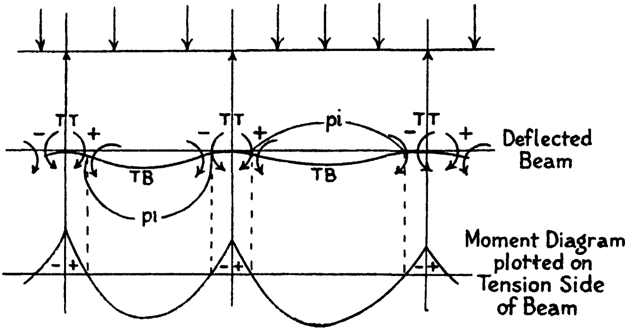


FIG. 206

diagram is given in Fig. 207 (*b*). *TL* indicates tension on the left side of a column, and *TR* tension in the right side of a column. In the diagram 207 (*a*), the joints *B* and *C* have

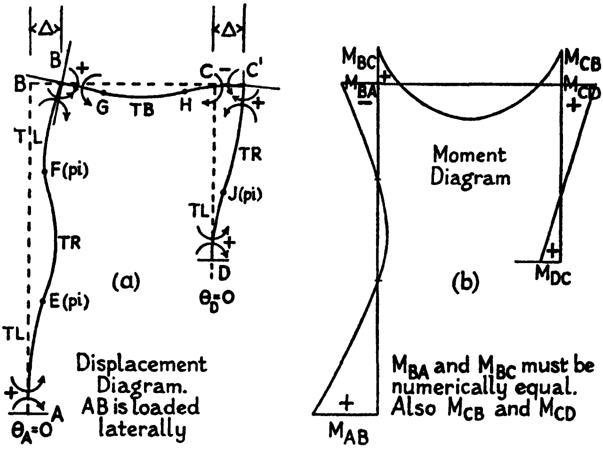

Δ
Δ
B
B'
C
C'
G
TB
H
T'L
F(pi)
TR
E(pi)
TL
TR
J(pi)
TL
D
θD=0
A
θA=0
(a)
Displacement Diagram. AB is loaded laterally
MBC
MBA
MCB
MCD
Moment Diagram
(b)
MDC
MAB
MBA and MBC must be numerically equal. Also MCB and MCD

FIG. 207

rotated as well as being translated. Figs. 208 (*a*) and (*b*) are for an unsymmetrical portal, when the ends *A* and *B* are hinged.

Fig. 209 (*a*) is of an unsymmetrical portal *ABCD* with the column bases *A* and *D* direction-fixed so that $\theta_{AB} = \theta_{DC} = 0$.

The frame has been displaced from left to right without any rotation of the joints B and C, so that $\theta_{BA} = \theta_{CD} = 0$. Let $AB = h$, and $DC = h_1$, then $\phi_{AB} = \phi_A = +\Delta/h$, and ϕ_{CD} $= \phi_C = +\Delta/h_1$. The lateral displacements of B and C are both equal to $+\Delta$, and the sway angles ϕ are both positive; also $\frac{\phi_A}{\phi_C} = \frac{h_1}{h}$. The moment diagram is given in Fig. 209 (b), and note that $M_{BA} = M_{AB}$ and $M_{CD} = M_{DC}$ and that they are all positive. These cases of sway correspond to that of the beam in Fig. 204A. If the portal sways from right to left, i.e. Δ is negative, the joints are translated from right to left and the angles ϕ_A and ϕ_C are negative. This case corresponds to that of the beam in Fig. 205A.

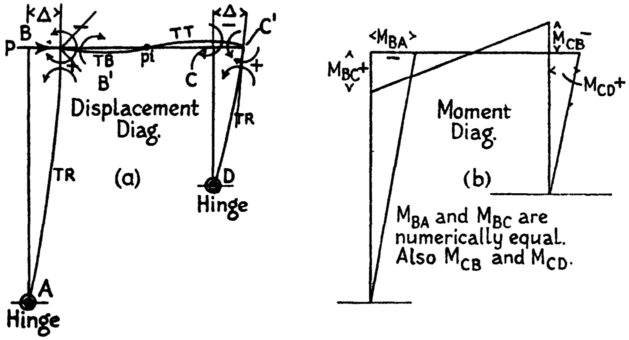


FIG. 208

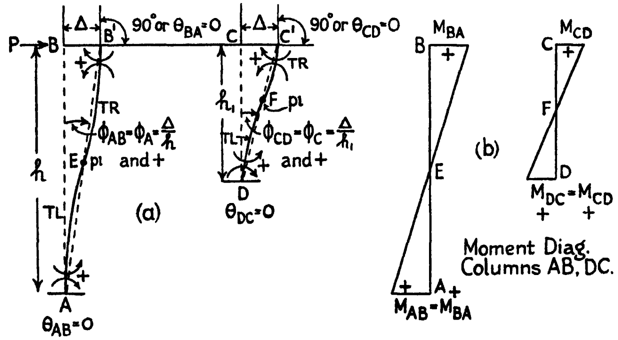


FIG. 209

The cases for the beam where one end is rotated and the other end is a hinge are given in Fig. 210. The member is, of course, free to rotate at the hinge. Fig. 211 indicates the displacement

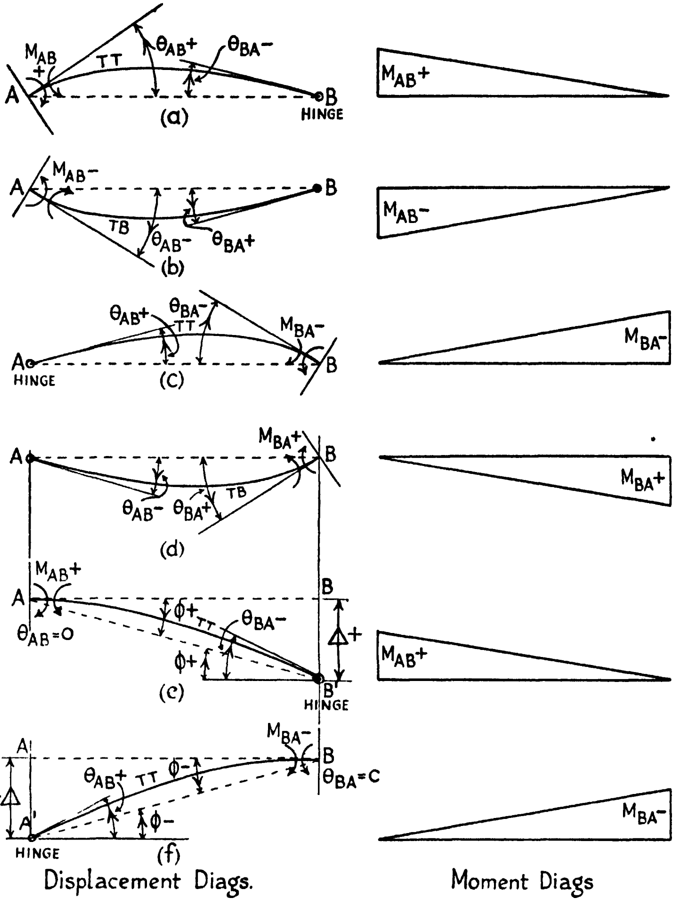


FIG. 210

of and the moment diagram for an unsymmetrical portal with the column bases hinged. The sway is from right to left.

Let the beams in the previous discussions be prismatic beams: i.e. the cross-section and therefore the cross-sectional area A and the moment of inertia I are constant. Consider the beam system given in Fig. 200A and the moment diagram for which is given in Fig. 200B to show that $M_{BA} = M_{AB}/2$.

The support B will be below the tangent at A by the amount $l\theta_{AB}$ where l is the length of the member. By the moment-area method, and taking moments of the moment diagram about the vertical through B, and letting I_{AB} be the moment of inertia,

then $$EI_{AB}l\theta_{AB} = M_{AB}\frac{l}{2}\cdot\frac{2l}{3} - M_{BA}\frac{l}{2}\cdot\frac{l}{3}$$

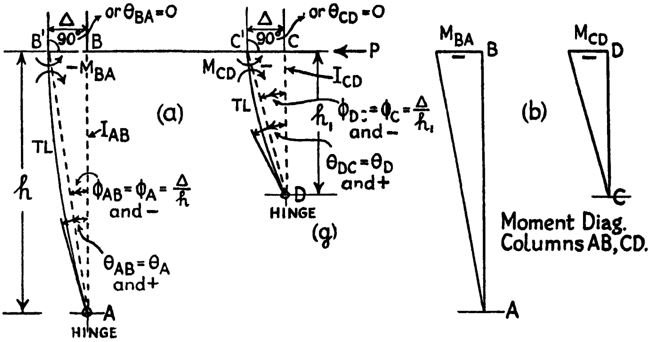


FIG. 211

The change of slope of the tangents from A to B is θ_{AB}

then $$EI_{AB}\cdot\theta_{AB} = M_{AB}\cdot\frac{l}{2} - M_{BA}\cdot\frac{l}{2}$$

(area of the moment diagram between A and B).

From these two equations, it can be shown

that $$\frac{M_{BA}}{M_{AB}} = \frac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the case considered both M_{BA} and M_{AB} are positive. Similarly, considering the beam in Fig. 202A it can be shown

that $$-M_{AB} = -M_{BA}/2 \quad . \quad . \quad . \quad (2)$$

Solving too for M_{AB} (case given in Fig. 200), it will be found

that $$M_{AB} = E\cdot\frac{4I_{AB}}{l}\cdot\theta_{AB} \quad . \quad . \quad . \quad (3)$$

In the moment-distribution method, the ratio of 1/2 given in equations (1) and (2) is known as the *carry-over* factor.

A rule is, therefore, that if a prismatic beam or member is locked at both ends, and if one end is released and rotated through an angle θ radians by a moment or couple M, a moment M is induced in the beam at the rotated end and a moment $M/2$ is induced in the member at the other end which has been kept locked in position. The sign of the two moments is the same: i.e. if the moment at the rotated end is $+M$, then the induced couple at the locked end is $+M/2$: if the moment at the rotated end is $-M$, then the induced couple at the locked end is $-M/2$.

(Note—For non-prismatic beams the carry-over factor is not 1/2 and reference must be made to other works.)

201. **Consideration of Equation** (3).

Let $$M_{AB} = E \, . \, \frac{4I_{AB}}{l} \, . \, \theta_{AB} = (M_s)_{AB}.$$

As θ_{AB} is in radians, then $E \, . \, \frac{4I_{AB}}{l}$ is the moment required to rotate the end A through unit angle, the end B remaining fixed. If A is kept fixed, and B is rotated through an angle $+\,\theta_{BA}$ radians, then $+M_{BA} = E \, . \, \frac{4I_{AB}}{l} \, . \, \theta_{BA} = (M_s)_{BA}$.

Imagine a number of prismatic members keying into a ringed joint (A) and with their other ends locked in position. Let the joint rotate (without translation) through an angle of $+\,\theta$ radians. The ends of the members at the joint will all rotate through this angle. The moments induced in the ends of the members at the joint will be

$$\begin{aligned}
+M_{AB} &= (E \, . \, 4I_{AB}/l_{AB})\theta, \\
+M_{AC} &= (E \, . \, 4I_{AC}/l_{AC})\theta, \\
+M_{AD} &= (E \, . \, 4I_{AD}/l_{AD})\theta, \text{ and so on.}
\end{aligned}$$

Assuming that E is the same for all the members, then

$$\begin{aligned}
+M_{AB} : +M_{AC} : +M_{AD} &= 4I_{AB}/l_{AB} : 4I_{AC}/l_{AC} : 4I_{AD}/l_{AD} \\
&= I_{AB}/l_{AB} : I_{AC}/l_{AC} : I_{AD}/l_{AD} \\
&= K_{AB} : K_{AC} : K_{AD}
\end{aligned}$$

One-half of the moments $+\,M_{AB}$, $+\,M_{AC}$, $+\,M_{AD}$, $+\,\ldots$ would be carried-over to their respective locked ends and they will be of the same sign as the moments at the joint ends.

i.e. $M_{BA} = +\,M_{AB}/2$; $M_{CA} = +\,M_{AC}/2$, etc.

If the joint rotation θ_{AB} had been negative, then

$$M_{AB} : M_{AC} : M_{AD} : \ldots$$
$$= -4I_{AB}/l_{AB} : -4I_{AC}/l_{AC} : -4I_{AD}/l_{AD} : \ldots$$

and $M_{BA} = M_{AB}/2$; $M_{CA} = M_{AC}/2$, etc.

i.e. $M_{BA} = -2I_{AB}/l_{AB}$; $M_{CA} = -2I_{AC}/l_{AC}$, etc.

In general $M = 4EI\theta/l$ and this product is called the "*Moment Stiffness Factor* (M_s)."

Consider the case of the prismatic beam or member AB, where A is fixed and B is hinged, Fig. 210 (*a*). Let the fixed end A be rotated (and without translation) by an external couple through an angle of $+\theta_{AB}$ radians. (There will be an induced rotation of the end B ($-\theta_{BA}$ radians). The induced moment in the beam at A is $+M_{AB}$; then considering Fig. 210 (*a*),

$$E \,.\, I_{AB} \,.\, l \,.\, \theta_{AB} = +M_{AB} \,.\, \frac{l}{2} \,.\, \frac{2l}{3},$$

$$\text{or } +M_{AB} = 3E \,.\, \frac{I_{AB}}{l} \,.\, \theta_{AB} = (M_s)_{AB} \quad . \quad (4)$$

No moment is carried over to the hinged end. In general $M = 3E \,.\, \frac{I}{l} \,.\, \theta$ and it is the "moment-stiffness factor (M_s)" for a beam fixed at one end and hinged at the other.

If A is hinged and B is rotated without translation through $+\theta_{BA}$ radians, then

$$+M_{BA} = 3E \,.\, \frac{I_{AB}}{l} \,.\, \theta_{BA} = (M_s)_{BA} \quad . \quad . \quad (5)$$

Suppose two prismatic members AB and AC key into a joint A. The end B is direction-fixed and the end C is hinged. Keeping B locked in position, and rotating A (without translation) through an angle of $+\theta$ radians, then the induced moments in the members at A will be—(both members have the same E)

$$+M_{AB} = 4E \,.\, \frac{I_{AB}}{l_{AB}} \,.\, \theta; \quad M_{AC} = 3E \,.\, \frac{I_{AC}}{l_{AC}} \,.\, \theta$$

At B the induced moment will be $+M_{BA} = \frac{2EI_{AB}}{l_{AB}} \,.\, \theta$

$$\text{Then } \frac{+M_{AB}}{+M_{AC}} = \frac{4I_{AB}/l_{AB}}{3I_{AC}/l_{AC}} = \frac{I_{AB}/l_{AB}}{0{\cdot}75I_{AC}/l_{AC}} = \frac{K_{AB}}{K_{AC}} \quad . \quad . \quad (6)$$

where K in general $= I/l$ for a member fixed at both ends and $K = 3I/4l$ for a member fixed at one end and hinged at the other.

Therefore, if one end of each of several members is rotated through an angle θ while the other end is held fixed, the required moment in each member will be proportional to a constant $K = I/l$, if the cross-section is uniform, and in any case to some constant K. If one end of each of several members is rotated through an angle θ, while the other end is fixed for part of the members and hinged for the rest, the required moment in those members which are fixed at the far end will be proportional to $K = I/l$, and for the others to $K = 3I/4l$, if the cross sections are uniform and E is the same for all the members.

Suppose, for example, that four prismatic members key into a rigid joint: two of them AB and AC are fixed and locked at B and C, and two of them AD and AE are hinged at D and E. Let joint A be translated through an angle of $+\theta$ radians (without translation) by a couple M. Then the induced moments in the members at A will be—

$$M_{AB} = M \cdot \frac{K_{AB}}{K_{AB} + K_{AC} + K_{AD} + K_{AE}} = M \cdot \frac{K_{AB}}{\Sigma K} \quad . \quad (7)$$

where $K_{AB} = I_{AB}/l_{AB}$; $K_{AC} = I_{AC}/l_{AC}$;
$K_{AD} = 3I_{AD}/4l_{AD}$; $K_{AE} = 3I_{AD}/4l_{AD}$

$$M_{AC} = M \cdot \frac{K_{AC}}{\Sigma K}; \; M_{AD} = M \cdot \frac{K_{AD}}{\Sigma K}; \; M_{AE} = M \cdot \frac{K_{AE}}{\Sigma K}$$

$M_{BA} = M_{AB}/2$ and both M_{BA} and M_{AB} are of the same sign;
$M_{CA} = M_{AC}/2$ and both M_{CA} and M_{AC} are of the same sign;
$M_{DA} = 0$; $M_{EA} = 0$.

NOTE—In the slope-deflection method it is assumed that all the members in the rigid frames are fixed at both ends. In the developed equations K will be equal to I/l for all the members, and for those ends which are hinged the moment here is equal to zero. In the moment-distribution method, for a one-hinged end member the other end being fixed, K is in general taken as it actually is, and the value of K for this member is then $3I/4l$.

202. Consider, now, the direction-fixed ended beam AB given in Fig. 204A and the bending-moment diagram in

Fig. 204B. Here the end B is displaced or translated relative to A by means of a lateral force through a distance of $+\Delta$. There is no rotation of the ends A and B (i.e. $\theta_{AB} = \theta_{BA} = 0$, after displacement).

There being no change in the slope of the tangents between A and B, then $EI_{AB} \, . \, \theta = 0$.

$$\therefore M_{AB} \, . \, \frac{l_{AB}}{2} - M_{BA} \, . \, \frac{l_{AB}}{2} = 0$$

$$\therefore M_{AB} = M_{BA}, \qquad . \quad . \quad . \quad (8)$$

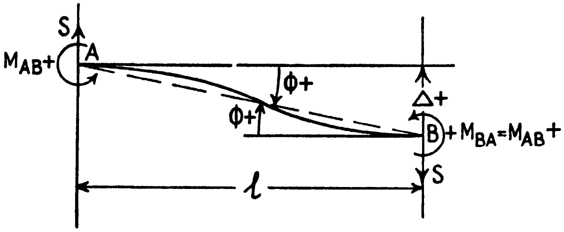


FIG. 212

and they are both of the same sign, positive in this case.

$$\text{Also,} \quad EI_{AB} \, . \, \Delta = M_{AB} \, . \, \frac{l_{AB}}{2} \, . \, \frac{2}{3} \, . \, l_{AB} - M_{BA} \, . \, \frac{l_{AB}}{2} \, . \, \frac{l_{AB}}{3}$$

$$= M_{AB} \, . \, \frac{l^2_{AB}}{6}$$

$$\therefore M_{AB} = M_{BA} = 6E \, . \, \frac{I_{AB}}{l^2_{AB}} \, . \, \Delta = (M^s_s) \qquad . \quad . \quad (9)$$

If Δ is negative, then M_{AB} and M_{BA} are both negative.

The product $6E \, . \, \frac{I}{l^2} \, . \, \Delta$ is known as the "sway-moment stiffness factor (M_s^s)."

The end shearing forces S due to the translation of B relative to A are as in the previous problem (refer to Fig. 212). Considering the equilibrium of the prismatic beam AB between the end cuts, we have $\Sigma M = 0$.

$$\therefore Sl_{AB} \, (\circlearrowright) + M_{AB} \, (\circlearrowleft) + M_{BA} \, (\circlearrowleft) = 0$$

$$\therefore S = 2\frac{M_{AB}}{l_{AB}} = 12E \, . \, \frac{I_{AB}}{l^3_{AB}} \, . \, \Delta = (S_s) \qquad . \quad . \quad (10)$$

In this case, the shearing force S at A will act vertically upwards and at B vertically downwards.

If Δ is negative, then the two equal end moments are negative, and S will act vertically downwards at A and vertically upwards at B.

The product $12E \cdot \frac{I}{l^3} \cdot \Delta$ is known as the "shear-stiffness factor (S_s)."

Consider the case of a vertical member such as the column of a portal. Referring to Fig. 213, both ends of the prismatic vertical member AB are direction-fixed, and B is translated relative to A through the distance $+\Delta$ but without any rotation of B. The shearing force S will act from left to right horizontally at the top of the member and from right to left at the bottom. The induced moments at A and B will both be equal and of the same sign, positive. If Δ is negative, i.e. B displaced relative to A from right to left, then the two equal end moments would be negative, and S would act from right to left at the top of the member and from left to right at the bottom. If h is the length of the member, then for the case given in Fig. 213,

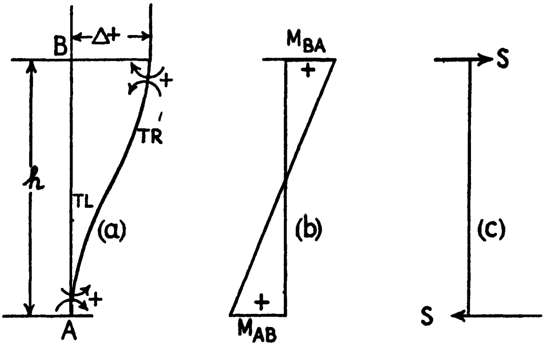


FIG. 213

$$Sh = 2M_{AB} = 2M_{BA} \quad . \quad . \quad . \quad (11)$$

Also, $$M_{AB} = S\frac{h}{2}, \text{ when } S = 12E \cdot \frac{I}{h^3}\Delta \quad . \quad . \quad (12)$$

and $$M_{AB} = M_{BA} = 6E\frac{I}{h^2} \cdot \Delta = (M_s') \text{ and is positive.} \quad (13)$$

If S is known in magnitude and direction, then the sign of the two equal end couples will depend upon the direction of S: S acting at the top of the member from left to right, the equal end couples are positive; S acting at the top of the column from right to left, the equal end couples are negative.

Consider the portal as displaced in Fig. 209, and considering the two vertical members AB and CD, whose lengths are h and h_1 respectively. Let E be the same for both members and I_{AB} and I_{CD} the respective moments of inertia. The upper ends B and C are displaced to the right by an amount $+\Delta$, but without any rotation of B and C.

Then $M_{BA} = 6E \,.\, \frac{I_{AB}}{h^2} \,.\, \Delta$, and $M_{CD} = 6E \,.\, \frac{I_{CD}}{h_1^2} \,.\, \Delta$

$$\therefore \frac{M_{BA}}{M_{CD}} = \frac{M_{AB}}{M_{DC}} = \frac{6I_{AB}/h^2}{6I_{CD}/h_1^2} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Also $S_{AB} = 12E \,.\, \frac{I_{AB}}{h^3} \,.\, \Delta$ and $S_{CD} = 12E \,.\, \frac{I_{CD}}{h_1^3} \,.\, \Delta$

$$\therefore \frac{S_{AB}}{S_{CD}} = \frac{12I_{AB}/h^3}{12I_{CD}/h_1^3} \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Referring now to Fig. 211, in which the vertical members of the portal are hinged at A and D, and the joints B and C are displaced to the left by an amount of $-\Delta$, but without any rotation of the joints. The members will rotate at the hinges. The lengths of AB and CD are respectively h and h_1, and I_{AB} and I_{CD} are the respective moments of inertia. E is the same for both members. In general, $EI\Delta = M \,.\, \frac{h}{2} \,.\, \frac{2}{3} h$ where M is the induced moment at the fixed end.

$$\therefore M_{BA} = -\,3E \,.\, \frac{I_{AB}}{h^2} \,.\, \Delta \text{ and } M_{CD} = -\,3E \,.\, \frac{I_{CD}}{h_1^2} \,.\, \Delta$$

and, in this case, they are both negative.

$$\therefore \frac{M_{BA}}{M_{CD}} = \frac{-\,3I_{AB}/h^2}{-\,3I_{CD}/h_1^3} \quad . \quad . \quad . \quad (16)$$

The end shearing forces S will, in general, be equal to $3E \,.\, \frac{I}{h^3} \,.\, \Delta$ for $M = Sh$.

$$\therefore \frac{S_{AB}}{S_{CD}} = \frac{3I_{AB}/h^3}{3I_{CD}/h_1^3} \quad . \quad . \quad . \quad (17)$$

Here, the shearing forces will act from left to right at the tops of the columns. The "Sway-Moment Stiffness Factor $(M_s^s)_1$" is here equal to $3E \cdot \frac{I}{h^2} \cdot \Delta$, and the shear stiffness factor (S_s) is equal to $3E \cdot \frac{I}{h^3} \cdot \Delta$

Thus for a beam *AB* direction-fixed at *B* and hinged at *A*,

$$M_{BA} = 3E \cdot \frac{I_{AB}}{l_{AB}^2} \cdot \Delta \quad . \quad . \quad . \quad (18)$$

$$\text{and } S_{BA} = 3E \cdot \frac{I_{AB}}{l_{AB}^3} \cdot \Delta \quad . \quad . \quad . \quad (19)$$

If *AB* and *CD* are the vertical members of a portal which is fixed at *A* and hinged at *D*, and *B* and *C* are translated but without rotation a distance of $+\Delta$, and letting $AB = h$ and $CD = h_1$.

$$\text{Then} \quad \frac{M_{AB}}{M_{CD}} = \frac{M_{BA}}{M_{CD}} = \frac{6EI_{AB}/h^2 \cdot \Delta}{3EI_{CD}/h_1^2 \cdot \Delta} = \frac{6I_{AB}/h^2}{3I_{CD}/h_1^2}$$

$$= \frac{I_{AB}/h^2}{0{\cdot}5I_{CD}/h_1^2} = \frac{K'_{AB}}{K'_{CD}} \quad . \quad . \quad . \quad (20)$$

where, in general, $K' = I/h^2$ for a member fixed at both ends and $K' = 0{\cdot}5\, I/h^2$ for a member fixed at one end and hinged at the other.

The ratio of the end shearing forces in the members is

$$\frac{S_{AB}}{S_{CD}} = \frac{12EI_{AB}/h^3 \cdot \Delta}{3EI_{CD}/h_1^3 \cdot \Delta} = \frac{I_{AB}/h^3}{0{\cdot}25I_{CD}/h_1^3} = \frac{K''_{AB}}{K''_{CD}} \quad . \quad (21)$$

where, in general, $K'' = I/h^3$ for a member fixed at both ends and $K'' = 0{\cdot}25\, I/h^3$ for a member fixed at one end and hinged at the other.

NOTE.—If the ends of several members having the same *E* are moved laterally with respect to each other through the same distance Δ, whilst the fixed ends are restrained against rotations, the induced equal end moments at the fixed ends will be proportional to a parameter $K' = I/l^2$ for members fixed at both ends and to $K' = I/2l^2$ for members fixed at one end and hinged at the other.

Again if the ends of several members having the same *E* are moved laterally with respect to each other through the same distance Δ, whilst the fixed ends are restrained against rotation, the required lateral or shearing force for each member will be proportional to a parameter $K'' = I/l^3$ for members fixed at both ends and to $K'' = I/4l^3$ for members fixed at one end and hinged at the other.

203. The student should carefully note the rotation and deflection conditions which have been discussed, and thoroughly familiarize himself with the necessary relations between the moments and forces and the rotations and deflections.

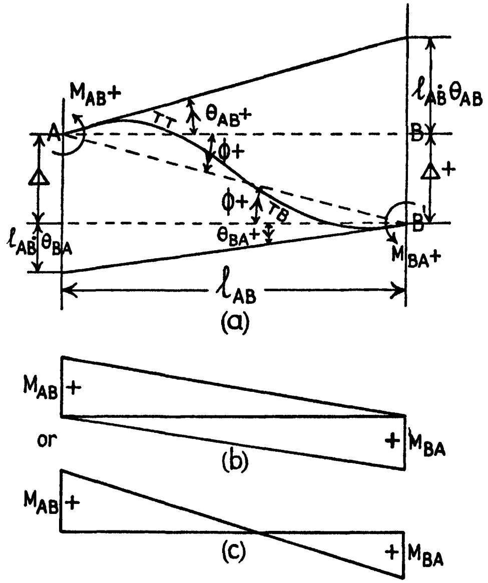
SUMMARY:

	Beam with two ends fixed	Beam with one end fixed and the other hinged
Moment-stiffness factor (M_s) . .	$4EI\theta/l$	$3EI\theta/l$
K	I/l	$3I/4l$
Carry-over factor (prismatic members) .	$1/2$	0
Sway-moment stiffness factor ($M_s{}'$) .	$6EI\Delta/l^2$	$3EI\Delta/l^2$
K'	I/l^2	$I/2l^2$
Shear stiffness factor (S_s) . . .	$12EI\Delta/l^3$	$3EI\Delta/l^3$
K''	I/l^3	$I/4l^3$

204. **The Slope-Deflection Method.** A method which can be used in the analysis of any continuous frame subjected to bending moments is commonly referred to as the Slope-Deflection method. It was developed by Otto Möhr in Germany (1892), and also by G. A. Maney, University of Minnesota (1915). It is an algebraic method, generally requiring the solution of two or more simultaneous equations.

One advantage of the method results from the selection of deflections (or translations) and rotation of joints as the redundants or unknowns, rather than unknown moments and shears, which can readily be expressed in terms of the deflections, hence the name slope-deflection.

Consider the case of prismatic members. Let such a member be AB, Fig. 214 (a), of length l_{AB}, moment of inertia I_{AB}, E = modulus of elasticity. Let $+M_{AB}$ be the couple at the end A of the member, and $+M_{BA}$ that at the end B. Let the end A rotate through a + angle θ_{AB} and the end B rotate through a + angle θ_{BA}. Let the end B be displaced relative to A by an amount $+\Delta$ so that the sway angle $\theta = \Delta/l_{AB}$ is positive. The alternative moment diagrams are given in Figs. 214 (*b*) and (*c*).



(b) and (c) Alternative Moment Diags

FIG. 214

Now we have: letting $l_{AB} = l$ and $I_{AB} = I$,

$$EI(l\theta_{AB} + \Delta) = M_{AB}\frac{l}{2}\cdot\frac{2}{3}l - M_{BA}\frac{l}{2}\cdot\frac{1}{3}l$$

and
$$EI(l\theta_{BA} + \Delta) = M_{AB}\frac{l}{2}\cdot\frac{l}{3} - M_{BA}\frac{l}{2}\cdot\frac{2}{3}l$$

$$EI\left(\theta_{AB} + \frac{\Delta}{l}\right) = M_{AB}\cdot\frac{2}{6}l - M_{BA}\frac{l}{6}$$

$$EI\left(2\theta_{BA} + 2\frac{\Delta}{l}\right) = -M_{AB}\frac{2}{6}l + M_{BA}\frac{4}{6}l$$

$$\therefore EI\left(2\theta_{BA} + \theta_{AB} + 3\frac{\Delta}{l}\right) = \frac{1}{2}M_{BA}l$$

or
$$2E\frac{I}{l}\left(2\theta_{BA} + \theta_{AB} + 3\frac{\Delta}{l}\right) = M_{BA}$$

or $$2EK(2\theta_{BA} + \theta_{AB} + 3\phi) = M_{BA} \quad . \quad . \quad (22)$$

where $$K = I/l$$

Similarly, $$2EK(2\theta_{AB} + \theta_{BA} + 3\phi) = M_{AB} \quad . \quad . \quad (23)$$

These are the fundamental slope-deflection equations.

Thus we see that M_{AB} consists of three moments—

(*a*) $4EK\theta_{AB} = (M_s)_{AB}$—the moment induced in the beam at the end A by rotating this end through an angle of θ_{AB} radians, whilst keeping the end B fixed or locked.

(*b*) $2EK\theta_{BA} = \frac{(M_s)_{BA}}{2}$—the moment induced in the beam at the end A whilst fixed or locked and the end B is rotated through an angle θ_{BA}.

(Note again, $2EK\theta_{BA}$ is one half of $4EK\theta_{BA}$, which is the moment induced in the beam at the end B when rotated through an angle θ_{BA}, the end A being fixed. *The sign of* $2EK\theta_{BA}$ *is the same as that of* $4EK\theta_{BA}$.)

(*c*) $6EK\phi = 6EK\Delta/l = (M_s{}^s)$—the moment induced at the ends of the beam when one end is displaced relative to the other through an amount Δ, whilst the ends are not allowed to rotate. The amount of sway ϕ is equal to $\frac{\Delta}{l}$. Similarly for the moment M_{BA}.

If loads are applied to the member, the total end moment will be equal to the moment for no end distortions (direction-fixed, or fixed beam moment), plus the moment caused by the end distortions: i.e. the beam is first restrained against end rotations and translation, when the end moments will be those for built-in beams, and afterwards allowed to be rotated and translated. Then the total moment at the end A of any member AB of constant cross section is,

$$M_{AB} = M_{FAB} + 2EK(2\theta_{AB} + \theta_{BA} + 3\phi) \quad . \quad (24)$$

and at the end B of the same member it is

$$M_{BA} = M_{FBA} + 2EK(\theta_{AB} + 2\theta_{BA} + 3\phi) \quad . \quad (25)$$

where the first term represents the proper fixed beam moment, and the second term the moment added or released by the joint deformations. *When using these general slope-deflection equations*, it must be kept in mind that moments acting on the ends of the beam are positive if they are anti-clockwise and negative if they are clockwise. For the usual cases M_{FAB} will be positive and M_{FBA} negative (see Fig. 215).

For a beam fixed or jointed at A and hinged at B, the moment at B will be zero. Then writing down the two slope-deflection equations—

$$M_{AB} = 2EK(2\theta_{AB} + \theta_{BA} + 3\phi) + (\pm)M_{FAB}$$
$$M_{BA} = 2EK(\theta_{AB} + 2\theta_{BA} + 3\phi) + (\pm)M_{FBA} = 0$$
$$2M_{AB} = 2EK(4\theta_{AB} + 2\theta_{BA} + 6\phi) + (\pm)2M_{FAB}$$

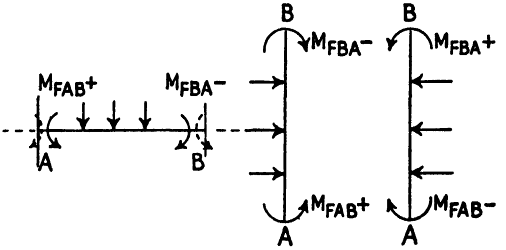


FIG. 215

Subtracting the expression for M_{BA} from that for $2M_{AB}$

$$\therefore\ 2M_{AB} = 2EK(3\theta_{AB} + 3\phi) + (\pm)2M_{FAB} - (\pm)M_{FBA}$$

$$\text{or } M_{AB} = 3EK\theta_{AB} + 3EK\phi + (\pm)M_{FAB} - (\pm)\frac{M_{FBA}}{2} \quad . \quad (26)$$

the correct signs being given to the direction fixed moments.

Assuming that the beam is not loaded, and

(*a*) that there is no sway, then

$$M_{AB} = 3EK\theta_{AB} \quad \text{—see equation (5)}$$

(*b*) that A is direction fixed and that there is sway, then,

$$M_{AB} = 3EK\phi \quad \text{—see equation (18)}$$

Consider equation (25). It will be seen that the moment at the fixed end A is made up of—

(*a*) the direction-fixed moment at A (if any),

(*b*) the moment due to the rotation of the end A through an angle θ_{AB}, the end B being locked,

(*c*) the carry-over moment from the end B, assuming that A is fixed when the end B is rotated: this carry-over moment is equal to one-half the moment at B due to rotation here for a beam of uniform section and is of the same sign as that at B, and (*d*) the moment due to the displacement of B relative to A corresponding to the sway angle $\phi = \Delta/l$, the ends A and B remaining locked.

These modifications of the direction-fixed moment due to the distortion of the member have been shown separately in the previous discussions.

The slope-deflection equations are perfectly general and apply equally to an isolated beam and to any member of a framework acting as a beam. Since they state the final value of the end moments in any given case in terms of the known fixed-beam moments and the changes in slope and the relative displacements of the points of supports, the equations are commonly known as the *slope-deflection* equations.

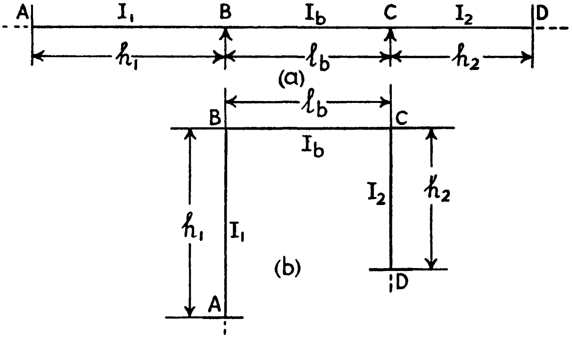


FIG. 216

By far the most important application of the slope-deflection method is in the analysis of stresses in multiple statically indeterminate structures under any given load conditions where the slopes and deflections or displacements are taken as the unknowns for which a solution is sought.

205. **Joint Equations.** Consider a continuous beam with rigid supports and a portal having two vertical members of different length and a horizontal beam, with no sway (see Fig. 216 (*a*) and (*b*). The length and moment of inertia symbols are given in the figures.

Let the ends A and D of the beam be direction-fixed and also the column bases A and D of the portal, direction-fixed.

Let the members of the frame be loaded in any manner and let there be no displacement of the supports at B and C; and let the rotation of the beam at B and C be θ_B and θ_C respectively.

θ_{AB} and θ_{DC} will be equal to zero; also let $\theta_{BA} = \theta_{BC} = \theta_B$, and they will be of the same sign.

Similarly, $\theta_{CB} = \theta_{CD} = \theta_C$.

Let the members of the portal be so loaded that there is only rotation of the joints *B* and *C* (i.e. there is no translation of the joints or sway of the members *AB* and *CD*). Also there will be no rotation of the axis of *BC* (neglecting the effect of axial strains in the column).

When the bars are sufficiently rigid so that the compressive axial forces are small in comparison with Euler's critical load, the influence of axial forces on bending can be neglected. The corresponding small shortening of the bars can be neglected and it can be assumed as a first approximation that the rigid joints only rotate by a certain angle due to bending of the bars. Any corrections necessary due to axial strains are usually small and can be disregarded in most practical calculations.

$\theta_{AB} = \theta_{DC} = 0$; $\theta_{BA} = \theta_{BC} = \theta_B$ and of the same sign; $\theta_{CB} = \theta_{CD} = \theta_C$.

Now write down the slope-deflection equations for all the members giving the correct sign to the direction-fixing couples.

Remember that $\theta_A = \theta_D = 0$.

Let $K_1 = I_1/h_1$; $K_b = I_b/l_b$; $K_c = I_c/l_c$.

$$M_{AB} = 2EK_1(\theta_B) + (\pm M_{FAB})$$
$$M_{BA} = 2EK_1(2\theta_B) + (\pm M_{FBA})$$
$$M_{BC} = 2EK_b(2\theta_B + \theta_C) + (\pm M_{FBC})$$
$$M_{CB} = 2EK_b(\theta_B + 2\theta_C) + (\pm M_{FCB})$$
$$M_{CD} = 2EK_2(2\theta_C) + (\pm M_{FCD})$$
$$M_{DC} = 2EK_2(\theta_C) + (\pm M_{FCD})$$

Now the end couples are a function of two unknowns θ_B and θ_C and of the known direction-fixing couples. We desire, therefore, only two simultaneous equations in θ_B and θ_C in order to find these values from which to calculate the six unknown *M*'s. These can be obtained by considering the equilibrium of the joints after rotation. These joints will be in equilibrium.

There are also shearing forces transmitted to the joints. The forces will be in equilibrium with the axial forces of the members and will not enter into the equations for calculating the angles θ. The moments acting on the joints are evidently equal and opposite to the corresponding end moments of the bent members of the structure.

The sum of the moments acting on the joints is zero. Thus we can write down for

joint B— $M_{BA} + M_{BC} = 0$, and for
joint C— $M_{CB} + M_{CD} = 0$

These equations are known as the *joint equations.*

Similarly, if there are a number of members meeting in a rigid joint, then $\Sigma M_n = 0$, where n stands for near moment, i.e. moment in a member at the joint end.

If the joint equations for B and C above are solved for θ_B and θ_C, then all the end-fixing couples are known. In the examples given we have only two unknowns, and therefore only two equations are required. Similarly if there were p joints, then only p equations would be required for solution.

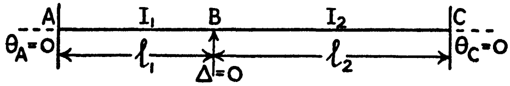


FIG. 217

In the particular example given, solving by the Theorem of Three Moments, there would be 4 unknowns—M_A; M_B; M_C; M_D—to be found and this would require 4 simultaneous equations for solution.

If, however, the ends A and D were hinged, then by the Theorem of Three Moments, there are only two unknowns M_B and M_C, and by the slope-deflection method four unknowns, θ_A, θ_B, θ_C and θ_D. However, we can, by the equations developed previously, eliminate θ_A and θ_D and solve for θ_B and θ_C from the two joint equations for joints B and C.

206. **Example.** Consider the continuous beam, direction-fixed at the ends, given in Fig. 217. The support at B is non-elastic or rigid and therefore no Δ occurs at B.

$\theta_A = 0$; $\Delta = 0$; $\theta_C = 0$; $K_1 = I_1/l_1$; $K_2 = I_2/l_2$.

AB, BC are loaded in any manner; then generally,

$$M_{BA} = 2E_1K_1(2\theta_B) + M_{FBA}$$
$$M_{BC} = 2E_2K_2(2\theta_B) + M_{FBC},$$

the correct signs being given to M_{FBA} and M_{FBC}.

Joint equation for B, $\Sigma M = 0$.

$$\therefore\ 2E_1K_1(2\theta_B) + 2E_2K_2(2\theta_B) + M_{FBA} + M_{FBC} = 0$$

$$\therefore\ \theta_B = -\frac{M_{FBA} + M_{FBC}}{4E_1K_1 + 4E_2K_2}$$

$$= -\frac{X}{4E_1K_1 + 4E_2K_2}$$

$$\therefore\ M_{BA} = -\frac{4E_1K_1}{4E_1K_1 + 4E_2K_2} \cdot X + M_{FBA}$$

Divide top and bottom of first factor by $4E_1K_1$,

$$M_{BA} = -\frac{X}{1 + \dfrac{E_2K_2}{E_1K_1}} + M_{FBA} = -\frac{X}{1 + n \,.\, m} + M_{FBA}$$

where $n = \dfrac{E_2}{E_1}$ and $m = \dfrac{K_2}{K_1}$.

Similarly for M_{BC}.

$\therefore$ In solving this type of problem, we may deal with relative values of E and K: i.e. the ratios n and m will not alter, so that M_{BA} will always have the same value. If the actual values are not used, but any relative values of E and K, then θ_B obtained will not be the true θ_B but a relative θ_B, which, when substituted in the M equation, will give the correct value of the M. In other words, θ is a function of the K's and E's used.

Similarly, if there are a large number of K's and E's, then in this type of problem stated we can deal with the relative values of E and K.

(See further notes in the "Mechanical Solution of Statically Indeterminate Structures.")

207. In other types of problems, such as those of finding moments due to temperature changes, and those of finding moments due to a known displacement of a support or supports in continuous beams, and known displacements (linear and angular) of the members of a portal, the actual values for E and I, and therefore K, must be inserted in the slope-deflection equations. These are cases of non-elastic deformation.

Let a portal consist of two vertical steel columns AB and

DC of lengths 100 in. and 150 in. respectively, and let their *I*'s be 200 in.-units and 250 in.-units respectively. Let the steel beam *BC* have a length of 120 in. and an *I* of 180 in.-units.

	Column *AB*	Beam *BC*	Column *CD*
Length (in.) . .	100	120	150
I (in.^{4}) . . .	200	180	250
$K = I/l$ . .	2	1·5	5/3
Relative K . .	1	0·75	0·83
Relative E . .	1	1	1

Therefore, we could use in the slope-deflection equations for this portal either,

	Column *AB*	Beam *BC*	Column *CD*
EK . . .	2·0	1·5	1·66
or EK . . .	1·0	0·75	0·83

or other corresponding relative quantities.

If, say, the beam of the portal was displaced left to right by a known amount $\Delta = 0{\cdot}10$ in., then $\phi_{AB} = 0{\cdot}10/100$, and $\phi_{CD} = 0{\cdot}10/150$. Then, as these are correctly known, we should have to use for *EK* the actual values for the members.

Let $E = 30 \times 10$- lb./in.^{2}

	Column *AB*	Beam *BC*	Column *CD*
EK (lb.-in. units) .	$30 \times 10^6 \times 2$ $= 60 \times 10^6$	$30 \times 10^6 \times 1{\cdot}5$ $= 45 \times 10^6$	$30 \times 10^6 \times 1{\cdot}66$ $= 50 \times 10^6$

208. **The Previous Problem when Sway is Considered.** (See Figs. 218 (*a*) and (*b*). With regard to the continuous beam, let the support *B* sink to *B'* below *A* by an amount of Δ; let the support *C* fall to *C'* by the same amount Δ so that *BC* remains horizontal. Let the support *D* fall to *D'*, 2Δ below *A*, or Δ below *C'*, so that $\phi_{CD} = \phi_C = \Delta/h_2$ and is positive.

Now $$\phi_A = \frac{\Delta}{h_1} \text{ and } \phi_C = \frac{\Delta}{h_2}$$

Therefore $$\phi_A h_1 = \phi_C h_2 \text{ or } \phi_C = \phi_A h_1/h_2$$

With regard to the portal, let the joint B, as well as rotating, be translated to B', where $BB' = \Delta$, and let the joint C, as well as rotating, be translated to C', where, if axial strain is neglected, $BB' = CC' = \Delta$.

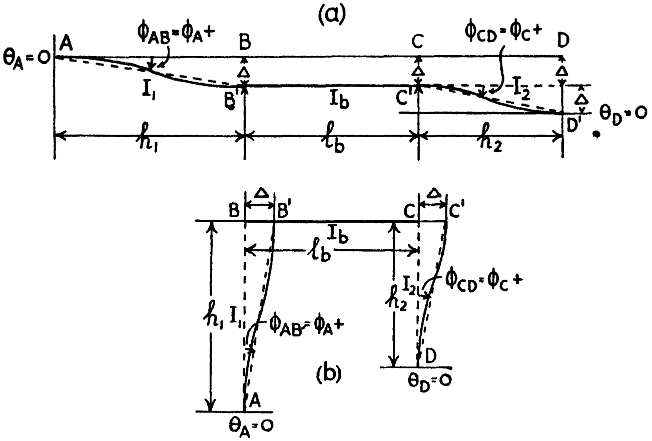


FIG. 218

$$\phi_{AB} = \frac{\Delta}{h_1} = \phi_A \text{ and } \phi_{CD} = \frac{\Delta}{h_2} = \phi_C$$

and both, for the case given, are positive (see Fig. 218 for the symbols).

Let the rotating at B be designated θ_B and at C, θ_C.

The slope-deflection equations are—

$$M_{AB} = 2EK_1(\theta_B + 3\phi_A) + (\pm M_{FAB})$$
$$M_{BA} = 2EK_1(2\theta_B + 3\phi_A) + (\pm M_{FBA})$$
$$M_{BC} = 2EK_b(2\theta_B + \theta_C) + (\pm M_{FBC})$$
$$M_{CB} = 2EK_b(\theta_B + 2\theta_C) + (\pm M_{FCB})$$
$$M_{CD} = 2EK_2(2\theta_C + 3\phi_C) + (\pm M_{FCD})$$
$$M_{DC} = 2EK_2(\theta_C + 3\phi_C) + (\pm M_{FDC})$$

In the above equations we have 4 unknown deformations, θ_B, θ_C, ϕ_A and ϕ_C, but ϕ_C is known in terms of ϕ_A, so that we have to find θ_B, θ_C and ϕ_A. We have only two joint equations, one for joint *B* and one for joint *C*. We therefore require a third equation so that we can solve for the three unknowns θ_B, θ_C and ϕ_A.

We can therefore consider the equilibrium of the columns of the portal, or the 2 end spans of the continuous beam. As a result we develop an equation known as a "bent" equation if it is in terms of the end moments of the columns, or a "shear" equation if we consider the shearing forces at the tops of the columns, or at the ends of the beam spans. With reference to Fig. 218 (*a*) and (*b*), and Figs. 219 (*a*) and (*b*).

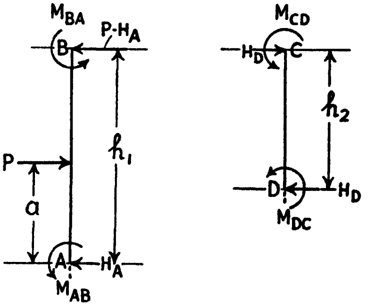


FIG. 219

Let the column *AB* of the portal be acted upon by a horizontal force *P* acting left to right and at a distance from *A* equal to *a*. There are no horizontal forces acting on *CD*. At the column bases there will be a horizontal force H_A acting at *A* in the right-to-left direction and at *D*, a horizontal force H_D also acting from right to left, such that $H_A + H_D = P$.

Let the moment couples at *A* and *B* be M_{AB} and M_{BA} in the portal, both positive.

Let the moment couples at *C* and *D* be M_{CD} and M_{DC} in the portal, both positive.

Both columns after distortion are in equilibrium. Then for column *AB* summing moments about top of column, i.e. about $B = 0$,

$$M_{AB} + \underbrace{(-H_A h_1)}_{\text{clockwise}} + M_{BA} + \underbrace{P(h_1 - a)}_{\text{anti-clockwise}} = 0 \quad . \quad (27)$$

$\therefore$ *P* acting from left to right will be taken as acting in the positive direction.

Considering column *DC*, and taking moments about *C*,

$$M_{DC} + \underbrace{(-H_D h_2)}_{\text{clockwise}} + M_{CD} = 0 \quad . \quad . \quad (28)$$

Eqn. (27) $\times h_2 = (M_{AB} + M_{BA})h_2 - H_A h_1 h_2 + Ph_1 h_2 - Pah_2$

$$= 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Eqn. (28) $\times h_1 = (M_{DC} + M_{CD})h_1 - H_D h_1 h_2$

$$= 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

adding (29) and (30)

$$(M_{AB} + M_{BA})h_2 + (M_{DC} + M_{CD})h_1 - h_1 h_2(H_A + H_D) + Ph_1 h_2 - Pah_2 = 0$$

But $$H_A + H_D = P$$

$$\therefore (M_{AB} + M_{BA})h_2 + (M_{DC} + N_{CD})h_1 = Pah_2$$

and, dividing by h_2,

$$(M_{AB} + M_{BA}) + \frac{h_1}{h_2}(M_{DC} + M_{CD}) = Pa \quad . \quad (31)$$

This is the "bent" equation for the given portal. Pa is called an *overturning moment.* Pa is positive on the right-hand side of the equation for P acting in the left-to-right direction. Pa is negative on the right-hand side of the equation if P acts in the right-to-left direction.

Dividing (31) by h_1,

$$\frac{(M_{AB} + M_{BA})}{h_1} + \frac{(M_{DC} + M_{CD})}{h_2} = P \cdot \frac{a}{h_1} \quad . \quad (32)$$

This is the "shear" equation for the given portal. $P\frac{a}{h_1}$ is of the same sign as P.

Now $\frac{M_{AB} + M_{BA}}{h_1} = \text{say } H_{AB} =$ the equal and opposite shearing forces acting at the ends A and B of the column AB,

and $\frac{M_{DC} + M_{CD}}{h_2} = \text{say } H_{CD} =$ the equal and opposite shearing forces acting at the ends C and D of the column CD.

If the fixing couples are positive, then H_{AB} acts in the direction left-to-right at B (top) and from right-to-left at A (bottom) of the column AB; and vice-versa if M_{AB} and M_{BA} are negative. (Remember that A and B are the cut ends of the column.)

An example of moment and shear signs, and the bent and shear equations are given in Fig. 220.

In problems where the members concerned are of the same length it will generally be more convenient to use the bent equation, but, where they are of different lengths, the shear equation.

Again note: that if the shearing forces represented by H act in the direction left-to-right at the top cut end of the column or member, then the corresponding end moments acting in the member (e.g. M_{AB} or M_{BA}) are positive. If H acts from right-to-left at the top cut end of the column or member, then the corresponding end moments are negative.

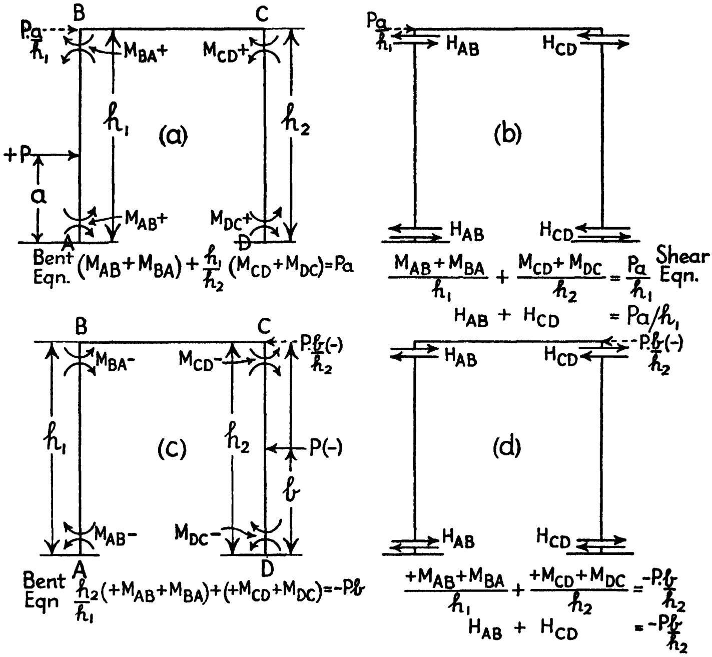

B
C
Pa/h₁
M_BA+
M_CD+
h₁
(a)
h₂
+P
a
M_AB+
M_DC+
A
D
Bent Eqn. (M_AB+M_BA) + h₁/h₂ (M_CD+M_DC) = Pa
Pa/h₁
H_AB
H_CD
(b)
H_AB
H_CD
(M_AB+M_BA)/h₁ + (M_CD+M_DC)/h₂ = Pa/h₁ Shear Eqn.
H_AB + H_CD = Pa/h₁
B
C
-Pb/h₂(-)
M_BA-
M_CD-
h₁
(c)
h₂
P(-)
b
M_AB-
M_DC-
A
D
Bent Eqn h₂/h₁(+M_AB+M_BA)+(+M_CD+M_DC)=-Pb
H_AB
H_CD
-Pb/h₂(-)
(d)
H_AB
H_CD
(+M_AB+M_BA)/h₁ + (+M_CD+M_DC)/h₂ = -Pb/h₂
H_AB + H_CD = -Pb/h₂

FIG. 220

Referring to Fig. 221, a portal consists of a number of bays, and the columns are all of the same length.

The bent equation is

$$\Sigma \text{ column end moments} = \Sigma Pa$$

On the right-hand side of this equation P is positive, acting from left to right, and negative acting from right to left: and

Pa is positive acting in a clockwise direction and negative when acting in a counter-clockwise direction.

If a portal consists of a number of bays and the columns are of different lengths,

then
$$\sum \frac{\text{column end moments}}{\text{column lengths}} = \sum \frac{Pa}{h}$$
$$= \sum \frac{(M_{AB} + M_{BA})}{l_{AB}} = \sum P \cdot \frac{a}{h} \quad . \quad (33)$$

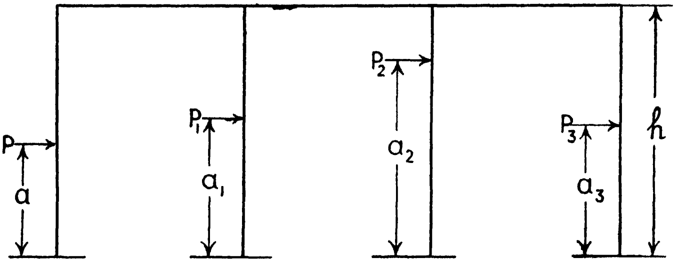


FIG. 221

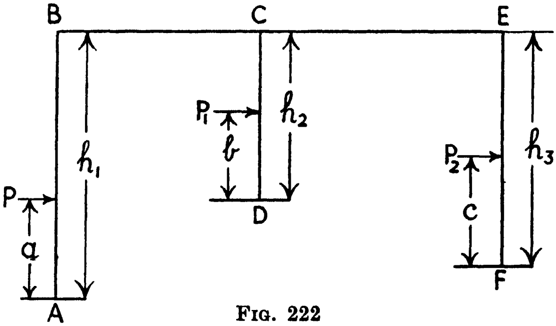


FIG. 222

P is positive in this expression on right-hand side of the equation when acting left to right, and negative when acting right to left. This is the shear equation. Thus for a portal of the type given in Fig. 222,

$$\frac{M_A+M_B}{h_1} + \frac{M_D+M_C}{h_2} + \frac{M_F+M_E}{h_3}$$
$$= P\frac{a}{h_1} + P_1\frac{b}{h_2} + P_2\frac{c}{h_3} \quad . \quad . \quad . \quad (34)$$

Where M_A, M_B, M_C, M_D, M_E and M_F are the end-column moments at *A*, *B*, *C*, *D*, *E*, and *F* respectively.

In the right-hand side of the equation (34)

P and Pa/h are positive acting left to right ⟶

P and Pa/h are negative acting right to left ⟵

The bent equation is

$$(M_A + M_B) + \frac{h_1}{h_2}(M_D + M_C) + \frac{h_1}{h_3}(M_F + M_E)$$

$$= Pa + P_1 b \,.\, \frac{h_1}{h_2} + P_2 c\frac{h_1}{h_3}$$

or $(M_A + M_B)\frac{h_2}{h_1} + (M_D + M_C) + (M_F + M_E)\frac{h_2}{h_3}$

$$= Pa\frac{h_2}{h_1} + P_1 b + P_2 c\frac{h_2}{h_3} \quad . \quad . \quad . \quad (35)$$

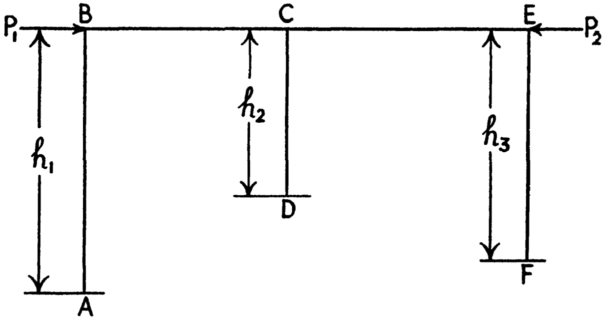


FIG. 223

Similarly for the portal loaded as in Fig. 223, shear equation—

$$\frac{M_A + M_B}{h_1} + \frac{M_D + M_C}{h_2} + \frac{M_F + M_E}{h_3} = P_1 + (-P_2)$$

$$= P_1 - P_2 \quad . \quad . \quad . \quad . \quad (36)$$

and bent equation—

$$M_A + M_B + \frac{h_1}{h_2}(M_D + M_C) + \frac{h_1}{h_3}(M_F + M_E)$$

$$= h_1(P_1 - P_2) = P_1 h_1 - P_2 h_1 = h_1(P_1 - P_2)$$

or $(M_A + M_B)\frac{h_3}{h_1} + (M_D + M_C)\frac{h_3}{h_2} + (M_F + M_E$

$$= h_3(P_1 - P_2) \quad . \quad . \quad . \quad . \quad (37)$$

NOTE—The same equations hold good even if there are hinges at the bases of the columns, there being, of course, no moments at the hinges.

Worked examples introducing all these equations and their method of use are given at the end of this chapter.

209. Interpretation of Equation (33), page 404.

$$\Sigma \text{ end couples/length} = \sum \frac{Pa}{h}$$

The right-hand side of the equation is equal in magnitude to the shearing forces at the top cut of the members, treating the members as simply supported, but its sign is opposite to that of the simple beam shears.

Therefore, the sum of the top shearing forces due to the end couples are equal to the sum of the simply-supported beam shears in magnitude but of the opposite sense of action: i.e. they act in the same direction as the sum or resultant of the external transverse horizontal forces: i.e.

Σ end couples/length $= - \Sigma$ simply-supported beam shearing forces at the top of the members $= + \sum \frac{Ph}{a}$.

If a vertical member of length h is loaded with a uniformly distributed load w per unit length over its whole length, then $+ \sum \frac{Ph}{a}$ would be replaced by $+ \frac{wh}{2}$ if the load acted from left to right and equal to $- \frac{wh}{2}$ if the load acted from right to left.

(See the illustrated problems, pages 473–6, for the case of a uniformly varying load on a loaded member.)

Again if $a = h$, $b = h$, etc., then Σ end couples/length $= \Sigma P$, i.e. the sum of the top shearing forces due to the end couples in the case of vertical members is equal to the sum of the transverse forces acting in the frame at the joints into which the members tie and they are of the same sense of action. If the frame is a multi-stories frame, then ΣP includes all transverse loads acting at and above the joints into which the members tie.

If a vertical member AB is transversely loaded by a load P at a distance a from the bottom (A) of the member and where a is $< h$, and if there are other loads P_1, P_2, etc., acting at the top joints and at other points above the joint at the top of the member, then the top shearing force at B due to the end couples in the member must be that due to $\frac{Pa}{h}$ together with P_1, P_2, etc. Or, for a number of vertical members,

$$\Sigma \text{ end couples/length} = \frac{Pa}{h} + P_1 + P_2, \text{ etc.}$$

(see equations (38) and (39), pages 408, 409).
For the horizontal members which sway in a horizontal framed girder see illustrative problem.

NOTE—The total shear at the ends of a member is finally equal to the sum of the simply-supported beam shear having its correct sense of action plus that due to the end couples.

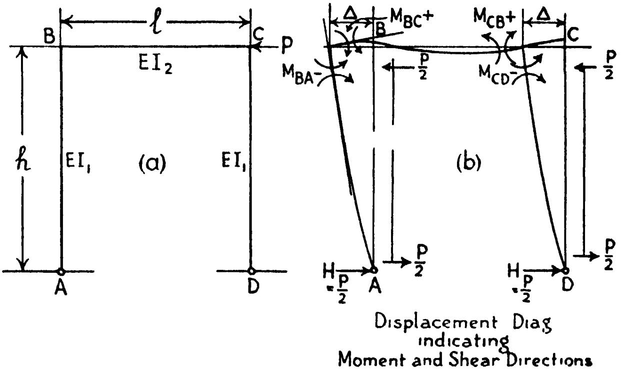


FIG. 224

210. **The Analysis of the Frame given in Fig. 224 by the Slope-Deflection Method.** The frame will sway from right to left so that $-\phi_{AB} = -\phi_{DC} = -\phi$. Also, $M_{BA} = M_{CD}$; $M_{AB} = M_{DC} = 0$ for AB and CD are similar members and they will be both displaced similarly.

Let $\theta_{BA} = \theta_{BC} = \theta_B$, and $\theta_{CB} = \theta_{CD} = \theta_C$; also, $\theta_B = \theta_C$

$$\text{Now} \quad M_{BA} = E\frac{I_1}{h}(3\theta_B - 3\phi) \text{ and } M_{BC} = 2E\frac{I_2}{l}(2\theta_B + \theta_C)$$

$$= EK_1(3\theta_B - 3\phi) \qquad = 2EK_2(3\theta_B)$$

Joint equation— $M_{BA} + M_{BC} = 0$

$$\therefore 3E\theta_B(K_1 + 2K_2) - 3EK_1\phi = 0$$

Bent equation— $2M_{BA} = -Ph$

$$\therefore 6E\theta_B \,.\, K_1 - 6EK_1\phi = -Ph$$

or
$$6E\theta_B K_1 - 6E\theta_B(K_1 + 2K_2) = -Ph$$

or
$$\theta_B = +\frac{Ph}{12EK_2}$$

$$\therefore M_{BC} = 2EK_2 \cdot \frac{Ph}{4EK_2} = \frac{Ph}{2}$$

$$M_{BA} = -\frac{Ph}{2}$$

$$Hh - P\frac{h}{2} = 0 \qquad \therefore H = \frac{P}{2}$$

and will act in the left-to-right direction at the column bases; or shearing force at the top of the columns $= P/2$ and acts in the direction right to left.

211. Similarly for a Multi-storeyed, Multi-bayed Frame.*

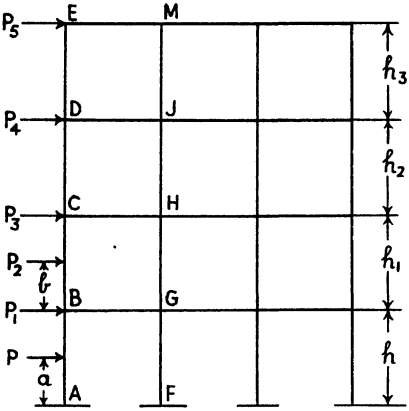


Fig. 225

Shear equations—Total shearing forces in the top of the columns in the ground floor

$$= \sum \frac{M_{AB} + M_{BA}}{h} = +P\frac{a}{h} + \Sigma \text{ horizontal forces above the ground floor}$$

$$= +P\frac{a}{h} + P_1 + P_2 + P_3 + P_4 + P_5 \quad . \quad (38)$$

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* See also, *Analysis of Rigid Frames* by Amerikian (distributed by Supt. of Documents, Washington, D.C., 1942), in which are given the solutions for a large number of single and multiple storey frames of both rectangular and trapezoidal form. The formulae are derived by the use of the slope-deflection equations.

For the 1st floor,

$$\sum \frac{M_{\mathrm{BC}} + M_{\mathrm{CB}}}{h_1} = + P_2\frac{b}{h} + \Sigma \text{ horizontal forces above this floor}$$

$$= + P_2\frac{b}{h} + P_3 + P_4 + P_5 \quad . \quad . \quad (39)$$

Due notice must be paid to the sign of the forces. The bent equation can be easily built up. (See the worked examples.)

If there are no horizontal forces acting in the frames, then the overturning moments are zero and obviously the terms Pa/h, etc., and the right-hand sides of the bent and shear equations are equal to zero.

212. The method of slope-deflection is one of the most general and most powerful of the special methods developed for the analysis of continuous framed structures. It is most advantageous when applied to structures where there are fewer joint rotations and movements than there are redundant forces and moments.

213. Consider a prismatic beam AB of length l with its axis horizontal and direction-fixed at the ends. Let it be loaded transversely so that the end fixing moments are $+ M_{\mathrm{FAB}}$ and $- M_{\mathrm{FBA}}$, when the ends are restrained against rotation and translation.

(*a*) Let the axis of the beam rotate through an angle $+ \phi$.

Keep $\theta_{\mathrm{AB}} = \theta_{\mathrm{BA}} = 0$

Then M_{FAB} will be modified by a moment $= + 6E\frac{I}{l}\phi_{\mathrm{AB}}$ $= M_s^s$, and M_{FBA} will be modified by a moment $= + 6E\frac{I}{l}\phi_{\mathrm{AB}}$

Then $M_{\mathrm{AB}} = M_{\mathrm{FAB}} + 6EI\phi_{\mathrm{AB}}/l$,
and $M_{\mathrm{BA}} = -M_{\mathrm{FBA}} + 6EI\phi_{\mathrm{AB}}/l$.

(*b*) Let the end A now be rotated through angle $+ \theta_{\mathrm{AB}}$ by a moment $= + E \,.\, 4I\theta_{\mathrm{AB}}/l = (M_s)_{\mathrm{AB}}$, θ_{BA} remaining $=$ zero.

Then a moment $= +$ E . $2I\theta_{\mathrm{AB}}/l$ will be carried over to the end B. θ_{AB} is with respect to the original position of AB.

Then the end moments at A and B will now be further modified by the above amounts.

Thus $M_{AB} = + M_{FAB} + 6EI\phi_{AB}/l + 4EI\theta_{AB}/l$,

and $M_{BA} = - M_{FBA} + 6EI\phi_{AB}/l + 2EI\theta_{AB}/l$.

(*c*) Now let the end B be rotated through an angle $+ \theta_{BA}$ (with respect to the original horizontal position of AB), by a moment $+ 4EI\theta_{BA}/l = (M_s)_{BA}$, θ_{AB} remaining at its previous value: i.e. A is locked against rotation. Then a moment of $+ 2EI\theta_{BA}/l$ will be carried over to the end A.

Then M_{AB} now becomes

$= + M_{FAB} + 6EI\phi_{AB}/l + 4EI\theta_{AB}/l + 2EI\theta_{BA}/l$

and M_{BA}

$= - M_{FBA} + 6EI\phi_{AB}/l + 2EI\theta_{AB}/l + 4EI\theta_{BA}/l$.

(*d*) These are the general slope-deflection equations. They can be modified correctly when the sense of the distortions is known (negative or positive).

NOTE. *Fixed-end Moments* can be calculated by the method given in Chapter IV. Values for different types of loading are tabulated in a number of references, e.g. Shepley, in *Continuous-Beam Structures*,* who gives tables of classified types of loading, bending-moment diagrams, and tables of fixed-end moments.

214. **The Moment-Distribution Method or the Calculation of End Moments by Successive Approximations.** We have seen that the deflection of axes and rotation of joints govern the distribution of moments, thrusts and shears in rigid frames. It is important always to visualize deflections and rotations in relation to the corresponding moments and shears.

A distinctive American method of analysis of continuous frames, the method of "Distributing Fixed End Moments," originated by Professor Hardy Cross,† will be used. This final form of the method of successive approximations was obtained in the paper by Professor Hardy Cross, published in Trans. A.C.E., Vol. 96, 1932.

This is also a most general and most powerful method for the analysis of rigid frames. It is advantageous for the same type of structures as those for which the slope-deflection is most useful. As the number of unknown deflections and rotations increases, requiring more simultaneous equations for a solution by the slope-deflection method, the relative advantage of the moment distribution method will become greater. The method may be modified in many ways and offers a number of short cuts in specific problems. Once the general principles of the method are understood, the student will have acquired a point of view which should enable him to arrive at a reasonable solution of any statically indeterminate problem involving a continuous type of structure, where the moments are the controlling factor in the design, without the necessity of any mathematical work except the simplest arithmetic. It is recommended, however, that certain work, especially when sway is concerned, is carried out in an orderly fashion on sheets which are kept numbered.

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* *Continuous-Beam Structures*, Shepley. (Concrete Publications, Ltd.)

† Method of Successive Approximations developed by O. Mohr, 1906. Extension to frames due to K. R. Cališev, 1923. Its final form given by Professor Hardy Cross, 1932.

In this chapter, the discussion will be concerned only with rectangular rigid frames whose members are prismatic (i.e. of the same cross-section throughout). The student should refer to other works for problems in connection with trapezoidal frames and frames whose members are of variable cross-section.

DEFINITIONS:

Fixed beam moments (F.B.M.) are the end moments due to loads on any member, which occur when the ends of the member are fully restrained against rotation and translation.

Moment-stiffness factor (M_s) is defined as the moment required to rotate one end of the member through an angle θ radians, when the other end of the member either is fully restrained against rotation, or has no restraint against rotation (e.g. a hinged end).

For members of constant cross-section,

$$M_s = 4EI\theta/l \text{ and is proportional to } K = I/l$$

if the far end of the member is fixed, whilst $M_s = 3EI\theta/l$ and is proportional to $K = 3I/4l$ if the far end of the member is hinged, assuming all members in a frame or structure have the same E—otherwise M_s will be proportional to $K = EI/l$ or $K = 3EI/4l$. (See equation (7), page 386.)

The *sway-moment stiffness factor* (M_s^s) may be defined as the moment produced at the ends of a member when one end is displaced laterally a distance Δ with respect to the other end and when both ends are restrained against rotation, or when one end only is restrained against rotation. For members of constant cross-section $M_s^s = 6EI\Delta/l^2$ and is proportional to $K' = I/l^2$ if both ends are fully restrained against rotation; while $M_s^s = 3EI\Delta/l^2$ and is proportional to $K' = I/2l^2$ if one end is fully restrained against rotation, and the other end is free to rotate. See equations (9), (13) on pages 387, 388.

The *shear-stiffness factor* (S_s) may be defined as the force required to displace one end of a member laterally a distance Δ with respect to the other end, when both ends are restrained against rotation, or when one end only is restrained against rotation. For members of constant cross-section, $S_s = 12EI\Delta/l^3$ and is proportional to $K'' = I/l^3$, if both ends are fully restrained against rotation, and is proportional to $K'' = I/4l^3$ if one end is fully restrained against rotation and the other end free to rotate. See equation (10) on page 387.

Carry-over factor (C.O.F.) is the ratio between the moment produced at the other (or far) end of a member when one (or near) end of the member is rotated through an angle θ radians, and the moment required to rotate the near end through the angle θ. For members of constant cross-section, the carry-over factor is plus one-half if the far end is fixed, and zero if the far end is hinged.

Positive end moments are those which tend to rotate the end of a member in a counter clockwise direction (or the joint in a clockwise direction). *Negative end moments* are those which tend to rotate the end of a member in a clockwise direction (or the joint in a counter-clockwise direction). The signs are the same as those used in the slope-deflection equations.

Also, as for other methods, it is only necessary to use relative values of E and I except when shears and moments are due to non-elastic effects such as changes in temperature, definite yielding of supports, shrinkage, etc.

215. It has been seen, in the slope-deflection method, that an end moment at a member is equal to the fixed or direction-fixed bending moment modified by the moment due to the end rotation, the other end being locked, plus a moment due to the rotation of the other end of the member, when the end considered is locked, and, if sway occurs, plus a moment due to the axial rotation of the member, the ends remaining locked.

In the moment-distribution method, it is assumed at first that all joints and supports are fixed against rotation and translation. Then the moments induced at the ends of the members are those due to the loads on them and these are the fixed bending moments, M_{FAB}, M_{FBA}, M_{FCD}, M_{FDC}, etc. These are then modified in turn due to (*a*) the rotation of a joint at the end of a member, (*b*) the rotation of the joint at the other end of the member, and (*c*) the axial rotation or sway of the member, if any.

Thus, if the rotation or translation of the joints is restrained, then apart from the transverse loads on the members, there are imposed locking couples at the joints and external forces restraining sway. Now to cause the joints to rotate, we apply to them releasing couples to get rid of the locking couples, but the sway restraints are kept on whilst the rotations take place. This corresponds to modifications (*a*) and (*b*) in the previous paragraph: when the members are allowed to sway (modification (*c*)) there must be no further rotation of the joints: i.e. after the joints are once freed, then before a translation of a joint takes place, it must be re-locked in its freed or new position.

216. The problem now is: how to find these modifications by the method of moment-distribution. The method is as follows—

(1) Calculate the proportional stiffness factor K for each member of the framework, $K = I/l$ for two fixed ends or $K = 3I/4l$ for one end fixed and one end hinged and for uniform cross-section.

(2) Calculate the carry-over factor for each member of the framework (equal to plus one-half if the cross-section is uniform).

(3) Calculate the fixed beam moment at the end of each loaded member of the framework. There will now exist in general an unbalanced moment at each joint.

> Allow one such joint to be released for rotation only, keeping all the other joints fixed. This joint will rotate until sufficient moment has been added to or released from each member entering that joint, to balance the moments around the joint. The amount of moment added to or released from each member will be proportional to K (stiffness factor) of the member, and the total of these correction moments will equal the unbalanced moment at the joint. All corrections will have the same algebraic sign, which will be opposite to that of the original unbalanced moment.

$\therefore$ (4) Distribute the resulting unbalanced fixed beam moment at *each* joint among the members entering the joint in proportion to their stiffness factors, as a correction of opposite sign to that of the unbalanced moment. This step is "Bal M" in the tables of the various problems.

> At the same time, if the various members are of constant cross-section, there will be produced at the other (or far) end of each member a correction moment (carry-over) moment of the same algebraic sign and one-half the magnitude of that produced at the rotated (or near) end.

∴ (5) Multiply the correction moments in stage (4) by the carry-over factor for the corresponding member, and carry over this result as a correction to the opposite end (denoted C.O. in the problem tables). As a result of these carry-over moments, there will again result at each joint an unbalanced moment, and therefore if there is no sway it is necessary to repeat steps (4) and (5) until the carry-over moments are sufficiently small to be neglected. If there is sway, after step (5) there must be added another step (6).

(6) Consider a vertical framework supported at its base. For sway due, say, to wind or horizontal loads on the vertical members, after step (5) complete the total corrected moments in the columns (or members) whose axes have rotated. Then, e.g. for columns of equal length, compute the total column moments in each storey. Add a correction moment in each storey so as to make this total moment equal to the overturning moment on the storey, distributing this correction moment between the columns in proportion to their $K' = I/l^2$ for two fixed ends or $K' = I/2l^2$ for one end hinged (or $K' = I/l$ or $I/2l$ as l is constant) and dividing the correction moment for each column equally between the top and bottom. This is denoted by Bal. S in the tables of the problems. (See notes on the bent equations and equation (20).

In cases where the column lengths are unequal, the shears must be computed (equal to the sum of the column end moments divided by the length), and this value corrected to equal the actual shear in the columns. (See notes on the shear equations.)

The correction shear force will be divided between the various columns in proportion to their $K'' = I/l^3$ for two fixed ends (or $K'' = I/4l^3$ for one end hinged) values, and will produce a total moment on each column equal to the shear on that column multiplied by its length, this moment to be divided equally between the two ends. As before, steps (4), (5) and (6) will be repeated until the corrections are small enough to be neglected. Note that when step (4) is repeated the corrections of both steps (5) and (6) must be balanced; similarly when step (6) is repeated. (See equation 21.)

Several examples are given illustrating these methods and of the tabulation of the work. The signs of the side forces and the directions of action of the shears are the same as those given in the previous discussions for the slope-deflection equations.

NOTE. In the tables used for the moment-distribution solutions, a line is drawn under the moments at a joint after every operation of balancing (Bal. M).

NOTES ON STEP (6): For a vertical framework supported at the bases and with sway occurring due to horizontal loads or unsymmetrical vertical loading, etc., write down the general shear equation for each storey; then for the vertical members of the ground floor.

Referring to the frame given in Fig. 222,

$$\frac{M_{AB} + M_{BA}}{h_1} + \frac{M_{CD} + M_{DC}}{h_2} + \frac{M_{EF} + M_{FE}}{h_3} + \ldots$$
$$= P\frac{a}{h_1} + P_1\frac{b}{h_2} + P_2\frac{c}{h_3} + \text{etc.} = X. \qquad (40)$$

After all distributions and carry-overs this equation must be satisfied ($X = 0$ if no horizontal forces acting). If there are more than, say, two vertical members (swaying), then it will be better to work with the above shear equation, the correct signs being used.

After balancing the fixing moments due to the first joint rotations, and the first carry-over moments, moments will be induced at the ends of the columns. Let these be M'_{AB}, M'_{BA}, M'_{CD}, M'_{DC}, etc. These moments will include the fixed bending moments (if any), the moment as a result of balancing a joint and a carry-over moment (if any). Now find the value of the shears at the tops of the columns due to these moments.

Then calculate—

$$\frac{M'_{AB} + M'_{BA}}{h_1} + \frac{M'_{CD} + M'_{DC}}{h_2} + \text{etc.} = Y \text{ (say)} \qquad (41)$$

Then in order to balance X after the first sway, shear at the tops of the columns must be introduced equal to $X - Y$. K''_{AB}, K''_{CD}, etc., are known.

Then $X - Y$ is proportioned to each column according to their K'' values.

Then to AB, $S'_{\text{AB}} = \dfrac{X - Y}{\Sigma K''} \times K''_{\text{AB}}$:

and note will be taken of the direction in which it acts . (42)

and to CD, $S'_{\text{CD}} = \dfrac{X - Y}{\Sigma K''} \times K''_{\text{CD}}$:

and note will be taken of the direction in which it acts . (43)

The direction of action of X will be known therefore, and also from the signs of the M'_{AB}, etc., the sense of action of Y (finally). Therefore sense of action of $X - Y$ will be known.

Then the couples to be introduced in the columns for satisfying the shear equation will be

$\pm \dfrac{S'_{\text{AB}} \cdot l_{\text{AB}}}{2}$, top and bottom of AB, if A and B are fixed (44)

$\pm \dfrac{S'_{\text{CD}} \cdot l_{\text{CD}}}{2}$, top and bottom of CD, if C and D are fixed (45)

and similarly for other members.

If a column is hinged at its base, then there will be a moment only at the top of the column, the correct K'' to be used for this case in relation to the K''s for the 2 fixed-ended columns or members. Then, e.g. $S_{\text{AB}} \cdot l_{\text{AB}} = M_{\text{BA}}$ if A is hinged.

These couples are inserted in the tables in the first Bal. S line. After the rebalancing of these and the carry-over couples at the joints, i.e. after the second balance, additional couples will be introduced into the columns, say M''_{AB}, M''_{BA}, etc.

Then using

$$\frac{M''_{\text{AB}} + M''_{\text{BA}}}{h_1} + \frac{M''_{\text{CD}} + M''_{\text{DC}}}{h_2} + \text{etc.} = X_1 \quad . \quad . \quad (46)$$

thus X_1 must be cancelled out by introducing, to the tops of the columns, shears in total value $= - X_1$; as we have already dealt with any external effects represented by X, then

$$S''_{\text{AB}} = - \frac{X_1}{\Sigma K''} + K''_{\text{AB}} \quad . \quad . \quad . \quad (47)$$

and similarly for the other columns.

The additional couples to be applied to the columns will then be found inserted in the tables for the 2nd Bal. *S*. These and the 2nd carry-overs are re-balanced and the work carried out to the end.

If there are only two vertical members we can easily use the bent equation.

$$\frac{M_{\text{AB}} + M_{\text{BA}}}{h_1} + \frac{M_{\text{CD}} + M_{\text{DC}}}{h_2} = P\frac{a}{h_1} + P\frac{b}{h_2}$$

$$M_{\text{AB}} + M_{\text{BA}} + \frac{h_1}{h_2}(M_{\text{CD}} + M_{\text{DC}}) = Pa + Pb \, . \, \frac{h_1}{h_2} = Z \quad . \quad (47\text{a})$$

If there are no horizontal external forces operating, $Z = 0$.

After the 1st Bal.*M* (Joint Bal.) and carry-over (C.O.) we find the column couples

$$M'_{\text{AB}} \text{ and } M'_{\text{BA}} \text{ etc.}$$

Then $$M'_{\text{AB}} + M'_{\text{BA}} + \frac{h}{h_1}(M'_{\text{CD}} + M'_{\text{DC}}) = Q \quad . \quad . \quad (48)$$

Now to balance Z, moments M''_{AB} equal to M''_{BA}, etc., must be added to the columns so that the bent equation $= Z - Q$.

K'_{AB} and K'_{CD} are known, so that

$$\frac{M''_{\text{AB}}}{M''_{\text{CD}}} = \frac{K'_{\text{AB}}}{K'_{\text{CD}}}$$

$$\therefore \; M''_{\text{AB}} + M''_{\text{BA}} + \frac{h_1}{h_2}\left(\frac{K'_{\text{CD}}}{K'_{\text{AB}}} \, . \, M''_{\text{AB}} + \frac{K'_{\text{CD}}}{K'_{\text{AB}}} \, . \, M''_{\text{AB}}\right)$$
$$= Z - Q \quad . \quad . \quad . \quad . \quad (49)$$

M''_{AB} can be found in magnitude and sign (see equation $Z - Q$), and consequently M''_{CD}. These couples are inserted in the 1st Bal. *S*., after these have been re-balanced with any carry-over moments that are introduced into the columns.

Find the bent equation total for these, say Z.

Then $$M''_{\text{AB}} + M''_{\text{AB}} + \frac{h_1}{h_2}\left(\frac{K'_{\text{CD}}}{K'_{\text{AB}}} \, . \, M''_{\text{AB}} + \frac{K'_{\text{CD}}}{K'_{\text{AB}}} \, . \, M''_{\text{AB}}\right)$$
$$= - Z_1 \quad . \quad . \quad . \quad . \quad (50)$$

The moments M'' thus found are placed in the table opposite the second Bal. *S*. The whole work is then carried on to finality.

From the tables for a particular end moment, we see that it is equal to

a fixed bending moment (if any),
a final moment due to rotation of the joint,
a final moment due to carry-over from the other end joint rotation, and
a final moment due to the sway of the member.

217. **Alternative for Step (6).*** Consider the portal and its loading indicated in Fig. 226. The structure is held against side sway, the fixed end moments due to the applied load calculated, and the distribution carried out until the joints are balanced by steps (4) and (5).

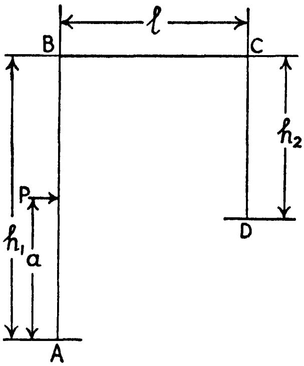

FIG. 226

The bent equation is

$$M_{AB} + M_{BA} + \frac{h_1}{h_2}(M_{CD} + M_{DC}) = Pa \quad . \quad (51)$$

Now without side-sway, values of the column end moments have been found. Let them be M'_{AB}; M'_{BA}; M'_{CD} and M'_{DC}.

Calculate the sum M'_s

where $$M'_s = M'_{AB} + M'_{BA} + \frac{h_1}{h_2}(M'_{CD} + M'_{DC}) \quad . \quad (52)$$

Any difference between M'_s and Pa is proportional to the moment introduced into the structure by holding it against side sway.

Let $$M'_s - Pa = M''_s$$

Calculate $K'(= I/h^2)$ for each of the columns.

When sway takes place, the horizontal movements of B and C will be the same; then the moments introduced into the columns will be proportional to their K' values, i.e.

$$M_{AB} = M_{BA} = \frac{K'_{AB}}{K'_{CD}} . M_{CD} = \frac{K'_{AB}}{K'_{CD}} . M_{DC}$$

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* See "Side-sway Correction for Portals, with Horizontal Loading Analysed by the Moment-Distribution Method"; W. T. Marshall. *Concrete and Constructional Engineering* (July, 1942).

Assume any convenient value for M_{AB} and M_{BA} and calculate the corresponding moments M_{CD} and M_{DC}. These moments are now distributed and the joints balanced as per steps (4) and (5). The sum

$$M''_{AB} + M''_{BA} + \frac{h_1}{h_2}(M''_{CD} + M''_{DC}) = M''_s \quad . \quad . \quad (53)$$

The moment due to side sway has already been found to be M'_s; the balanced moments for the second distribution are therefore altered in the ratios $\frac{M'_s}{M''_s}$, and these moments are added to those obtained from the original distribution (no sway distribution) to get the true moments. Therefore, e.g.,

$$M_{AB}\text{ (true)} = M'_{AB} + M''_{AB} \cdot \frac{M'_s}{M''_s} \quad . \quad . \quad (54)$$

and similarly for the other moments.

A number of worked examples are now given introducing the various results given in the discussion in previous paragraphs.

218. **The Procedure in the Moment-Distribution Method.**

(*a*) All joints are locked against rotation, and all members are restrained from axial rotation or swaying.

(*b*) Joints are unlocked in turn and moments distributed to the members at the joint ends and carried over to the opposite ends. The restraints against axial rotation of the members and consequently the translation of the joints have remained on the structure.

(*c*) The joints are re-locked in their new positions before the sway restraints are removed. These sway restraints are now removed and the axes of members are allowed to rotate with resulting joint translation, there being no rotation of the joints. The resulting moments at the ends of the rotated members are now calculated.

(*d*) As a result of these operations, locking moments and further sway restraints are introduced and consequently steps (*b*) and (*c*) are repeated until finally the locking moments and sway restraints are completely eliminated.

MEMBERS WITH VARIABLE I. The discussion of the methods has been based upon the assumption that the moment of inertia remains constant throughout the length of each member. Though seldom strictly true, the assumption gives results sufficiently exact for designing purposes for a large class of problems in continuous girders, building frames, secondary stresses in riveted structures, etc. There are many cases, however, in which the deviation from a constant I value is too great to ignore if the results are to be practically usable, and if using the slope-deflection method, modified slope-deflection equations are required. Similarly for the moment-distribution method, this must also be modified for members whose moment of inertia varies over all or part of the length. The fixed-end moments and the distribution and carry-over factors will differ markedly from the case of that for prismatic members. Tables, in other works of reference, have been prepared from which the modified values for these terms can be obtained, and when they are determined, the solution is carried out in the same way as for members with constant I (see references (2) and (4)).

REFERENCES

(1) See Chapter IX, page 219.
(2) *Continuous Frames of Reinforced Concrete,* Cross and Morgan.
(3) *Theory of Structures,* Timoshenko and Young. (McGraw-Hill & Co.)
(4) *Fundamentals of Indeterminate Structures,* F. L. Plummer. (Pitman Publishing Corporation.)
(5) *Relaxation Methods in Engineering Science,* R. V. Southwell. (Oxford.)
(6) *Continuous Beam Structures,* Shepley. (Concrete Publications, Ltd.)
(7) "The Problem of Sway in Complicated Rigid Frames," J. L. Matheson. *Inst. of C.E.*: Paper No. 6, April, 1945.
(8) "The Sidesway Correction for Portals with Horizontal Loading, etc." W. T. Marshall. *Concrete and Constructional Engineering,* Vols. 37 and 38.
(9) "Frames subjected to Multiple Sway," A. J. Francis, *Concrete and Constructional Engineering,* Nov. and Dec., 1949.
(10) "The Column Analogy," Hardy Cross. *University of Illinois Bulletin,* No. 215, October, 1930. A further method, presenting some theorems, dealing with the elastic analysis of continuous frames.
(11) *Structural Analysis,* W. Fisher Cassie. (Longmans.)
(12) "Analysis of Rigid Frames by an Electronic Digital Computer," R. K. Livesey. *Engineering,* Aug., 21 and 28, 1953.
(13) *An Introduction to Distribution Methods of Structural Analysis,* R. W. Steed. (Pitman.)
(14) *Calcul des Constructions Hyperstatiques: La Methode de Hardy Cross et ses Simplifications,* Serge Zaytzeff. (Dunod, Paris, 6c.)

219. **Illustrative Examples.** In these problems, when solving by the moment-distribution method,

Bal. = Proportionate amount of unbalanced moment to be given to a particular beam. It is represented by $K/\Sigma K$ at the joint, or the ratio $\dfrac{\text{relative } K}{\Sigma \text{ relative } K}$ at the joint.
C.O.F. = Carry-over factor.
F.B.M. = Fixed bending moment.
Bal. S = Balancing couples due to sway.
Bal. M = Balancing joint moment.

Illustrative Problem 15. (See Chapter IV, p. 98.)

Beam continuous over two spans of length l, and loaded with a uniformly distributed load of w tons per unit length. Find the moment at the central support. (Fig. 227.)

Assume EI constant for the two spans and that the ends are simply supported, and that the supports are rigid.

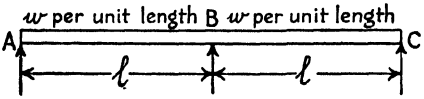


FIG. 227

Solving by the Slope-deflection Method. K is the same for both beams. Considering span AB,

$$M_{AB} = 0 = 2EK(2\theta_{AB} + \theta_{BA}) + \frac{wl^2}{12}$$

$$M_{BA} = 2EK(\theta_{AB} + 2\theta_{BA}) - \frac{wl^2}{12}$$

Due to symmetry of the beam and the loading about a vertical axis through the support B,

$$\theta_{BA} = \theta_{BC} = 0$$

$$\therefore 2M_{BA} = 2EK(2\theta_{AB}) - \frac{2wl^2}{12}$$

$$\therefore 2M_{BA} - M_{AB} = 2M_{BA} = -\frac{2wl^2}{12} - \frac{wl^2}{12}$$

$$\therefore M_{BA} = -\frac{wl^2}{8}$$

NOTE. In all slope-deflection equations K for all members is equal to I/l, whether they are fixed at both ends or fixed at one end and hinged at the other. In the latter case, the moment at the hinge is zero: but the right-hand side of the equation includes the fixed bending moment, treating the beam as fixed at both ends.

Moment-Distribution Method—No Sway. (a) Assume that A and C are fixed initially: as they are hinged the final moments M_{AB} and M_{CB} are = zero: ∴ Bal. will be 1 at A and C.

$$M_{FAB} = +\frac{1}{12}wl^2; \quad M_{FBA} = -\frac{1}{12}wl^2;$$

$$M_{\mathrm{FCB}} = -\frac{1}{12}wl^2; \quad M_{\mathrm{FBC}} = +\frac{1}{12}wl^2.$$

$$K_{\mathrm{AB}} = K_{\mathrm{BC}} = I/l. \ \therefore K_{\mathrm{AB}} : K_{\mathrm{BC}} = 1 : 1$$

Member	*AB*	*BA*	*BC*	*CB*
K	1		1	
C.O.F.	$\frac{1}{2}$		$\frac{1}{2}$	
Bal.	1	$\frac{K_{AB}}{K_{AB}+K_{BC}} = \frac{1}{2}$	$\frac{1}{2} = \frac{K_{BC}}{K_{AB}+K_{BC}}$	
Multiple wl^2 F.B.M.	$+\frac{1}{12}$	$-\frac{1}{12}$	$+\frac{1}{12}$	$-\frac{1}{12}$
Bal. M.	$-\frac{1}{12}$	0	0	$+\frac{1}{12}$
C.O.F.	0	$-\frac{1}{24}$	$+\frac{1}{24}$	0
ΣM	0	$-\frac{1}{8}$	$+\frac{1}{8}$	0

$$\therefore M_{\mathrm{BA}} = -M_{\mathrm{BC}} = -\frac{wl^2}{8}$$

Or—(*b*) Initially *A* and *C* are hinged:

$$\therefore M_{\mathrm{FAB}} = M_{\mathrm{FCD}} = 0 \text{ and } M_{\mathrm{FBA}} = -M_{\mathrm{FBC}} = -\frac{wl^2}{8}$$

Member	*AB*	*BA*	*BC*	*CB*
K	$\frac{3}{4} \times 1$		$\frac{3}{4} \times 1$	
Rel. K	1		1	
Bal.	0	$\frac{1}{2}$	$\frac{1}{2}$	0
C.O.F.	0		0	
Multiple wl^2: F.B.M.	–	$-\frac{1}{8}$	$+\frac{1}{8}$	–
C.O.	–	0	0	–
ΣM	0	$-\frac{1}{8}$	$+\frac{1}{8}$	0

$$\therefore M_{\mathrm{BA}} = -M_{\mathrm{BC}} = -\frac{wl^2}{8}$$

NOTE. In the first solution above, each member has been treated initially as a beam fixed at both ends, and therefore K for each member is I/l (taken relatively as 1) and at each end (including the hinged end) there is the fixed bending moment due to the load on a beam fixed at both ends. As the final moment at the hinge is zero, then the Bal. factor at the hinge is unity. There is thus a Bal. M at the hinge and also a carry over from the hinge to the other end of the member. In the second solution above, the members have initially been treated as hinged at one end and fixed at the other. The K value is then $= \frac{3}{4} \times \frac{I}{l}$ (relatively $\frac{3}{4} \times 1$): the F.B.M. at the hinge is zero and the correct F.B.M. is taken for a loaded beam hinged at one end and fixed at the other. There is obviously no balancing, in this case, at the hinge and no carrying over to the fixed end.

Illustrative Problem 16. (See Chapter IV, p. 100.)

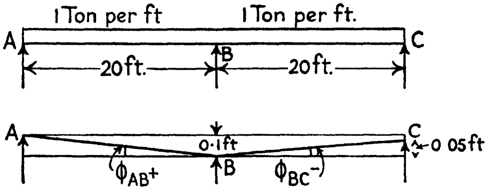


FIG. 228

$EI = 40{,}000$ (ft.^{2}-ton) units, and, as it is the same for both AB and BC, then $K = I/l$ is the same for both members.

$$\phi_{\text{AB}} \text{ is positive} = \frac{0{\cdot}1}{20}\,; \quad \phi_{\text{BC}} \text{ is negative} = \frac{0{\cdot}05}{20}$$

Slope-deflection Method.

$$M_{\text{AB}} = M_{\text{CB}} = 0$$

$$M_{\text{BA}} + M_{\text{BC}} = 0 \text{ or } M_{\text{BA}} = -M_{\text{BC}}$$

$$M_{\text{FS}} \text{ all equal to } \frac{wl^2}{12} = \frac{1 \times 400}{12} = \frac{100}{3} \text{ tons-ft.}$$

$$M_{\text{AB}} = 2EK(2\theta_{\text{AB}} + \theta_{\text{BA}} + 3\phi_{\text{AB}}) + 100/3 = 0$$

$$M_{\text{BA}} = 2EK(\theta_{\text{AB}} + 2\theta_{\text{BA}} + 3\phi_{\text{AB}}) - 100/3$$

$$2M_{\text{BA}} = 2EK(2\theta_{\text{AB}} + 4\theta_{\text{BA}} + 6\phi_{\text{AB}}) - 200/3$$

$$\therefore\ 2M_{\text{BA}} = 2EK(3\theta_{\text{BA}} + 3\phi_{\text{AB}}) - 100 \quad . \quad . \quad . \qquad \text{(i)}$$

$$2M_{\text{BC}} = 2EK(4\theta_{\text{BC}} + 2\theta_{\text{CB}} + 6\phi_{\text{BC}}) + 200/3$$

$$M_{\text{CB}} = 2EK(\theta_{\text{BC}} + 2\theta_{\text{CB}} + 3\phi_{\text{BC}}) - 100/3 = 0$$

$$\therefore\ 2M_{\text{BC}} = 2EK(3\theta_{\text{BC}} + 3\phi_{\text{BC}}) + 100 \quad . \quad . \quad . \qquad \text{(ii)}$$

Now $\theta_{\text{BA}} = \theta_{\text{BC}}$

Adding (i) and (ii),

$$2EK(6\theta_{BA} + 3\phi_{AB} + 3\phi_{BC}) = 0$$

$$6\theta_{BA} + \frac{0{\cdot}3}{20} - \frac{0{\cdot}15}{20} = 0$$

$$\therefore \theta_{BA} = -\frac{0{\cdot}15}{120}$$

$$\therefore 2M_{BA} = \frac{80{,}000}{20}\left(-\frac{0{\cdot}45}{120} + \frac{0{\cdot}3}{20}\right) - 100$$

$$M_{BA} = \frac{40{,}000}{20}\left(+\frac{1{\cdot}35}{120}\right) - 50$$

$$= \frac{450}{20} - 50$$

$$= 22{\cdot}5 - 50 = -27{\cdot}5 \text{ tons-ft.}$$

$$\text{and } M_{BC} = +27{\cdot}5 \text{ tons-ft.}$$

Moment Distribution Solution. (See Fig. 229.)

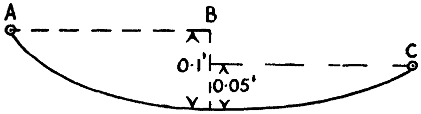


FIG. 229

Assume $\frac{I}{l}$ = unity for both members:

Member	*AB*	*BA*	*BC*	*CB*
K	¾ × 1		¾ × 1	
Rel. *K*	1		1	
Bal.	0	½	½	0
C.O.F.	0		0	
No sway. F.B.M.	0	− 50	+ 50	0
Bal.	0	0	0	0
Sway fixing moment	0	+ 30	− 15	0
* Bal. *M*.	0	− 7·5	− 7·5	0
ΣM (tons-ft.)	0	− 27·5	+ 27·5	0

* Bal. + 30 − 15 = + 15. Amount to *BA* = ½ × 15 and is negative; Amount to *BC* = ½ × 15 and is negative.

Note. Sway Moments.

Moment M_{BA} due to sway and no rotation of B is

$$+\,3EI\Delta/l^2 = 3 \times 40{,}000 \times 0{\cdot}1/20^2 = 30 \text{ tons-ft.}$$

Moment M_{BC} due to sway and no rotation of B is

$$-\,3EI\Delta/l^2 = -\,3 \times 40{,}000 \times 0{\cdot}05/20^2 = -\,15 \text{ tons-ft.}$$

$$\text{Finally } M_{BA} = -\,27{\cdot}5 \text{ tons-ft.}$$

$$M_{BC} = +\,27{\cdot}5 \text{ tons-ft.}$$

For a beam hinged at one end A and direction fixed at the end B, and carrying a uniformly distributed load,

$$M_{FBA} = \frac{wl^2}{12} + \frac{wl^2}{24} = \frac{wl^2}{8}$$

In this example $M_{FAB} = -50{\cdot}0$ tons-ft.

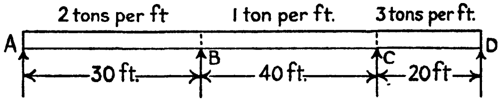


Fig. 230

Illustrative Problem 17. (See page 102.) See diagram Fig. 70 and Fig. 230.

Slope-deflection Method: No Sway.

$$\begin{array}{rccccc} EK \text{ for} & AB & & BC & & CD \\[6pt] \text{equals} & \dfrac{EI}{30} & & \dfrac{EI}{40} & & \dfrac{EI}{20} \\[10pt] \therefore EK \text{ for} & AB & : & BC & : & CD \\[6pt] \text{equals} & \dfrac{1}{30} & : & \dfrac{1}{40} & : & \dfrac{1}{20} \\[10pt] \text{equals} & 4 & : & 3 & : & 6 \end{array}$$

These are the relative values we shall use in the slope-deflection equations; the values of the θ's obtained will be the corresponding relative ones.

Let the θ terms be θ_A, θ_B, θ_C and θ_D.

Considering beam *AB*—

$$M_{AB} = 0 = 2EK_{AB}(2\theta_A + \theta_B) + 150 \quad . \quad . \quad . \quad \text{(i)}$$

$$M_{BA} = 2EK_{AB}(\theta_A + 2\theta_B) - 150 \quad . \quad . \quad . \quad \text{(ii)}$$

Considering beam *BC*—

$$M_{BC} = 2EK_{BC}(2\theta_B + \theta_C) + 133\tfrac{1}{3} \quad . \quad . \quad . \quad \text{(iii)}$$

$$M_{CB} = 2EK_{BC}(\theta_B + 2\theta_C) - 133\tfrac{1}{3} \quad . \quad . \quad . \quad \text{(iv)}$$

Considering beam *CD*—

$$M_{CD} = 2EK_{CD}(2\theta_C + \theta_D) + 100 \quad . \quad . \quad . \quad \text{(v)}$$

$$M_{DC} = 0 = 2EK_{CD}(\theta_C + 2\theta_D) - 100 \quad . \quad . \quad . \quad \text{(vi)}$$

Now we have that—

joint *B* equation: $M_{BA} + M_{BC} = 0$ . . . (vii)

joint *C* equation: $M_{CB} + M_{CD} = 0$ . . . (viii)

Considering equations (i) and (ii) and eliminating θ_A

we find $M_{BA} = 3EK_{AB}(\theta_B) - 225$ . . . (ix)

Similarly for equations (v) and (vi)

$$M_{CD} = 3EK_{CD}(\theta_C) + 150 \quad . \quad . \quad . \quad \text{(x)}$$

Considering equations (vii), (viii), (ix) and (x),

we find $\theta_B = + 4{\cdot}16$, and $\theta_C = - 1{\cdot}38$

Substituting in equations (ix) and (x),

$$M_{BA} = - 175{\cdot}08 \text{ tons-ft.}$$

and $M_{CD} = + 125{\cdot}16$ tons-ft.

These agree well with the corresponding values found by the three-moment method.

Moment-Distribution Method: No Sway. Assume E = same for all members = 1, and I = 1 for all members.

	AB	*BA*	*BC*	*CB*	*CD*	*DC*
K	$\frac{3}{4} \times \frac{1}{30}$		$\frac{1}{40}$		$\frac{3}{4} \times \frac{1}{20}$	
Rel. K	6		6		9	
Bal.	0	$\frac{1}{2} = \frac{6}{12}$	$\frac{6}{12} = \frac{1}{2}$	$\frac{2}{5} = \frac{6}{15}$	$\frac{9}{15} = \frac{3}{5}$	0
C.O.F.	0		$\frac{1}{2}$		0	
F.B.M.	0	− 225·0	+133·33	−133·33	+150	0
Bal. M.	–	+ 45·83	+ 45·83	− 6·66	− 10	–
C.O.	–	0	− 3·33	+ 22·91	0	–
Bal. M.	–	+ 1·67	+ 1·67	− 9·16	− 13·74	–
C.O.	–	0	− 4·58	+ 0·84	0	–
Bal. M.	–	+ 2·29	+ 2·29	− 0·34	− 0·51	–
C.O.	–	0	− 0·17	+ 1·15	0	–
Bal. M.	–	+ 0·09	+ 0·09	− 0·46	− 0·69	–
C.O.	–	0	− 0·23	+ 0·05	0	–
Bal. M.	–	+ 0·12	+ 0·12	− 0·02	− 0·03	–
C.O.	–	0	− 0·01	+ 0·06	0	–
Bal. M.	–	0	0	− 0·03	− 0·03	–
ΣM (tons-ft.)	0	− 175·00	+175·03	−125·00	+125·00	0
		(M_{BA})			(M_{CD})	

Continuous beam results:	175·00 tons-ft.	125·00 tons-ft. (Slide rule working.)
Slope-deflection:	175·08 tons-ft.	125·16 tons-ft.
Moment-distribution:	175·00 tons-ft.	125·00 tons-ft.

Illustrative Problem 18. (See page 107, Fig. 71 and Fig. 231.) Fixed ends at A and D, therefore no rotation of the beam ends at these points, and therefore no balancing moment is required, i.e. Bal. = 0.

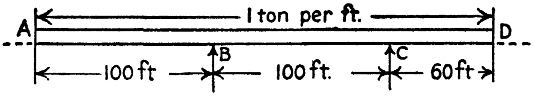


FIG. 231

EI is constant.

$$EI/l = EK \text{ for } AB : BC : CD$$

$$\text{is } \frac{EI}{100} : \frac{EI}{100} : \frac{EI}{60}$$

Relative EK is therefore 3 : 3 : 5

$$\theta_A = \theta_D = 0$$

Let $\theta_{BA} = \theta_B$; $\theta_{CB} = \theta_{CD} = \theta_C$

$$M_{FAB} = -M_{FBA} = 10{,}000/12 \text{ tons ft.};$$

$$M_{BC} = M_{CB} = 10{,}000/12 \text{ tons-ft.};$$

$$M_{FCD} = M_{FDC} = 300 \text{ tons-ft.}$$

$$M_{AB} = 6(\theta_B) + 10{,}000/12$$

$$M_{BA} = 6(2\theta_B) - 10{,}000/12$$

$$M_{BC} = 6(2\theta_B + \theta_C) + 10{,}000/12$$

$$M_{DC} = 10(\theta_C) - 300$$

$$M_{CD} = 10(2\theta_C) + 300$$

$$M_{CB} = 6(2\theta_C + \theta_B) - 10{,}000/12$$

Joint B equation: $M_{BA} + M_{BC} = 0$

Joint C equation: $M_{CB} + M_{CD} = 0$

Solving all the above equations for θ_B and θ_C,

$$\theta_B = -4{\cdot}37 \text{ and } \theta_C = +17{\cdot}48$$

It is found that—

$$M_{AB} = \quad +807{\cdot}11 \text{ tons-ft.}$$

$$M_{BA} = -M_{BC} = -885{\cdot}77 \text{ tons-ft.}$$

$$M_{CD} = -M_{CB} = -649{\cdot}60 \text{ tons-ft.}$$

$$M_{DC} = \quad -125{\cdot}20 \text{ tons-ft.}$$

Moment-Distribution Method: No Sway.

Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
Rel. *K*'s	3		3		5	
Bal.	0	$\frac{3}{3+3}=\frac{1}{2}$	$\frac{1}{2}=\frac{3}{3+3}$	$\frac{3}{3+5}=\frac{3}{8}$	$\frac{5}{8}=\frac{5}{3+5}$	
C.O.F.	½		½		½	
F.B.M.	+833·33	−833·33	+833·33	−833·33	+300·00	−300·00
Bal. M.	0	0	0	+200·00	+333·33	0
C.O.	0	0	+100·00	0	0	+166·67
Bal. M.	0	− 50·00	− 50·00	0	0	0
C.O.	− 25·00	0	0	− 25·00	0	0
Bal. M.	0	0	0	+ 9·38	+ 15·62	0
C.O.	0	0	+ 4·69	0	0	+ 7·81
Bal. M.	0	− 2·35	− 2·35	0	0	0
C.O.	− 1·18	0	0	− 1·18	0	0
Bal. M.	0	0	0	+ 0·45	+ 0·73	0
C.O.	0	0	+ 0·23	0	0	+ 0·37
Bal. M.	0	− 0·12	− 0·12	0	0	0
C.O.	− 0·06	0	0	− 0·06	0	0
Bal. M.	0	0	0	+ 0·02	+ 0·04	0
ΣM (tons-ft.)	+807·07 (M_{AB})	−885·80 (M_{BA})	+885·80	−649·72	+649·72 (M_{CD})	−125·15 (M_{DC})

Results	M_{AB} (tons-ft.)	M_{BA} (tons-ft.)	M_{CD} (tons-ft.)	M_{DC} (tons-ft.)
Slope-deflection method	807·11	885·77	649·60	125·20
Moment distribution	807·07	885·80	649·72	125·15
Continuous beam .	807·13	885·73	649·93	125·14

Illustrative Problem 33. (See page 206 and Fig. 232.)

EI is the same for all members, ∴ $K = I/l$ is the same for all members.

$H_A = H_D$ in magnitude but opposite in sign, as no horizontal forces acting on the frame.

Slope-deflection Method. The beam is symmetrical about a vertical axis through the load point and is symmetrically loaded.

$$M_{FBC} = +\frac{Wl}{8} = \frac{1 \times 15}{8} = 1\tfrac{7}{8} \text{ tons-ft.};$$

$$M_{FCB} = -\frac{Wl}{8} = -1\tfrac{7}{8} \text{ tons-ft.}$$

$$M_{BA} = \qquad 2EK(\theta_{AB} + 2\theta_{BA}) + 0$$

$$M_{AB} = 0 = 2EK(2\theta_{AB} + \theta_{BA}) + 0$$

$$2M_{BA} = \qquad 2EK(2\theta_{AB} + 4\theta_{BA}) + 0$$

$$\therefore\ 2M_{BA} = 2EK(2\theta_{AB} + 4\theta_{BA}) + 0$$

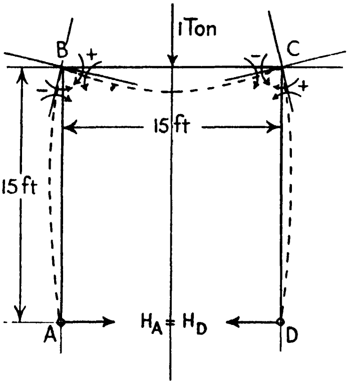


FIG. 232

Subtracting the expression for M_{AB} from that for $2M_{BA}$,

$$2M_{BA} = 2EK(3\theta_{BA})$$

$$M_{BC} = 2EK(2\theta_{BC} + \theta_{CB}) + 15/8$$

Due to symmetry, $\theta_{CB} = -\theta_{BC}$

$$\therefore\ M_{BC} = 2EK(\theta_{BC}) + 15/8$$

$$M_{BA} + M_{BC} = 0$$

$$\therefore\ EK(3\theta_{BA}) + 2EK(\theta_{BC}) + 15/8 = 0$$

$$\theta_{BC} = \theta_{BA}$$

$$\therefore\ 5EK(\theta_{BA}) + 15/8 = 0$$

$$\therefore \theta_{BA} = -\frac{3}{8EK}$$

$$\therefore M_{BA} = EK\left(-\frac{9}{8EK}\right) + 0$$

$$= -1{\cdot}125 \text{ tons-ft.}$$

$$\therefore H_A = \frac{1{\cdot}125}{15} = 0{\cdot}075, \text{ and acts from left to right.}$$

Moment-Distribution Method: No Sway.

Let $K = I/l$ be represented by unity for all the members.

	AB	*BA*	*BC*	*CB*	*CD*	*DC*
Cols. hinged at base. K	$\frac{3}{4} \times 1$		1		$\frac{3}{4} \times 1$	
Rel. K	3		4		3	
Bal.	0	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{4}{7}$	$\frac{3}{7}$	0
C.O.F.	0		$\frac{1}{2}$		0	
F.B.M.	0	0	+ 1·875	− 1·875	0	0
Bal. M.	0	− 0·804	− 1·072	+ 1·072	− 0·804	0
C.O.	0	0	+ 0·536	− 0·536	0	0
Bal. M.	0	− 0·230	− 0·307	+ 0·307	+ 0·230	0
C.O.	0	0	+ 0·154	− 0·154	0	0
Bal. M.	0	− 0·066	− 0·088	+ 0·088	+ 0·066	0
C.O.	0	0	+ 0·044	− 0·044	0	0
Bal. M.	0	− 0·019	− 0·025	+ 0·025	+ 0·019	0
C.O.	0	0	+ 0·073	− 0·013	0	0
Bal. M.	0	− 0·006	− 0·007	+ 0·007	+ 0·006	0
C.O.	0	0	+ 0·004	− 0·004	0	0
Bal. M.	0	− 0·002	− 0·002	+ 0·002	+ 0·002	0
ΣM (tons-ft.)	0	− 1·127 (M_{BA})	+ 1·125	− 1·125	+ 1·127 (M_{CD})	0

These results agree with the slope-deflection method.

$K = \frac{3}{4} \times 1$ for the columns because they are hinged at the bases.

Illustrative Problem 48. (See Figs. 233, 234 and 235.)

$$(I/l);\ K_{AB} : K_{BC} : K_{CD} = 20 : 20 : 20$$
$$= 1 : 1 : 1$$
$$K'(= I/l^2);\ K'_{AB} : K'_{CD} = 2 : 4$$
$$= 1 : 2$$
$$K''(= I/l^3);\ K''_{AB} : K''_{CD} = 0{\cdot}2 : 0{\cdot}8$$
$$= 1 : 4$$

E is the same for all members $AB = h_1$; $CD = h_2$

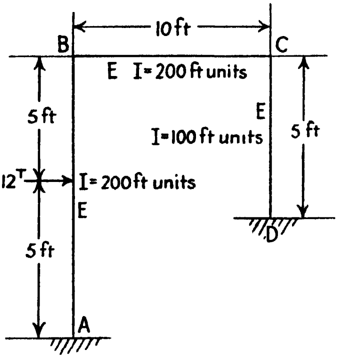


FIG. 233

$$\text{Bent equation: } M_{AB} + M_{BA} + \frac{h_1}{h_2}(M_{CD} + M_{DC})$$
$$= 12 \times 5 = +\ 60 \text{ tons-ft.}$$

$$\text{Shear equation: } \frac{M_{AB} + M_{BA}}{h_1} + \frac{M_{CD} + M_{DC}}{h_2} = \frac{12 \times 5}{10}$$
$$= +\ 6 \text{ tons (equivalent to acting at } B)$$

Distribution of the sway couples, assuming all joints locked after a balancing (Bal. M) and carry-over (C.O.)—

$$(M'_{AB} = M'_{BA}) : (M'_{CD} = M'_{DC}) = K'_{AB} : K'_{CD} = 1 : 2$$

Corresponding shears at the tops of the columns—

$$H_{AB} : H_{CD} = K''_{AB} : K''_{CD} = 1 : 4$$
$$M_{FAB} = 15{\cdot}00 \text{ tons-ft.};\ M_{FBA} = -\ 15{\cdot}00 \text{ tons-ft.}$$

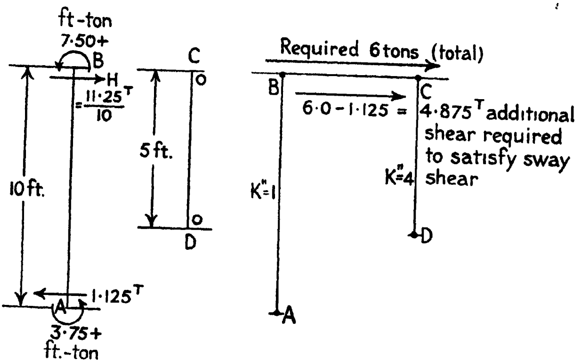


FIG. 234

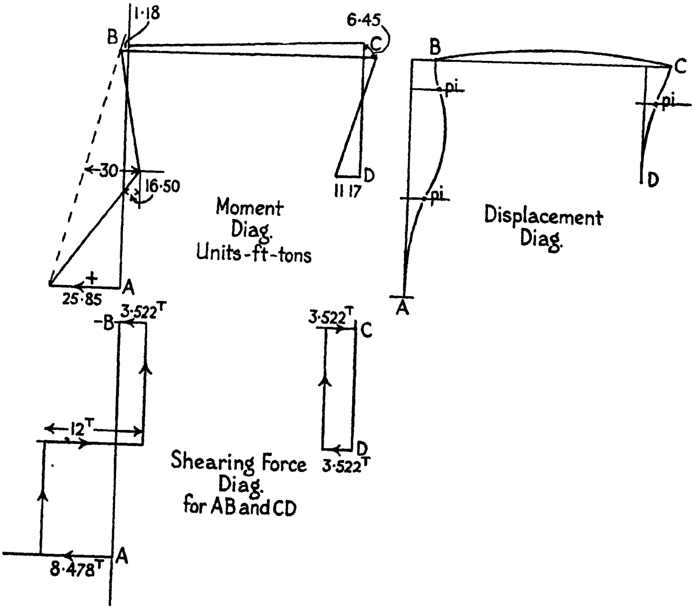


FIG. 235

Moment-Distribution Method, by Step (6)*: Sway Problem.*

Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
Rel. *K*	1		1		1	
Bal.	0	$\frac{1}{2} = \frac{1}{1+1}$	$\frac{1}{2}$	$\frac{1}{2} = \frac{1}{1+1}$	$\frac{1}{2}$	0
C.O.F.	½		½		½	
Rel. *K′*	1		0		2	
Rel. *K″*	1		0		4	
F.B.M.	+ 15·00	− 15·00	0	0	0	0
Bal. M.	0	+ 7·50	+ 7·50	0	0	0
C.O.	+ 3·75	0	0	+ 3·75	0	0
Bal. S.	+ 4·88	+ 4·88	0	0	+ 9·75	+ 9·75
Bal. M.	0	− 2·44	− 2·44	− 6·75	− 6·75	0
C.O.	− 1·22	0	− 3·38	− 1·22	0	− 3·38
Bal. S.	+ 2·39	+ 2·39	0	0	+ 4·78	+ 4·78
Bal. M.	0	+ 0·50	+ 0·50	− 1·78	− 1·78	0
C.O.	+ 0·25	0	− 0·89	+ 0·25	0	− 0·89
Bal. S.	+ 0·46	+ 0·46	0	0	+ 0·92	+ 0·92
Bal. M.	0	+ 0·22	+ 0·22	− 0·58	− 0·58	0
C.O.	+ 0·11	0	− 0·29	+ 0·11	0	− 0·29
Bal. S.	+ 0·14	+ 0·14	0	0	+ 0·28	+ 0·28
Bal. M.	0	+ 0·08	+ 0·08	− 0·20	− 0·20	0
C.O.	+ 0·04	0	− 0·10	+ 0·04	0	− 0·10
Bal. S.	+ 0·05	+ 0·05	0	0	+ 0·10	+ 0·10
Bal. M.	0	+ 0·03	+ 0·03	− 0·07	− 0·07	0
ΣM (tons-ft.)	+ 25·85 (M_{AB})	− 1·19 (M_{BA})	+ 1·23	− 6·45	+ 6·45 (M_{CD})	+ 11·17 (M_{DC})

$$M_{AB} + M_{BA} + 2(M_{CD} + M_{DC}) = +\ 59\cdot9 \simeq 60\cdot0 \text{ tons-ft.}$$

1*st Bal. S.* After the first balancing of the joint moments and the carry-over, there have been induced in the columns additional moments of + 7·5 and + 3·75 tons-ft., whose sum is 11·25 tons-ft.

(The sum of the fixing moments on column *AB* = zero.)

$$\text{Now } M_{AB} + M_{BA} + \frac{h_1}{h_2}(M_{CD} + M_{DC})$$

$$= Pa = 12 \times 5 = 60 \text{ tons-ft.}$$

Therefore the sway of the frame has to introduce a further + 48·75 tons-ft. into the columns. (60 — 11·25 = 48·75.)

Keeping the joints *B* and *C* locked, and swaying the frame, extra moments $M'_{AB} = M'_{BA}$ and $M'_{CD} = M'_{DC}$ will be introduced into the columns.

Now
$$\frac{M'_{AB}}{M'_{CD}} = \frac{K'_{AB}}{K'_{CD}} = \frac{1}{2}$$

$$\therefore M'_{CD} = 2M'_{AB}$$

Now
$$\frac{h_1}{h_2} = \frac{10}{5} = 2$$

$$\therefore M'_{AB} + M'_{AB} + 2(2M'_{AB} + 2M'_{AB}) = 60 - 11{\cdot}25 = 48{\cdot}75$$

$$\text{i.e } 10M'_{AB} = 48{\cdot}75 \quad \therefore M'_{AB} = +\ 4{\cdot}88 \text{ tons-ft.}$$
$$M'_{CD} = +\ 9{\cdot}75 \text{ tons-ft.}$$

The joints are now out of balance again, so they must be re-balanced and the moments carried-over. Additional moments are thus introduced into the columns and these have to be cancelled out by a further swaying of the frame.

2nd Bal. S. Additional couples in *AB* are — 2·44 and — 1·22 ft.-tons, and in *CD* are — 6·75 and — 3·38 tons-ft.

Then the bent summation becomes — 23·91 tons-ft.

Having already balanced the external moment of 60 tons-ft., this moment of — 23·91 tons-ft. must be cancelled out by a moment of + 23·91 tons-ft. divided amongst the column ends by swaying.

$$\therefore 10M'_{AB} = 23{\cdot}91 \text{ tons-ft.}$$

$$\therefore M'_{AB} = M'_{BA} = +\ 2{\cdot}39 \text{ tons-ft. and}$$
$$M'_{CD} = M'_{DC} = +\ 4{\cdot}78 \text{ tons-ft.}$$

Balancing the column end couples for the column end shears (see Fig. 234): after the 1st Bal. M and C.O., we have in the columns (neglecting F.B.M.s),

$$H \text{ additional at } B = \frac{(4{\cdot}875)1}{1+4} = 0{\cdot}975^T, \text{ and}$$

$$\text{at } C = 0{\cdot}975 \times 4 = 3{\cdot}9^T$$

These forces acting left to right and at the top introduce equal couples at B and A of

$$+ 0{\cdot}975 \times \tfrac{10}{2} = + 4{\cdot}88 \text{ tons-ft.}$$

and at C and D $= + 3{\cdot}90 \times \tfrac{5}{2} = + 9{\cdot}75$ tons.-ft.

Similarly for balancing of the other additional moments which will be introduced into the columns due to balancing the joints and carrying-over.

Moment-Distribution Method when Using the Alternative Step (6)*:*

1*st Distribution—No Sway Allowed.*

Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
K's	1		1		1	
Bal.	0	½	½	½	½	0
C.O.F.	½		½		½	
F.B.M.	+ 15·00	− 15·00	0	0	0	0
Bal. M.	0	+ 7·50	+ 7·50	0	0	0
C.O.	+ 3·75	0	0	+ 3·75	0	0
Bal. M.	0	0	0	− 1·88	−1·88	0
C.O.	0	0	− 0·94	0	0	− 0·94
Bal. M.	0	+ 0·47	+ 0·47	0	0	0
C.O.	+ 0·24	0	0	+ 0·24	0	0
Bal. M.	0	0	0	− 0·12	− 0·12	0
C.O.	0	0	− 0·06	0	0	− 0·06
Bal. M.	0	+ 0·03	+ 0·03	0	0	0
C.O.	+ 0·02	0	0	+ 0·02	0	0
Bal. M.	0	0	0	− 0·01	− 0·01	0
ΣM (tons-ft.)	+ 19·01 (M_{AB})	− 7·00 (M_{BA})	+ 7·00	+ 2·00	− 2·01 (M_{CD})	− 1·00 (M_{DC})

$$M_{AB} + M_{BA} + 2(M_{CD} + M_{DC}) = + 6{\cdot}00 \text{ tons-ft.}$$

$+ Pa = + 60{\cdot}0$ tons-ft., $\therefore$ $60{\cdot}0 - 6{\cdot}00 = + 54$ tons-ft., introduced by sway.

For sway all joints locked, $M'_{CD} = 2M'_{AB}$,

or let $M'_{AB} = + 10$ tons-ft. and $M'_{CD} = + 20$ tons-ft.

2nd Distribution of the Proportionate Sway Moments.

Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
K	1		1		1	
Bal.	0	½	½	½	½	0
C.O.F.	½		½		½	
Sway F.B.M.	+ 10·00	+ 10·00	0	0	+ 20·00	+ 20·00
Bal. M.	0	− 5·00	− 5·00	− 10·00	− 10·00	0
C.O.	− 2·50	0	− 5·00	− 2·50	0	− 5·00
Bal. M.	0	+ 2·50	+ 2·50	+ 1·25	+ 1·25	0
C.O.	+ 1·25	0	+ 0·63	+ 1·25	0	+ 0·63
Bal. M.	0	− 0·32	− 0·32	− 0·63	− 0·63	0
C.O.	− 0·16	0	− 0·32	− 0·16	0	− 0·32
Bal. M.	0	+ 0·16	+ 0·16	+ 0·08	+ 0·08	0
C.O.	+ 0·08	0	+ 0·04	+ 0·08	0	+ 0·04
Bal. M.	0	− 0·02	− 0·02	− 0·04	− 0·04	0
C.O.	− 0·07	0	− 0·02	− 0·01	0	− 0·02
Bal. M.	0	+ 0·01	+ 0·01	0	0	0
ΣM (tons-ft.)	+ 8·66 (M_{AB})	+ 7·33 (M_{BA})	− 7·34	− 10·67	+ 10·66 (M_{CD})	+ 15·32 (M_{DC})

$$\Sigma \text{ column moments} = M_{AB} + M_{BA} + 2(M_{CD} + M_{DC})$$
$$= +\ 67{\cdot}95 \text{ tons-ft.}$$

But Σ column moments are to equal + 54·00 tons-ft., ∴ the end column moments obtained must be reduced in the ratio of $\dfrac{+\ 54{\cdot}00}{+\ 67{\cdot}95} = 0{\cdot}795$

M_{AB} is then $= +\ 8{\cdot}66 \times 0{\cdot}795 = +\ 6{\cdot}88$ tons-ft.
M_{BA} is then $= +\ 7{\cdot}33 \times 0{\cdot}795 = +\ 5{\cdot}82$ tons-ft.
M_{CD} is then $= +\ 10{\cdot}66 \times 0{\cdot}795 = +\ 8{\cdot}48$ tons-ft.
M_{DC} is then $= +\ 15{\cdot}32 \times 0{\cdot}795 = +\ 12{\cdot}15$ tons-ft.

The final table is—

Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
No-sway Moment	+ 19·01	− 7·00			− 2·01	− 1·00
Sway Moment	+ 6·88	+ 5·82			+ 8·48	+ 12·15
Final Values (tons-ft)	+ 25·89	− 1·18	+ 1·18	− 6·47	+ 6·47	+ 11·15
	(M_{AB})	(M_{BA})	(M_{BC})	(M_{CB})	(M_{CD})	(M_{DC})

$$M_{AB} + M_{BA} + 2(M_{CD} + M_{DC}) = +\ 59{\cdot}95 \simeq 60{\cdot}00 \text{ tons-ft.}$$

Slope-Deflection Method.

$$\theta_{AB} = \theta_{DC} = 0; \;\; \phi_{AB} = \phi_A = \frac{\Delta}{10}; \;\; \phi_{DC} = \phi_D = \frac{\Delta}{5}$$

Sway is left to right, $\therefore\ 10\phi_A = 5\phi_D$ or $\phi_D = 2\phi_A$ and both are positive.

$$EK_{AB} : EK_{BC} : EK_{CD} = 1 : 1 : 1$$

$$M_{AB} = 2(\theta_B + 3\phi_A) + 15{\cdot}00$$

$$M_{BA} = 2(2\theta_B + 3\phi_A) - 15{\cdot}00$$

$$M_{BC} = 2(2\theta_B + \theta_C)$$

Joint equation: $M_{BA} + M_{BC} = 0$

$$\therefore\ 8\theta_B + 2\theta_C + 6\phi_A - 15{\cdot}00 = 0$$

$$M_{CB} = 2(\theta_B + 2\theta_C)$$

$$M_{CD} = 2(2\theta_C + 6\theta_A)$$

Joint equation: $M_{CB} + M_{CD} = 0$

$$\therefore\ 2\theta_B + 8\theta_C + 12\phi_A = 0$$

Bent equation: $M_{AB} + M_{BA} + 2(M_{CD} + M_{DC}) = +\ 60$

$$6\theta_B + 12\theta_C + 60\phi_A = +\ 60$$

or $\theta_B + 2\theta_C + 10\phi_A = +\ 10$

Joint *B* equation: $4\theta_B + \theta_C + 3\phi_A = +\ 7{\cdot}5$

Joint *C* equation: $\theta_B + 4\theta_C + 6\phi_A = 0$

$$\theta_B = +\ 1{\cdot}52; \;\; \theta_C = -\ 2{\cdot}36; \;\; \phi_A = 1{\cdot}32$$

Thus $M_{AB} = 25{\cdot}96$ tons-ft.; $M_{BA} = -\ 1{\cdot}00$ tons-ft.

$M_{CD} = +\ 6{\cdot}40$ tons-ft.; $M_{DC} = +\ 11{\cdot}12$ tons-ft.

Note on the Moment-Distribution Method Alternative Step (6) *for Shearing Forces on the Top and Bottom Cut Ends of the Members AB and DC.*

(*a*) Shears due to the end couples. The only transverse force is the force of 12 tons acting from left to right on *AB* and at 5 ft. from *A*.

$$\therefore\ \frac{Pa}{h} = \frac{12 \times 5}{10} = 6 \text{ tons}$$

Now $\dfrac{M_{AB} + M_{BA}}{l_{AB}} + \dfrac{M_{CD} + M_{DC}}{l_{CD}}$ has to equal 6 tons.

$$\text{But } \frac{+\,25{\cdot}89 + (-\,1{\cdot}18)}{10} + \frac{6{\cdot}47 + 11{\cdot}15}{5}$$

$$= 2{\cdot}478\ (\rightarrow) \text{ at } A + 3{\cdot}52\ (\rightarrow) \text{ at } C$$

$= 5{\cdot}998$ tons, nearly equal to 6·00 tons, and both shearing forces act from left to right which is the direction of Pa/h.

The forces of 2·478 tons and 3·52 tons are the shears due to the end couples.

(*b*) The total shears in the members at A, B, C and D.

At A, the shear must be equal to the simple beam shear of 6 tons acting from right to left plus the end couple shear of 2·478 tons also acting from right to left, to give a total of 8·478 tons acting from right to left. At B, the shear must be equal to the simple beam shear of 6 tons acting from right to left together with the end couple shear of 2·478 tons acting from left to right, giving a resultant of 3·522 tons acting from right to left.

The total end shears of 8·478 tons and 3·522 tons are equal to 12 tons and act from right to left, thereby balancing the external force of 12 tons which acts from left to right.

At C and D, as there is no external force on CD, the end shears are simply those due to the end couples, i.e. 3·52 tons acting left to right at D, and 3·52 tons acting left to right at C.

NOTE. The sum of the actual shears at B and C must equal zero; and it can be seen that this is so.

The sum of the actual shears at A and D are equal to 12 tons acting from right to left and thus balancing the external force of 12 tons. The bending moment, shearing force (column), and displacement diagrams are given in Fig. 235.

Illustrative Problem 49. (See Example 2, pp. 217 and 220, and Fig. 236.)

Referring to Fig. 236,

$$K_{AB} : K_{BC} : K_{CD}$$

$$= \tfrac{375}{8} : \tfrac{315}{10} : \tfrac{375}{12}$$

$$= 1{\cdot}5 : 1{\cdot}0 \text{ (nearly)} : 1{\cdot}0$$

$$E = \text{unity (say) for all members.}$$

This is a sway problem: Let the frame sway to the right so that B and C both move horizontally to the right by an amount Δ.

Let $\theta_{BA} = \theta_{BC} = \theta_B$ and $\theta_{CB} = \theta_{CD} = \theta_C$

" $\phi_{AB} = \phi$ and $\phi_{DC} = \phi_1$

" $\Delta = 8\phi = 12\phi_1, \quad \therefore \phi_1 = \frac{2}{3}\phi$

From the slope-deflection equations for $M_{AB} = 0$ and M_{BA} it can be shown that

$$M_{BA} = \tfrac{2}{3}(3\theta_B + 3\phi)$$

and similarly $M_{CD} = (3\theta_C + 2\phi)$

$M_{BC} = 2(2\theta_B + \theta_C) + 45$ and $M_{CB} = 2(2\theta_C + \theta_B) - 45$

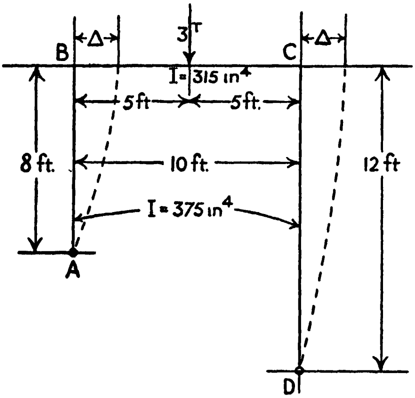


FIG. 236

Joint equations give $M_{BA} + M_{BC} = 0$ . . . (i)

$M_{CB} + M_{CD} = 0$ . . . (ii)

Shear equation: $\frac{M_{BA}}{8} + \frac{M_{CD}}{12} = 0$ . . . (iii)

or bent equation: $M_{BA} + \frac{2}{3} M_{CD} = 0$

Substituting in, and solving, equations (i), (ii) and (iii),

$$\theta_B = -9{\cdot}66; \quad \theta_C = +7{\cdot}83; \quad \phi = +5{\cdot}25 \text{ (nearly)}$$

$M_{BC} = -M_{BA} = +22{\cdot}08$ tons-in. $= +1{\cdot}84$ tons-ft.

$M_{CB} = -M_{CD} = -33{\cdot}12$ tons-in. $= -2{\cdot}76$ tons-ft.

H at $A = -1{\cdot}84/8 = -0{\cdot}23$ tons and acts from left to right.

H at $B = 2{\cdot}76/12 = 0{\cdot}23$ tons and acts from right to left.

Moment-Distribution Method: A Sway Problem Introducing Step (6). Effective K_{AB} and K_{CD}, as the columns are hinged at A and D,

are $\frac{3}{4} \times 1{\cdot}5$ and $\frac{3}{4} \times 1{\cdot}0$

$\therefore$ for this method, and no fixing couples at A and D,

$$K_{AB} : K_{BC} : K_{CD} = \frac{4 \cdot 5}{4} : 1 : \frac{3}{4} = 9 : 8 : 6$$

Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
K	9		8		6	
Bal.	0	$\frac{9}{17}$	$\frac{8}{17}$	$\frac{4}{7}$	$\frac{3}{7}$	0
C.O.F.	0		$\frac{1}{2}$		0	
Rel. K'	9		0		4	
K''	$\frac{1}{2} \times \frac{375}{8^3}$		0		$\frac{1}{2} \times \frac{375}{12^3}$	
Rel. K''	27		0		8	
F.B.M.	0	0	+ 45·00	− 45·00	0	0
Bal. M.	0	− 23·85	− 21·15	+ 24·75	+ 20·25	0
C.O.	0	0	+ 12·38	− 10·58	0	0
Bal. S.	0	+ 8·00	0	0	+ 3·33	0
Bal. M.	0	− 10·80	− 9·60	+ 4·14	+ 3·11	0
C.O.	0	0	+ 2·07	− 4·80	0	0
Bal. S.	0	+ 6·72	0	0	+ 2·98	0
Bal. M.	0	− 4·66	− 4·13	+ 1·04	+ 0·78	0
C.O.	0	0	+ 0·52	− 2·07	0	0
Bal. S.	0	+ 3·12	0	0	+ 1·44	0
Bal. M.	0	− 1·93	− 1·71	+ 0·36	+ 0·27	0
C.O.	0	0	+ 0·18	− 0·86	0	0
Bal. S.	0	+ 1·36	0	0	+ 0·60	0
Bal. M.	0	− 0·81	− 0·73	+ 0·15	+ 0·11	0
C.O.	0	0	+ 0·08	− 0·37	0	0
Bal. S.	0	+ 0·56	0	0	+ 0·24	0
Bal. M.	0	− 0·34	− 0·30	+ 0·07	+ 0·06	0
C.O.	0	0	+ 0·04	− 0·15	0	0
Bal. S.	0	+ 0·24	0	0	+ 0·10	0
Bal. M.	0	− 0·15	− 0·13	+ 0·03	+ 0·02	0
C.O.	0	0	+ 0·02	− 0·07	0	0
Bal. S.	0	+ 0·15	0	0	0	0
Bal. M.	0	− 0·08	− 0·07	+ 0·04	+ 0·03	0
ΣM (tons-in.)	0	− 22·47	+ 22·47	− 33·32	+ 33·32	0
(tons-ft.)	0	− 1·87 (M_{BA})	+ 1·87	− 2·77	+ 2·77 (M_{CD})	0

Bent equation: $M_{BA} + \frac{2}{3}(M_{CD}) = 0$

$\therefore - 1 \cdot 87 + 1 \cdot 85 = 0$ (very nearly).

After 1st balance,

$$M_{AB} + \tfrac{2}{3}(M_{CD}) = -23{\cdot}85 + 13{\cdot}5 = -10{\cdot}35$$

$$M'_{AB} + \tfrac{2}{3}(M'_{CD}) = +10{\cdot}35$$

$$M'_{AB} = 2{\cdot}25M'_{CD}$$

$$\therefore 2{\cdot}91M'_{CD} = +10{\cdot}35$$

$$M'_{CD} = +3{\cdot}54$$

$$M'_{AB} = +8{\cdot}00$$

and so onwards; or work from shear equations and shearing forces and couples.

Moment-Distribution Introducing the Alternative Step (6).

(*a*) Distribution restraining frame against sway.

Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
Rel. *K*		9		8		6
Bal.	-	$\frac{9}{17}$	$\frac{8}{17}$	$\frac{4}{7}$	$\frac{3}{7}$	-
C.O.F.		0		$\frac{1}{2}$		0
F.B.M.	0	0	+ 45·00	− 45·00	0	0
Bal. M.	0	− 23·85	− 21·15	+ 24·75	+ 20·25	0
C.O.	0	0	+ 12·38	− 10·58	0	0
Bal. M.	0	− 6·54	− 5·84	+ 6·04	+ 4·54	0
C.O.	0	0	+ 3·02	− 2·92	0	0
Bal. M.	0	− 1·60	− 1·42	+ 1·68	+ 1·24	0
C.O.	0	0	+ 0·84	− 0·72	0	0
Bal. M.	0	− 0·45	− 0·39	+ 0·40	+ 0·32	0
C.O.	0	0	− 0·20	− 0·20	0	0
Bal. M.	0	− 0·11	− 0·09	+ 0·12	+ 0·08	0
C.O.	0	0	+ 0·06	− 0·05	0	0
Bal. M.	0	− 0·03	− 0·03	+ 0·03	+ 0·02	0
ΣM	0	− 32·58			+ 26·45	0

(*b*) Keeping joints *B* and *C* fixed and *BC* unloaded. Let the frame sway; then the moments induced at the top of the columns *AB* and *CD* will be in the ratios of their *K'* values.

$$K'_{AB} = \frac{375}{8 \times 8}; \; K'_{CD} = \frac{375}{12 \times 12}$$

$$\therefore K'_{AB} : K'_{CD} = \tfrac{144}{64} : \tfrac{144}{144} = 2{\cdot}25 : 1 = 9 : 4$$

Therefore, assuming a value of K'_{SAB} as + 90 tons-in.; then K_{SCD} = + 40 tons-in. Plus sign, because sway assumed from left to right (positively).

These moments are now distributed in the usual way.

Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
K	9		8		6	
Bal.	0	$\frac{9}{17}$	$\frac{8}{17}$	$\frac{4}{7}$	$\frac{3}{7}$	0
C.O.F.	0		$\frac{1}{2}$		0	
Sway M.	0	+ 90·00	0	0	+ 40·00	0
Bal. M.	0	− 42·30	− 47·70	− 22·29	− 17·10	0
C.O.	0	0	− 11·45	− 23·85	0	0
Bal. M.	0	+ 6·74	+ 4·73	+ 13·64	+ 10·21	0
C.O.	0	0	+ 6·82	+ 2·82	0	0
Bal. M.	0	− 3·60	− 3·20	− 1·60	− 1·22	0
C.O.	0	0	− 0·80	− 1·60	0	0
Bal. M.	0	+ 0·45	+ 0·35	+ 0·92	+ 0·68	0
C.O.	0	0	+ 0·44	+ 0·18	0	0
Bal. M.	0	− 0·25	− 0·19	− 0·10	− 0·08	0
ΣM		+ 51·04			+ 32·49	

Shear equation: $$\frac{M_{BA}}{8} + \frac{M_{CD}}{12} = 0$$

or bent equation: $3M_{BA} + 2M_{CD} = 0$

In the first distribution (no sway)

$$3M_{BA} + 2M_{CD} = -\ 97{\cdot}74 + 52{\cdot}90 = -\ 44{\cdot}84 \text{ units.}$$

∴ to allow for sway, a moment of + 44·84 units must be impressed on the columns so that $3M_{BA} + 2M_{CD} = 0$.

Now from the Second Distribution of the Sway Moments (+ 90 and + 40 units), $3M_{BA} + 2M_{CD} = +\ 153{\cdot}12 - 64{\cdot}98 = +\ 218{\cdot}10$ units, i.e. + 218·10 units due to sway will produce at the tops of the columns, after distribution, a moment + 51·04 units at *A* and + 32·49 units at *C*.

∴ + 44·84 units will produce after distribution:

$$\frac{+\ 57{\cdot}04 \times 44{\cdot}84}{218{\cdot}10} \text{ units at } A = +\ 10{\cdot}48 \text{ units} \quad .$$

and $$\frac{+\ 32{\cdot}49 \times 44{\cdot}84}{218{\cdot}10} \text{ units at } C = +\ 6{\cdot}68 \text{ units}$$

Therefore, allowing for side sway,

$$M_{BA} = -\,32{\cdot}58 + 10{\cdot}48 = -\,22{\cdot}10 \text{ tons-in.}$$
$$= -\,1{\cdot}84 \text{ tons-ft.}$$
$$M_{CD} = +\,26{\cdot}45 + 6{\cdot}68 = +\,33{\cdot}13 \text{ tons-in.}$$
$$= +\,2{\cdot}76 \text{ tons-ft.}$$

Check: $3M_{BA} + 2M_{CD} = -\,66{\cdot}30 + 66{\cdot}26 = -\,0{\cdot}04$ lb.-in. (nearly equal to zero.)

$$\underset{\longrightarrow}{H_A} = -\,\underset{\longleftarrow}{H_D} = \frac{1{\cdot}84}{8} = 0{\cdot}23 \text{ tons } \left(\text{check: } \frac{2{\cdot}76}{12} = 0{\cdot}23.\right)$$

The displacement and moment diagrams are given in Fig. 237.

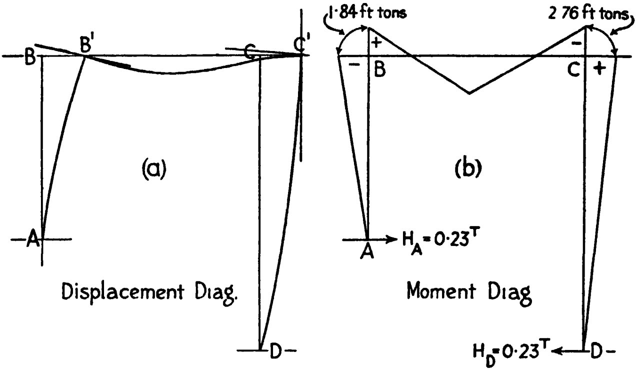


FIG. 237

Illustrative Problem 50

Problem (*c*), page 21 worked by the moment distribution method (see Fig. 238).

Let E for all members be equal to unity.

„ $I_1 = I_2 = 100$ ft. units.

„ $h = l = 10$ ft.

„ $P = 1000$ lb., then $Ph = 10{,}000$ lb.-ft.

Bent equation: $M_{BA} + M_{CD} = -\,Ph = -\,10{,}000$. (i)

In this problem, due to sway, as AB and DC are identical, then

$$M_{BA} = M_{CD}$$

$$\therefore\ 2M_{BA} = -\,10{,}000 \text{ units or } M_{BA} = -\,5000 \text{ units.}$$

When distributing these moments, let 1000 units be the moment unit and we shall then distribute the multiples.

There are no end-fixing couples due to transverse loading.

Bal. M (step 4) indicates the balancing of the moments at a joint.

Note (*step* 6), 1*st Operation.* From the bent equation it was found that $M_{BA} = M_{CD} = -5 \times 1000$ units. So we place -5×1000 units at the upper column ends. This is indicated by Bal. S.

Now the joints *B* and *C* are out of balance. It is therefore necessary to distribute at *B* the unbalanced couple of $+\frac{3}{7} \times 5 \times 1000$ units to *BA* when joint *B* is now rotated, and $+\frac{4}{7} \times 5 \times 1000$ units goes to *BC*.

Then joint *B* is balanced: for $-5000 = +2100 + 2900$.

Similarly for joint C.

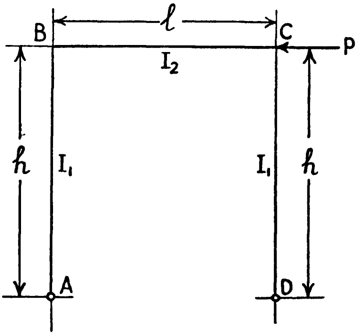


FIG. 238

When *B* is rotated and *C* kept fixed, then $\frac{1}{2}$ of the new moment at *BC* must be transferred to end *C* of *BC*, i.e. + 1450 units are carried-over to *C*. Similarly the effect at *B* for the rotation of *C*. There is no carry-over of *BA* from *B* to *A* as *A* is hinged. Due to balancing of the moments at *B* and *C*, we have introduced new couples equal to $+2{\cdot}1 \times 1000$ units at the tops of the columns *BA* and *CD*. As we have already balanced for sway, we must multiply these additions by equal and opposite corresponding sway moments or couples. Therefore we must, in this particular problem, add $-2{\cdot}1 \times 1000$ units to *BA* and *BC* (2nd step 6) and (2nd Bal. S).

The joints *B* and *C* are again thrown out of balance and these new additions of $-2{\cdot}10 \times 1000$ units together with $+1{\cdot}45 \times 1000$ units at *B* in *BC* due to carry-over must be re-balanced.

$$(-2{\cdot}10 + 1{\cdot}45)1000 = -0{\cdot}65(1000)$$

$-0{\cdot}65$ has been distributed as $+0{\cdot}28$ to *BA* and $+0{\cdot}37$ to *BC*. A line has been drawn under these figures to show that the joint has again been balanced. All the stages are repeated and the work completed after balancing the joints for the fourth time.

In this method of sway analysis, one should always complete the work after a balance of the joint moments (Bal. M) or after a (Bal. S).

Member	AB	BA	BC	CB	CD	DC
K	$\frac{3}{4}\times\frac{100}{10}$		$\frac{100}{10}$		$\frac{3}{4}\times\frac{100}{10}$	
Rel. K	3		4		3	
Bal.	0	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{4}{7}$	$\frac{3}{7}$	0
Equivalent K'	$\frac{1}{2}\times\frac{100}{100}$		0		$\frac{1}{2}\times\frac{100}{100}$	
Rel. K'	1		0		1	
C.O.F.	0		$\frac{1}{2}$		0	
F.B.M.	0	0	0	0	0	0
Bal. S.	0	− 5·00	0	0	− 5·00	0
Bal. M.	0	+ 2·10	+ 2·90	+ 2·90	+ 2·10	0
C.O.	0	0	+ 1·45	+ 1·45	0	0
Bal. S.	0	− 2·10	0	0	− 2·10	0
Bal. M.	0	+ 0·28	+ 0·37	+ 0·37	+ 0·28	0
C.O.	0	0	+ 0·19	+ 0·19	0	0
Bal. S.	0	− 0·28	0	0	− 0·28	0
Bal. M.	0	+ 0·04	+ 0·05	+ 0·05	+ 0·04	0
C.O.	0	0	+ 0·03	+ 0·03	0	0
Bal. S.	0	− 0·04	0	0	− 0·04	0
Bal. M.	0	0	+ 0·01	+ 0·01	0	0
ΣM	0	− 5·00 (M_{DA})	+ 5·00	+ 5·00	− 5·00 (M_{CD})	0

The couples at the joints are all therefore in magnitude

$$= 5000 \text{ units} = \frac{10{,}000}{2} = 5000,$$

which confirms the values for the solutions given by the slope-deflection method that $M_{AB} = P\frac{h}{2}$.

Illustrative Problem 51.

Rectangular Bent ABC (Fig. 239). It is direction-fixed at A and C and jointed at B. There are no transverse loads on the members, but the bent is subjected to a change of temperature.

Let E for AB, $BC = 30 \times 10^6$ lb. per sq. in.

For simplicity, let $I_{AB} = I_{BC} = 200$ in.^{4}, and let

$$l_{AB} = l_{BC} = 100 \text{ in.}$$

$$K_{AB} = K_{BC} = 200/100 = 2.$$

Let the coefficient of thermal expansion be 0·000006 per degree Fahrenheit, and let there be a rise in temperature from 58° F. to 88° F.

AB and *BC* will both expand by an amount

$$100 \times 0{\cdot}000006 \times 30 = 0{\cdot}0018 \text{ in.}$$

Due to this expansion *AB* will sway to the left and *BC* upwards, i.e. *AB* in a counter-clockwise movement, and *BC* in a clockwise movement.

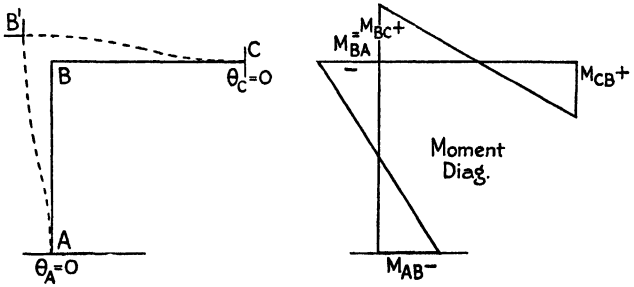


FIG. 239

$$\therefore \phi_{AB} = \phi_A = -\frac{0{\cdot}0018}{100} \text{ and will be negative.}$$

$$\therefore \phi_{BC} = \phi_B = +\frac{0{\cdot}0018}{100} \text{ and will be positive.}$$

The small increases in the lengths of the members are neglected.

Let $\theta_{BA} = \theta_{BC} = \theta_B$

Slope-deflection Solution.

$$M_{AB} = 2EK(\theta_B + 3\phi_A)$$
$$M_{BA} = 2EK(2\theta_B + 3\phi_A)$$
$$M_{BC} = 2EK(2\theta_B + 3\phi_B)$$
$$M_{CB} = 2EK(\theta_B + 3\phi_B)$$

Joint equation B: $M_{BA} + M_{BC} = 0$

$$\therefore \theta_B = 0$$

$$\therefore M_{AB} = -2 \times 30 \times 10^6 \times 2 \times 3 \times \frac{0{\cdot}0018}{100} = -6480 \text{ lb.-in.}$$

$$M_{BA} = M_{AB} = -M_{BC} = -6480 \text{ lb.-in.}$$

$$M_{CB} = -M_{AB} = +6480 \text{ lb.-in.}$$

Moment-Distribution Method. Keeping the ends direction-fixed, the induced moments due to the displacement Δ are,

$$\left.\begin{aligned} M'_{AB} = M'_{BA} &= -\ 6EI\Delta/l^2 = -\ 6480 \text{ lb.-in.} \\ M'_{BC} = M'_{CB} &= +\ 6EI\Delta/l^2 = +\ 6480 \text{ lb.-in.} \end{aligned}\right\} = \text{sway F.B.M.}$$

Member	*AB*	*BA*	*BC*	*CB*
Sway F.B.M.	− 6480	− 6480	+ 6480	+ 6480
Bal. M.	0	0	0	0
ΣM	− 6480	− 6480	+ 6480	+ 6480

NOTE. In these examples, the frame members sway only due to the change in temperature: once this has occurred there is no further sway, i.e. in the moment-distribution method, the joint will simply rotate and the frame is fixed against translation.

EXERCISES. (*a*) For the frame in illustrative problem 51,

let $l_{AB} = l_{BC} = 100$ in.

" $I_{AB} = 200 \text{ in.}^4$; $I_{BC} = 100 \text{ in.}^4$

" $E_{AB} = E_{BC} = 30 \times 10^6$ lb. per sq. in.

Then $K_{AB} = 2$; $K_{BC} = 1$.

$$\phi_{AB} = \phi_A = -\frac{0{\cdot}0018}{100} \text{ and } \phi_{BC} = \phi_B = +\frac{0{\cdot}0018}{100}$$

$$= +\ 18 \times 10^{-6}$$

Slope-deflection equations:

$$M_{AB} = 2 \times E \times 2\left(\theta_B - 3 \times \frac{0{\cdot}0018}{100}\right)$$

$$M_{BA} = 2 \times \text{E} \times 2\left(2\theta_B - 3 \times \frac{0{\cdot}0018}{100}\right)$$

$$M_{BC} = 2 \times E \times 1\left(2\theta_B + 3 \times \frac{0{\cdot}0018}{100}\right)$$

$$M_{CB} = 2 \times E \times 1\left(\theta_B + 3 \times \frac{0{\cdot}0018}{100}\right)$$

$$\text{Joint equation } B\colon\ M_{\mathrm{BA}} + M_{\mathrm{BC}} = 0\ \therefore\ 6\theta_{\mathrm{B}} = +\,3 \times \frac{0{\cdot}0018}{100}$$

$$\theta_{\mathrm{B}} = +\,0{\cdot}000009$$

$$= +\,9 \times 10^{-6}$$

$$M_{\mathrm{AB}} = 2 \times 30 \times 10^{6} \times 2\,(9 \times 10^{-6} - 54{\cdot}0 \times 10^{-6}) = -\,5400\ \text{lb.-in.}$$

$$M_{\mathrm{BA}} = 2 \times 30 \times 10^{6} \times 2\,(180 \times 10^{-6} - 54{\cdot}0 \times 10^{-6}) = -\,4320\ \text{lb.-in.}$$

$$M_{\mathrm{BC}} = 2 \times 30 \times 10^{6} \times 1\,(18{\cdot}0 \times 10^{-6} + 54{\cdot}0 \times 10^{-6}) = +\,4320\ \text{lb.-in.}$$

$$M_{\mathrm{CB}} = 2 \times 30 \times 10^{6} \times 1\,(9 \times 10^{-6} + 54{\cdot}0 \times 10^{-6}) = +\,3780\ \text{lb.-in.}$$

Moment-Distribution Method.

Member	*AB*	*BA*	*BC*	*CB*
K	2		1	
Bal.	0	$\frac{2}{3}$	$\frac{1}{3}$	0
C.O.F.	$\frac{1}{2}$		$\frac{1}{2}$	
Sway F.B.M.	− 6480	− 6480	+ 3240	+ 3240
Bal. M.	0	+ 2160	+ 1080	0
C.O.	+ 1080	0	0	+ 540
Bal. M.	0	0	0	0
ΣM	− 5400	− 4320	+ 4320	+ 3780

These values for the moments agree with those obtained by the slope-deflection method.

The sway F.B.M.

$$M'_{\mathrm{BA}} = M'_{\mathrm{AB}} = -\,6E\frac{I}{l}\cdot\frac{\Delta}{l} = -\,6480 \text{ and is negative.}$$

$$M'_{\mathrm{BC}} = M'_{\mathrm{CB}} = \ 6E\frac{I_1}{l}\cdot\frac{\Delta}{l} = +\,3240 \text{ and is positive.}$$

EXERCISE (*b*). Sketch the bending moment and distortion diagrams for the rectangular bent *ABC*, if it is hinged at *A* and *C*.

$l_{\mathrm{AB}} = 100$ in.; $I_{\mathrm{AB}} = 200$ in.^{4}:
$l_{\mathrm{BC}} = 60$ in.; $I_{BC} = 180$ in.^{4}:
$E_{\mathrm{AB}} = E_{\mathrm{BC}} = 30 \times 10^6$ lb. per sq. in.: and
the rise in temperature is from 58° F. to 88° F.

$\theta_{\mathrm{B}} = 0{\cdot}0000137$; $\phi_{\mathrm{AB}} = -\,0{\cdot}0000108$; $\phi_{\mathrm{BC}} = +\,0{\cdot}00003$
$M_{\mathrm{BA}} = -\,4425$ lb.-in.; $M_{\mathrm{BC}} = +\,4425$ lb.-in.

Illustrative Problem 52.

Rectangular bent ABC (Fig. 240), direction fixed at A and C.

$$\text{Let } l_{AB} = l_{BC} = 100 \text{ in.}$$
$$,, \; I_{AB} = I_{BC} = 200 \text{ in.}^4$$
$$,, \; E_{AB} = E_{BC} = 30 \times 10^6 \text{ lb. per sq. in.}$$

Let A be displaced vertically to A' by an amount of 0·05 in.

Then $\phi_{BC} = \phi = 0{\cdot}05/100$ radians, and is negative.

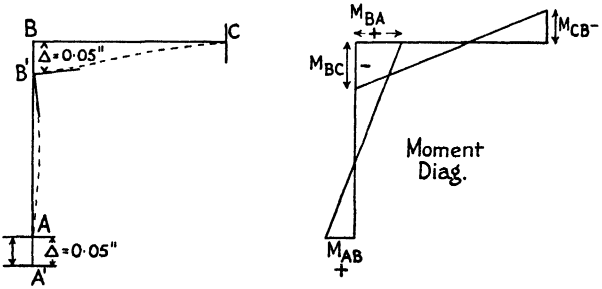


FIG. 240

There will be no sway of AB, and neglecting any shortening of BC,

$$\theta_{AB} = \theta_{CB} = 0; \; \theta_{BA} = \theta_{BC} = \theta_B$$
$$K_{AB} = K_{BC} = \tfrac{200}{100} = 2$$
$$M_{AB} = 2 \times E \times 2(\theta_B); \quad M_{CB} = 2 \times E \times 2(\theta_B \times 3\phi);$$
$$M_{BA} = 2 \times E \times 2(2\theta_B); \quad M_{BC} = 2 \times E \times 2(2\theta_B + 3\phi)$$

Joint equation B: $M_{BA} + M_{BC} = 0$

$$\therefore 8\theta_B + 8\theta_B + 12\phi = 0$$
$$\therefore \theta_B = -0{\cdot}75\phi \quad = -\frac{3}{4} \times \left(\frac{-0{\cdot}05}{100}\right)$$
$$= +0{\cdot}15/400$$
$$\phi = 500 \times 10^{-6} \quad = 375 \times 10^{-6}$$

$$M_{AB} = 4 \times 30 \times 10^6 \times 375 \times 10^{-6} = +\;45{,}000 \text{ lb.-in.}$$
$$M_{BA} = 8 \times 30 \times 10^6 \times 375 \times 10^{-6} = +\;90{,}000 \text{ lb.-in.}$$
$$M_{BC} = 4 \times 30 \times 10^6 \times 10^{-6}(750 - 1500) = -\;90{,}000 \text{ lb.-in.}$$
$$M_{CB} = 4 \times 30 \times 10^6 \times 10^{-6}(375 - 1500) = -\;135{,}000 \text{ lb.-in.}$$

Moment-Distribution Solution. Sway *BC*, but keep *B* fixed against rotation. Then the sway-fixing couples

$$M_{FBC} = M_{FCB} = -\,6EI\Delta/l^2;\quad \Delta/l^2 = 0{\cdot}05/1000 = 5 \times 10^{-6}$$

$$\therefore\ 6EI\Delta/l^2 = -\,6 \times 30 \times 10^6 \times 200 \times 5 \times 10^{-6}$$

$$= -\,180{,}000 \text{ lb.-in.}$$

There will be no further sway, therefore the table is as follows.

Member	*AB*	*BA*	*BC*	*BC*
K	2		2	
Bal.	0	½	½	0
C.O.F.	½		½	
Sway F.B.M.	0	0	$-1{\cdot}8 \times 10^5$	$-1{\cdot}8 \times 10^5$
Bal. M.	0	$+0{\cdot}9 \times 10^5$	$+0{\cdot}9 \times 10^5$	0
C.O.	$+0{\cdot}45 \times 10^5$	0	0	$+0{\cdot}45 \times 10^5$
Bal. M.	0	0	0	0
ΣM (lb.-in.)	+ 45,000 (M_{AB})	+ 90,000 (M_{BA})	− 90,000 (M_{BC})	− 135,000 (M_{CB})

If the support *A* was displaced horizontally to the left, say, by the same amount $\Delta = 0{\cdot}005$ in., then it can easily be shown that for this case

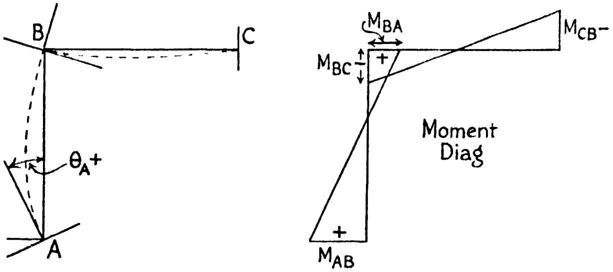


FIG. 241

$$M_{AB} = +\ 135{,}000 \text{ lb.-in.}$$
$$M_{BA} = +\ \ 90{,}000 \text{ lb.-in.}$$
$$M_{BA} = -\ \ 90{,}000 \text{ lb.-in.}$$
$$M_{CB} = -\ \ 45{,}000 \text{ lb.-in.}$$

Illustrative Problem 53

Rectangular Bent *ABC* (see Fig. 241). The dimensions and properties of the members are as given in the previous problem.

Let the end *A* of *AB* be rotated through an angle of $+\,\theta_{AB} = +\,\theta_A = +\,0{\cdot}05/100$ radians $= 500 \times 10^{-6}$ radians.

There will only occur a rotation of joint B.

$K_{AB} = K_{BC} = K = 2$; $\theta_{BA} = \theta_{BC} = \theta_B$; $E = 30 \times 10^6$ lb. per sq. in.

$$M_{AB} = 2EK(2\theta_A + \theta_B);\quad M_{CB} = 2EK(\theta_B);$$
$$M_{BA} = 2EK(\theta_A + 2\theta_B);\quad M_{BC} = 2EK(2\theta_B)$$

$$M_{BA} + M_{BC} = 0 \therefore \theta_A + 4\theta_B = 0$$

$$\therefore \theta_B = -\frac{\theta_A}{4} = -125 \times 10^{-6} \text{ radians}$$

$$\begin{aligned}
M_{AB} &= 4 \times 30 \times 10^6 \times 10^{-6}(1000 - 125) &&= +\ 105{,}000 \text{ lb.-in.}\\
M_{BA} &= 4 \times 30 \times 10^6 \times 10^{-6}(500 - 250) &&= +\ 30{,}000 \text{ lb.-in.}\\
M_{BC} &= 4 \times 30 \times 10^6 \times 10^{-6}(-250) &&= -\ 30{,}000 \text{ lb.-in.}\\
M_{CB} &= 4 \times 30 \times 10^6 \times 10^{-6}(-125) &&= -\ 15{,}000 \text{ lb.-in.}
\end{aligned}$$

The Solution by the Moment-Distribution Method. Keeping B fixed against rotation initially, then F.B.M.s in AB will be:

$$M_{FAB} = +\ 4EK\theta_A = +\ 4 \times 30 \times 10^6 \times 2 \times 500 \times 10^{-6} = +\ 120{,}000 \text{ lb.-in.}$$
$$M_{FBA} = +\ 2EK\theta_A = +\ 60{,}000 \text{ lb.-in.}$$

Member	*AB*		*BA*	*BC*		*CB*
K		2			2	
Bal.	0		½	½		0
C.O.F.		½			½	
F.B.M.	$+ 1{\cdot}2 \times 10^5$		$+ 0{\cdot}6 \times 10^5$	0		0
Bal. M.	0		$- 0{\cdot}3 \times 10^5$	$- 0{\cdot}3 \times 10^5$		0
C.O.	$- 0{\cdot}15 \times 10^5$		0	0		$- 0{\cdot}15 \times 10^5$
Bal. M.	0		0	0		0
ΣM (lb.-in.)	+ 105,000		+ 30,000	− 30,000		− 15,000

If A was displaced vertically downwards and horizontally by the amounts given in the problems, and rotated through the given angle θ_A, all simultaneously, then the moments at the supports and the joints would be the sum of those found by considering the deformations singly.

For exercises the student is recommended to vary the lengths and I's, and the types of supports, and deform similarly as given in the worked problems.

Illustrative Problem 54.

Rectangular bent ABC (Fig. 242), with hinges at A and C and rigidly jointed at B. The lengths of the members, their E's and I's are the same as in the two preceding problems.

Now rotate the end A of the member AB through an angle

$$\theta_A = 0{\cdot}05/100 \text{ radians.}$$
$$= 500 \times 10^{-6} \quad ,,$$

This, in effect, makes the hinged end A, a fixed end, because in order to induce this rotation of $+\theta_A$, an effective couple must be applied to the member $= M_{AB}$. The hinge C will, of course, remain a hinge and there will be no moment at C.

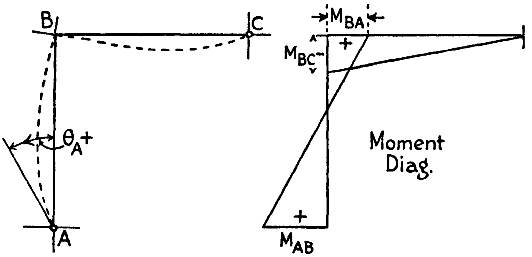


FIG. 242

$$\therefore\ M_{AB} = 2EK(2\theta_A + \theta_B)$$
$$M_{BA} = 2EK(\theta_A + 2\theta_B)$$
$$M_{BC} = 2EK(2\theta_B + \theta_C)$$
$$M_{CB} = 2EK(\theta_B + 2\theta_C) = 0 \ \therefore\ \theta_C = -\frac{\theta_B}{2}$$
$$M_{BA} + M_{BC} = 0 \ \therefore\ \theta_A + 4\theta_B + \theta_C = 0$$
$$\therefore\ \tfrac{7}{2}\theta_B = -\theta_A$$
$$\therefore\ \theta_B = -\tfrac{2}{7}\theta_A$$

$$\therefore\ M_{AB} = 4 \times 30 \times 10^6 \times 10^{-6}(1000 - 1000/7)$$
$$= +\ 102{,}857 \text{ lb.-in.}$$

$$M_{BA} = 4 \times 30 \times 10^6 \times 10^{-6}(500 - 2000/7)$$
$$= +\ 25{,}714 \text{ lb.-in.}$$

$$M_{BC} = 4 \times 30 \times 10^6 \times 10^{-6}(-\ 1500/7)$$
$$= -\ 25{,}714 \text{ lb.-in.}$$

$$M_{CB} = 0$$

Moment-Distribution Solution: No Sway.

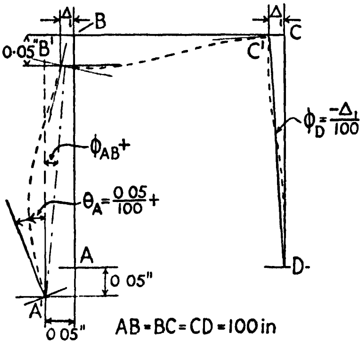
Member	*AB*	*BA*	*BC*	*CB*
K	2		1·5 (pinned at *C*)	
Bal.	0	$\frac{4}{7}$	$\frac{3}{7}$	0
C.O.F.	$\frac{1}{2}$		0	
F.B.M. (lb.-in.)	$+ 1·2 \times 10^5$	$+ 0·6 \times 10^5$	0	0
Bal. M.	0	$- 0·343 \times 10^5$	$- 0·257 \times 10^5$	0
C.O.	$- 0·172 \times 10^5$	0	0	0
Bal. M.	0	0	0	0
ΣM (lb.-in.)	+ 102,800	+ 25,700	− 25,700	0

$M_{AB} = + 102{,}800$ lb.-in.: $M_{BA} = 25{,}700$ lb.-in.:

$M_{BC} = - 25{,}700$ lb.-in.

Illustrative Problem 55.

Consider a simple portal *ABCD* (Fig. 243), with direction-fixed bases *A* and *D*, and rigidly jointed at *B* and *C*.



Δ
B
0·05"
B'
Δ
C
C'
$\phi_D = \frac{-\Delta_1}{100}$
$\phi_{AB}+$
$\theta_A = \frac{0\ 05}{100} +$
A
0 05"
D
A'
0 05"
AB=BC=CD=100in

FIG. 243

Let $l_{AB} = l_{BC} = l_{CD} = 100$ in.

„ $I_{AB} = I_{BC} = I_{CD} = 200$ in.^{4}

„ E for all members $= 30 \times 10^6$ lb. per sq. in.

K's for all members $= 2$.

Let the fixed base *A* be displaced vertically downwards and to the left by amounts $\Delta = 0·05$ in., and let the base *A* be rotated positively through an angle $\theta_{AB} = \theta_A = + 0·05/100 = 500 \times 10^{-6}$ radians.

Due to these deformations, C will move to the left to C' through an amount Δ, and B to B' to the left through a horizontal movement of Δ_1. Vertical displacement of B will be 0·05 in. = the vertical displacement of A.

Horizontally B' will be relative to A by an amount of $(0{\cdot}05 - \Delta_1)$ in., assuming $\Delta_1 < 0{\cdot}05$ in., the horizontal displacement of A.

The sway angle of DC will be $\phi_{DC} = \phi_D = -\Delta_1/100 = -10{,}000\Delta \times 10^{-6}$ radians. The sway angle of AB will be $+(0{\cdot}05 - \Delta_1)/100$, and will be positive, i.e. $A'B'$ with respect to the vertical through A'.

Let it be $\phi_A = +(500 - 10{,}000\Delta_1) \times 10^{-6}$.

BC rotates to position $B'C'$ through a sway angle

$$\phi_{C'B'} = \phi_C = -0{\cdot}05/100$$
$$= -(500 \times 10^{-6}) \text{ and is negative.}$$

$\theta_D = 0$; $\theta_A = +500 \times 10^{-6}$ radians.

$$M_{AB} = 2EK(2\theta_A + \theta_B + 3\phi_A)$$
$$M_{BA} = 2EK(\theta_A + 2\theta_B + 3\phi_A)$$
$$M_{BC} = 2EK(2\theta_B + \theta_C + 3\phi_C)$$
$$M_{CB} = 2EK(\theta_B + 2\theta_C + 3\phi_C)$$
$$M_{CD} = 2EK(2\theta_C + 3\phi_D)$$
$$M_{DC} = 2EK(\theta_C + 3\phi_D)$$

Joint equation B: $M_{BA} + M_{BC} = 0$

Joint equation C: $M_{BC} + M_{CD} = 0$

Bent equation: $M_{AB} + M_{BA} + M_{CD} + M_{DC} = 0$ (no external horizontal forces acting).

Solving the above equations,

$$\theta_A = +500 \times 10^{-6};\ \theta_B = +130{\cdot}0 \times 10^{-6};$$
$$\theta_C = +798{\cdot}0 \times 10^{-6};\ \Delta_1 = +10{\cdot}0607 \times 10^{-6},$$

and then the values of the moments are found to be—

$$M_{AB} = +120 \times 809 = +\ 97{,}080 \text{ lb.-in.}$$
$$M_{BA} = +120 \times 439 = +\ 52{,}680 \text{ lb.-in.}$$
$$M_{BC} = -120 \times 448 = -\ 53{,}760 \text{ lb.-in.}$$
$$M_{CB} = +120 \times 226 = +\ 27{,}120 \text{ lb.-in.}$$
$$M_{CD} = -120 \times 225 = -\ 27{,}000 \text{ lb.-in.}$$
$$M_{DC} = -120 \times (-1023) = +\ 122{,}760 \text{ lb.-in.}$$

The Moment-Distribution Solution. Imagine the portal *ABCD* as in Fig. 244 (*a*) and (*d*).

A is displaced vertically only by $\Delta = 0{\cdot}05$, Fig. 244 (*b*), and the joints *B* and *C* fixed against rotation. As in Fig. 244 (*c*), move *A* to *A″* through a horizontal distance, left to right, $= 0{\cdot}05$ in. Then *B* moves to *B″* an amount $0{\cdot}05/2$, and *C* to *C″* an amount $0{\cdot}05/2$. There is no rotation of the joints *B* and *C*.

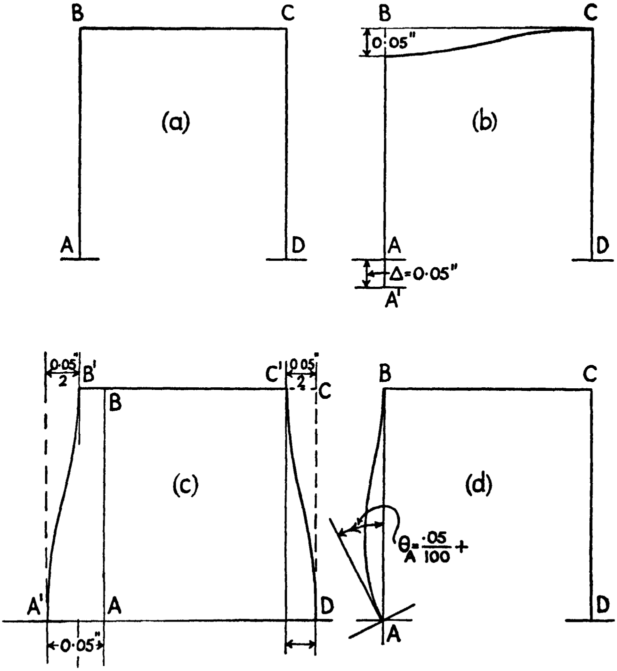


FIG. 244

Next, as in Fig. 244 (*d*), rotate *A* positively through an angle $0{\cdot}05/100$ radians, keeping *B* and *C* fixed and with no translation.

Case (*b*), Fig. 244 (*b*);

$$M_{FBC} = M_{FCB} = -\,6EI\Delta/l^2 = -\,180{,}000 \text{ lb.-in.} \ (\Delta = 0{\cdot}05)$$

Case (*c*), Fig. 244 (*c*);

$$M_{FAB} = M_{FBA} = +\,90{,}000 \text{ lb.-in.} \ (\Delta = 0{\cdot}05/2)$$
$$M_{FCD} = M_{FDC} = -\,90{,}000 \text{ lb.-in.}$$

Case (*d*), Fig. 244 (*d*);

$M_{FAB} = +$ 120,000 lb.-in. (see problem 53).
$M_{FBA} = +$ 60,000 lb.-in. (see problem 53).

Case (*c*) and case (*d*) together $\begin{cases} \text{Total } M_{FAB} = +\ 210{,}000 \text{ lb.-in.} \\ \quad\text{,,}\quad M_{FBA} = +\ 150{,}000 \text{ lb.-in.} \end{cases}$

Bent equation: $M_{AB} + M_{BA} + M_{CD} + M_{DC} = 0.$

∴ For balancing sway couples,

$4M'_{AB} = \pm X$; where $\pm X$ is the sum of the couples on the ends of the columns after a balancing and a carry-over.

M'_{AB} will be the distributed Bal. S. couple for each end A, B, C and D, as $K'_{AB} = K'_{CD}$.

Moments in Multiples of 10^5 lb.-in.

Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
K	2		2		2	
Bal.	0	½	½	½	½	0
C.O.F.	½		½		½	
K′	1		0		1	
F.B.M.	+ 2·10	+ 1·50	− 1·80	− 1·80	− 0·90	− 0·90
Bal.M.	0	+ 0·15	+ 0·15	+ 1·35	+ 1·35	0
C.O.	+ 0·08	0	+ 0·68	+ 0·08	0	+ 0·68
Bal. S.	− 1·02	− 1·02	0	0	− 1·02	− 1·02
Bal. M.	0	+ 0·17	+ 0·17	+ 0·47	+ 0·47	0
C.O.	+ 0·09	0	+ 0·24	+ 0·09	0	+ 0·24
Bal. S.	− 0·24	− 0·24	0	0	− 0·24	− 0·24
Bal. M.	0	0	0	+ 0·08	+ 0·08	0
C.O.	0	0	+ 0·04	0	0	+ 0·04
Bal. S.	− 0·03	− 0·03	0	0	− 0·03	− 0·03
ΣM (lb.-in.)	+98,000 (M_{AB})	+53,000 (M_{BA})	−52,000	+27,000	−29,000 (M_{CD})	+123,000 (M_{DC})

EXERCISE.

Consider the same portal *ABCD* as in the previous problem 55. Find the moments at *A*, *B*, *C*, and *D* due to a uniform rise in temperature of 30° F. Coefficient of thermal expansion = 0·000006 per °F.

All the members have the same properties and *E* as before. See Fig. 245 (p. 458) for hints.

Ans: $M_{AB} = - M_{DC} = -$ 21,600 lb.-in.: $M_{BA} = - M_{CD} = -$ 10,800 lb.-in.

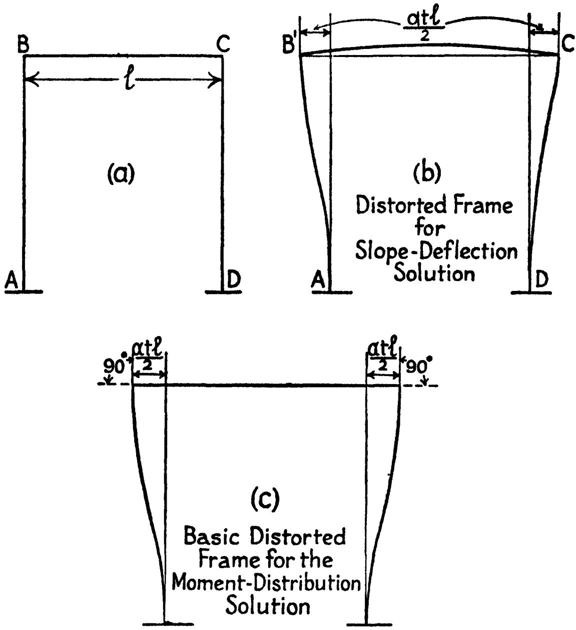

B
C
l
(a)
A
D
B'
C'
$\frac{\alpha t l}{2}$
(b)
Distorted Frame for Slope-Deflection Solution
A
D
90°
$\frac{\alpha t l}{2}$
$\frac{\alpha t l}{2}$
90°
(c)
Basic Distorted Frame for the Moment-Distribution Solution

FIG. 245

EXERCISE.

Referring to the frame in Fig. 233, Illustrative Problem 48, determine the moments at A, B, C, and D due to a uniform rise in temperature of 30° F. of the whole frame. $\alpha = 0{\cdot}000006$ per °F.: $E = 30 \times 10^6$ lb. per sq. in. This is a sway problem.

AB will extend vertically upwards by twice the amount of *DC*: therefore there will be a positive rotation of *BC*. *BC* will also expand and therefore subject the whole frame to bending.

Using the slope-deflection method, let *B* move to the left by an amount d so that there is a negative rotation of *AB*: then as d will be less than the thermal expansion Δ of *BC*, *C* will move to the right by an amount $\Delta - d$, and there will be a positive rotation of *CD*. There will be two joint equations and one bent (or shear) equation to be developed and solved for the joint rotations and d. The end moments can then be calculated from the appropriate slope-deflection equation.

Solution by the Moment-distribution Method. Imagine that *AB* is restrained against rotation, so that *BC* will extend to the right by the thermal expansion Δ. *BC* will rotate positively.

BC will sway positively as indicated in the previous paragraph. Imagine joints *B* and *C* have been restrained against rotation, whilst the members have extended. Calculate the end-fixing couples $M_{FBC} = M_{FCB}$, and $M_{FCD} = M_{FDC}$, using the formula $M_F = \frac{6EI\Delta}{l^2}$, giving *E* and $\frac{I}{l^2}$ their correct values. As indicated in the examples, balance these moments at their respective joints, carry over Bal. S, and continue as in the illustrated examples. The moments are found to be, approximately,

$M_{AB} = -\ 28{,}000$ lb.-in.; $M_{BA} = -\ 24{,}000$ lb.-in.;
$M_{CD} = +\ \ 4{,}000$ lb.-in.; $M_{DC} = +\ 22{,}000$ lb.-in.

These values satisfy the shear equation,

$$\frac{M_{AB} + M_{BA}}{120} + \frac{M_{CD} + M_{DC}}{60} = 0$$

This method of analysis can be applied also to investigate the effect that the elongations of the members of a frame due to axial forces may have on the bending of a frame. The analyses used have been based on the assumption that axial deformations of structural members can be neglected. If we wish to take these into account, we treat the previous results as an approximation, and apply them to calculate the axial forces and axial deformations in all members of the structure. The cross-sectional areas of the members must be known. The changes in length obtained in this way can be used in exactly the same manner as we have just used the thermal changes in length, and we can calculate the additional moments at the joints due to axial deformations. These moments are usually negligible in frame structures.

Illustrative Problem 56. (See Figs. 246 and 247.)

The analysis of a two-storied, single bay frame *ABCDEF*, supported at the fixed bases *A* and *F*. Horizontal loads of 1000 lb., acting from left to right are applied at the joints *B* and *C*.

Let the lengths of all the members be 100 in., and their *I*s equal to 100 in.^{4}

All *K*s = 1; Let *E* for all members = 1.

$K_{AB} : K_{EF} : K_{BC} : K_{ED} = 1 : 1 : 1 : 1.$

The frame will swing or sway in a clockwise direction.

Then $\phi_{AB} = \phi_{FE} = +\ \phi_A$
$\phi_{BC} = \phi_{ED} = +\ \phi_B$

There will be no sway of the horizontal members.

$$\therefore \theta_A = \theta_F = 0$$

$$\theta_{BA} = \theta_{BE} = \theta_{BC} = \theta_B$$

$$\theta_{CB} = \theta_{CD} = \theta_C$$

As all the members have the same lengths and properties, then, $\theta_E = \theta_B$; and $\theta_D = \theta_C$; and $M_{FE} = M_{AB}$; $M_{BA} = M_{EF}$; $M_{BE} = M_{EB}$ and similarly for the top storey.

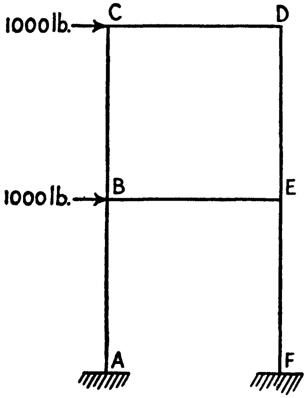


FIG. 246

Solution by the Slope-Deflection Method.

$$M_{AB} = 2(\theta_B + 3\phi_A)$$

$$M_{BA} = 2(2\theta_B + 3\phi_A)$$

$$= M_{BE} = 2(3\theta_B)$$

$$M_{BC} = 2(2\theta_B + \theta_C + 3\phi_B)$$

$$M_{CB} = 2(\theta_B + 2\theta_C + 3\phi_B)$$

$$= M_{CD} = 2(3\theta_C)$$

Joint equation (B): $M_{BA} + M_{BE} + M_{BC} = 0$

Joint equation (C): $M_{CB} + M_{CD} = 0$

Bent equation, lower storey:

$$2(M_{AB} + M_{BA}) = +\ 2000 \times 100 \text{ lb.-in.}$$

Bent equation, upper storey:

$$2(M_{CB} + M_{BC}) = +\ 1000 \times 100 \text{ lb.-in.}$$

We thus have 4 equations for solving 4 unknowns, θ_B; θ_C; ϕ_A and ϕ_B.

Solving

$$\theta_B = -\ 19{,}900$$

$$\theta_C = -\ 5000$$

$$\phi_A = +\ 26{,}633$$

$$\phi_B = +\ 11{,}650$$

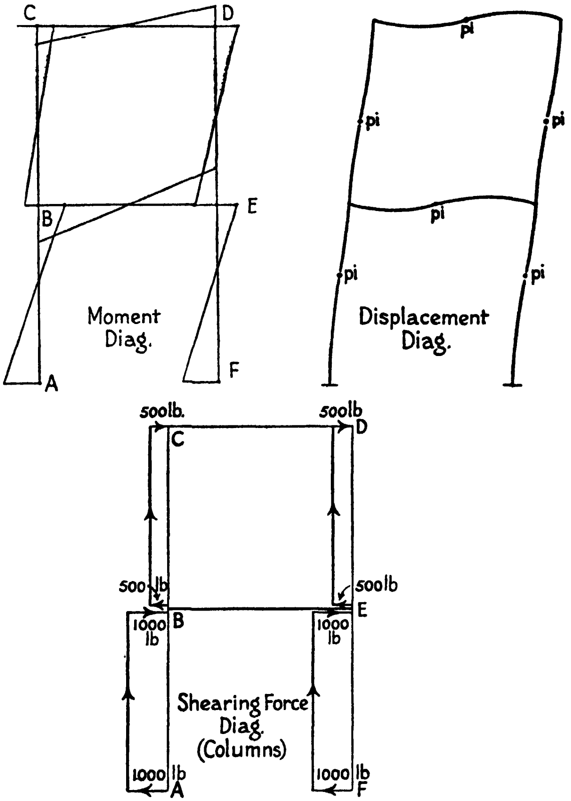


FIG. 247

Substitution of these values in the necessary equations gives,

$M_{AB} = +\ 60{,}100$ lb.-in.

$$\left.\begin{array}{ll} M_{BA} = +\ 40{,}100 & ,, \\ M_{BE} = -\ 59{,}700 & ,, \\ M_{BC} = +\ 20{,}100 & ,, \\ M_{CB} = +\ 30{,}000 & ,, \end{array}\right\} \Sigma Ms \text{ at joint } B = +\ 500 \text{ lb.-in. (nearly equal to zero).}$$

$M_{CD} = -\ 30{,}000$,,

Bent equation, lower storey, gives 2(100,200)

$= 200{,}400$ lb.-in. $\simeq 200{,}000$ lb.-in. reqd.

Bent equation, upper storey, gives 2(50,100)

$$= 100{,}200 \text{ lb.-in.} \simeq 100{,}000 \text{ lb.-in. reqd.}$$

Solution by Moment-Distribution Method.

K for all members = 1.

K' for all vertical members proportional to 1.

The sway fixing moments:

$$M'_{AB} + M'_{BA} + M'_{EF} + M'_{FE} = 200{,}000 \text{ lb.-in.}$$

Now if these lower members AB and EF are swayed without any rotation of the joints B and E, then the above moments are all equal, and equal to + 200,000/4 = + 50,000 lb.-in.

Similarly, swaying the members BC and DE, without rotating the joints B, C, D and E, the 4 end column moments induced will be equal to M'_{BC} = + 100,000/4 = + 25,000 lb.-in.

Moments in multiples of 10^4 = 10,000 lb.-in.

It is the multiple which will be balanced, etc.

Member	*AB*	*BA*	*BE*	*EB*	*EF*	*FE*
K	1		1		1	
		(with	BC $\frac{1}{3}$)	(with	ED $\frac{1}{3}$)	
Bal.	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	
C.O.F.	$\frac{1}{2}$		$\frac{1}{2}$		$\frac{1}{2}$	
K'	1		—		1	
Sway F.B.M.	+ 5·00	+ 5·00	0	0	+ 5·00	+ 5·00
Bal. M.	0	− 2·50	− 2·50	− 2·50	− 2·50	0
C.O.	− 1·25	0	− 1·25	− 1·25	0	− 1·25
Bal. S.	+ 1·88	+ 1·88	0	0	+ 1·88	+ 1·88
Bal. M.	0	− 0·96	− 0·96	− 0·96	− 0·96	0
C.O.	− 0·48	0	− 0·48	− 0·48	0	− 0·48
Bal. S.	+ 0·72	+ 0·72	0	0	+ 0·72	+ 0·72
Bal. M.	0	− 0·36	− 0·36	− 0·36	− 0·36	0
C.O.	− 0·18	0	− 0·18	− 0·18	0	− 0·18
Bal. S.	+ 0·27	+ 0·27	0	0	+ 0·27	+ 0·27
Bal. M.	0	− 0·14	− 0·14	− 0·14	− 0·14	0
C.O.	− 0·07	0	− 0·07	− 0·07	0	− 0·07
Bal. S.	+ 0·10	+ 0·10	0	0	+ 0·10	+ 0·10
Bal. M.	0	− 0·05	− 0·05	− 0·05	− 0·05	0
ΣM (lb.-in.)	+59,900	+39,600	−59,900	−59,900	+39,600	+59,900

Bent equation, lower storey, + 59,900 + 39,600 + 39,600 + 59,900 = 199,000 ≏ 200,000 lb.-in.

Member	*BC*	*CB*	*CD*	*DC*	*DE*	*ED*
K	1		1		1	
Bal.	$\frac{1}{3}$ (with *BA* and *BE*)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	(with *EB* and *EF*) $\frac{1}{3}$
C.O.F.	$\frac{1}{2}$		$\frac{1}{2}$		$\frac{1}{2}$	
K′	1		0		1	
Sway F.B.M.	+ 2·50	+ 2·50	0	0	+ 2·50	+ 2·50
Bal. M.	− 2·50	− 1·25	− 1·25	− 1·25	− 1·25	− 2·50
C.O.	− 0·63	− 1·25	− 0·63	− 0·63	− 1·25	− 0·63
Bal. S.	+ 2·88	+ 2·88	0	0	+ 2·88	+ 2·88
Bal. M.	− 0·96	− 0·05	− 0·50	− 0·50	− 0·50	− 0·96
C.O.	− 0·25	− 0·48	− 0·25	− 0·25	− 0·48	− 0·25
Bal. S.	+ 1·10	+ 1·10	0	0	+ 1·10	+ 1·10
Bal. M.	− 0·36	− 0·19	− 0·19	− 0·19	− 0·19	− 0·36
C.O.	− 0·10	− 0·18	− 0·10	− 0·10	− 0·18	− 0·10
Bal. S.	+ 0·42	+ 0·42	0	0	+ 0·42	+ 0·42
Bal. M.	− 0·14	− 0·07	− 0·07	− 0·07	− 0·07	− 0·14
C.O.	− 0·04	− 0·07	− 0·04	− 0·04	− 0·07	− 0·04
Bal. S.	+ 0·16	+ 0·16	0	0	+ 0·16	+ 0·16
Bal. M.	− 0·05	− 0·03	− 0·03	− 0·03	− 0·03	− 0·05
ΣM	+20,300	+30,400	−30,600	−30,600	+30,400	+20,300

Bent equation, upper storey:

2(20,300 + 30,400) = 101,400 lb.-in. ≏ 100,000 lb.-in.

Joint equation (*B*): + 39,600 + 20,300 — 59,900 = 0

Joint equation (*C*): + 30,400 — 30,600 = — 200·0 ≏ 0

The moment, shearing force, and displacement diagrams are given in Fig. 247.

Shearing Forces in the Members.

(*a*) *AB* and *EF*. At *A* and *F*, both = 1000 lb. acting right to left, and their sum balances the external forces of 1000 lb. and 1000 lb. acting left to right.

At *B* and *E*, the shearing forces must be 1000 lb. in each member acting from left to right: i.e. their sum is equal to the 2 external loads and of the same sign.

(*b*) *BC* and *ED*. At *B* and *E* the shearing forces are both = 500 lb. acting right to left and they balance the 1000 lb. load acting at *C* from left to right.

At *C* and *D*, both shearing forces are equal to 500 lb. each acting from left to right, and equal to the external force of 1000 lb. at *C* and of the same sign.

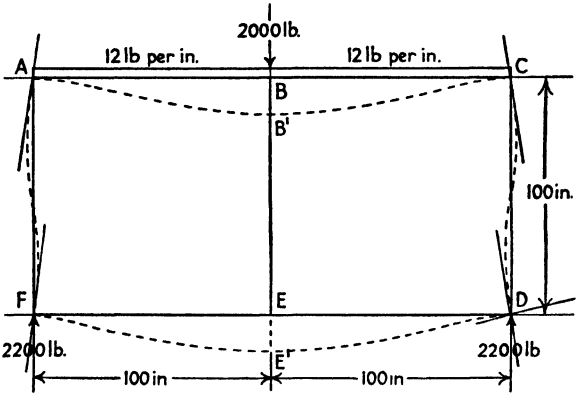


FIG. 248

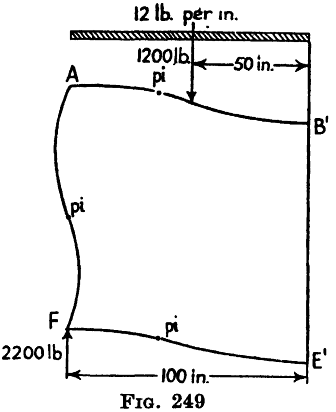


FIG. 249

From the actual calculated moments the above shearing force values are very nearly equal to those given above.

Illustrative Problem 57. (Figs. 248, 249, 250, 251.)

A framed bridge span or open webbed girder, consisting of two rectangular frames simply-supported at the ends. Let the frame be *ABCDEF*, consisting of the 2 continuous beams *ABC* and *DEF* and the 3 vertical members *FA*, *EB* and *DC*. The whole structure is monolithic. It is supported at *F* and *D*. All the members have the same length of 100 in., and the same *E* and *I*.

So let $2EK = 1$ for all members in the slope-deflection equations. The upper members *AB*, *BC* are loaded with a uniform load of 12 lb. per in., and there is a point load of 2000 lb. at the joint *B*.

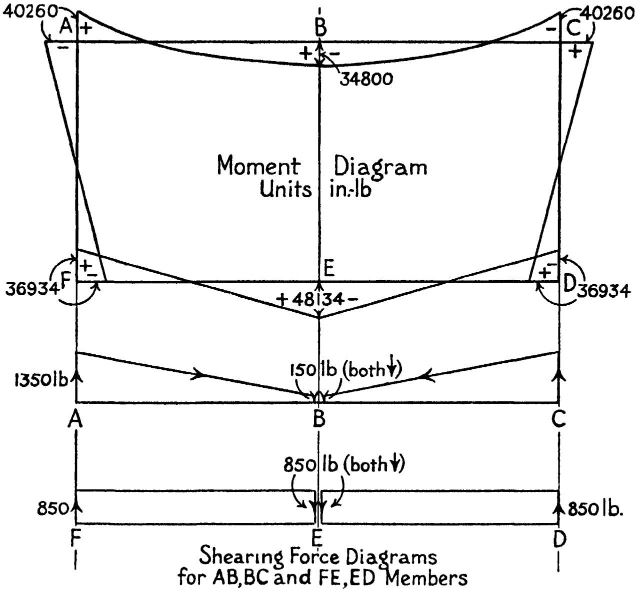


FIG. 250

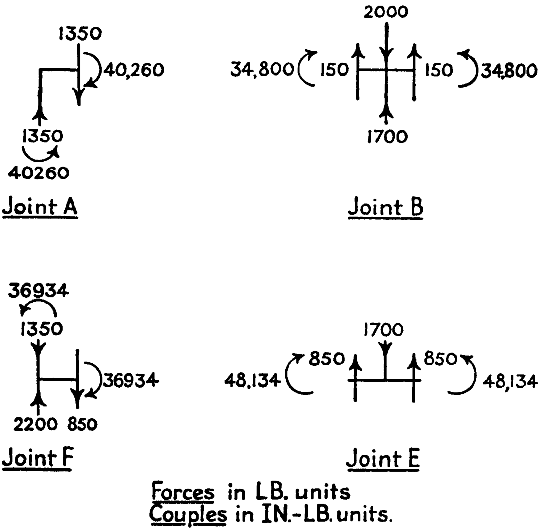


FIG. 251

The reactions at F and D will both be 2200 lb., as the beam is symmetrical about the vertical member BE, and, as it is symmetrically loaded about this member, then under the loading there will be no moments M_{BE} or M_{EB}; the beam will deform as shown; members AB and BC swaying to AB', CB', and the members FE and DE to FE' and DE'.

There will be no sway of the members AF and CD.

After distortion, and due to symmetry and symmetry of loading,

$$\theta_{BA} = \theta_{BC} = 0; \text{ and } \theta_{EF} = \theta_{FE} = 0$$

The frame $ABEF$ sways to the position $AB'E'F$ (Fig. 248), where the angles at B' or E' are zero. Therefore imagine that the frame rotates positively about $E'B'$. There is no change in the lengths AF and BE, therefore

$$\phi_{AB} = \phi_{B'A} = + \phi_A; \ \phi_{EF} = \phi_{E'F} = + \phi_A, \text{ say } + \phi$$

Let $\theta_{AB} = \theta_{AF} = \theta_A$; $\theta_{FA} = \theta_{FE} = \theta_F$.

Slope-Deflection Solution.

$$M_{AB} = 2\theta_A + 3\phi + 10{,}000; \quad M_{BA} = \theta_A + 3\phi - 10{,}000$$
$$M_{AF} = 2\theta_A + \theta_F; \quad M_{FA} = 2\theta_F + \theta_A$$
$$M_{FE} = 2\theta_F + 3\phi; \quad M_{EF} = \theta_F + 3\phi$$

Joint equation (A): $M_{AB} + M_{AF} = 0$

Joint equation (F): $M_{FA} + M_{FE} = 0$

Bent equation: $M_{BA} + M_{AB} + M_{FE} + M_{EF}$

$$= (+ 2200 \times 100) + (- 1200 \times 50)$$
$$= + 160{,}000 \text{ lb.-in.}$$

Solving the resulting 3 simultaneous equations for ϕ, θ_A and θ_B, we find $\phi = 19{,}778$; $\theta_A = - 14{,}534$ and $\theta_B = - 11{,}200$. The moments are then:

$$M_{AB} = + 40{,}260 \text{ lb.-in.} \quad M_{BA} = + 34{,}800 \text{ lb.-in.}$$
$$M_{AF} = - 40{,}260 \text{ lb.-in.} \quad M_{EF} = + 48{,}134 \text{ lb.-in.}$$
$$M_{FA} = - 36{,}934 \text{ lb.-in.} \quad M_{FE} = + 36{,}934 \text{ lb.-in.}$$

NOTE. The joint equations are satisfied.

The bent equation: $+ 40{,}260 + 34{,}800 + + 36{,}934 + 48{,}134$

$$= 160{,}128 \simeq 160{,}000 \text{ lb.-in.}$$

Moment-Distribution Method.

Member	*FA*		*AF*	*AB*		*BA*	*BC*		*CB*	*CD*		*DC*
K		1			1			1			1	
Bal.	½ (with *FE*)		½	½		⅓	⅓ (with *BE*)	⅓	½	½		(with *DE*) ½
C.O.		½			½			½			½	
K′ or *K″*		0			1			1			0	
F.B.M.	0		0	+ 10,000		− 10,000	+ 10,000		− 10,000	0		0
Bal. M.	0		− 5000	− 5000		0	0		+ 5000	+ 5000		0
C.O.	− 2500		0	0		− 2500	+ 2500		0	0		+ 2500
Bal. S.	0		0	+ 41,875		+ 41,875	− 41,875		− 41,875	0		0
Bal. M.	− 19,188		− 20,938	− 20,938		0	0		+ 20,938	+ 20,938		+ 19,188
C.O.	− 10,469		− 9594	0		− 10,469	+ 10,469		0	− 9594		+ 10,469
Bal. S.	0		0	+ 15,047		+ 15,047	− 15,047		− 15,047	0		0
Bal. M.	− 2789		− 2727	− 2727		0	0		+ 2727	+ 2727		+ 2789
C.O.	− 1364		− 1395	0		− 1364	+ 1364		0	+ 1395		+ 1364
Bal. S.	0		0	+ 2069		+ 2069	− 2069		− 2069	0		0
Bal. M.	− 353		− 337	− 337		0	0		+ 337	+ 337		+ 353
C.O.	− 169		− 177	0		− 169	+ 169		0	+ 177		+ 169
Bal. S.	0		0	+ 259		+ 259	− 259		− 259	0		0
Bal. M.	− 45		− 41	− 41		0	0		+ 41	+ 41		+ 45
C.O.	− 21		− 23	0		− 21	+ 21		0	+ 23		+ 21
Bal. S.	0		0	+ 33		+ 33	− 33		+ 33	0		0
Bal. M.	− 6		− 5	− 5		0	0		+ 5	+ 5		+ 6
ΣM (lb.-in.)	− 36,904		− 40,237	+ 40,235		+ 34,760	− 34,760		− 40,235	+ 40,235		+ 36,904

Member	*FE*	*EF*	*EB*	*BE*	*ED*	*DE*
K	1		1		1	
Bal.	½ (with *FA*)	(with ⅓	*EO* ⅓) ⅓	(with ⅓ *BC*, *BA*) ⅓	(with ⅓ *EF*, *EB*) ⅓	(with *DC*) ½
C.O.F.	½		½		½	
K′ or *K″*	1		0		1	
F.B.M.	0	0	0	0	0	0
Bal. M.	0	0	0	0	0	0
C.O.	0	0	0	0	0	0
Bal. S.	+41,875	+41,875	0	0	−41,875	−41,875
Bal. M.	− 19,188	0	0	0	0	+19,188
C.O.	0	− 9594	0	0	+ 9594	0
Bal. S.	+15,047	+15,047	0	0	−15,047	−15,047
Bal. M.	− 2789	0	0	0	0	+ 2789
C.O.	0	− 1395	0	0	+ 1395	0
Bal. S.	+ 2069	+ 2069	0	0	− 2069	− 2069
Bal. M.	− 353	0	0	0	0	+ 353
C.O.	0	− 177	0	0	+ 177	0
Bal. S.	+ 259	+ 259	0	0	− 259	− 259
Bal. M.	− 45	0	0	0	0	+ 45
C.O.	0	− 23	0	0	+ 23	0
Bal. S.	+ 33	+ 33	0	0	− 33	− 33
Bal. M.	− 6	0	0	0	0	+ 6
Σ*M* lb.-in.	+36,902	+48,094	0	0	−48,094	−36,902

Bent equation: $M_{BA} + M_{AB} + M_{FE} + M_{EF}$

$$= + 40{,}235 + 34{,}760 + 36{,}902 + 48{,}094$$

$$= + 159{,}991 \simeq 160{,}000 \text{ lb.-in.}$$

After the 1st balance and carry-over, the sum of the moments at the ends *AB* and *FE*

$$= - 7500 + \Sigma M\text{s from the bent equation}, = + 160{,}000$$

∴ In the 1st balance S (Bal. S), $4M'_{AB} = + 167{,}500$ lb.-in and $M'_{AB} = + 41{,}875$ lb.-in.

∴ joints *A*, *B*, *F*, *E* are kept fixed and *AB*, *FE* swayed equally; then the moments induced at the ends *A*, *B*, *F* and *E* will all be equal to M'_{AB}.

After the 2nd balance and carry-over, additional moments at the ends of the horizontal members *AB* and *FE* are—

for *AB*, $-$ 20,938 and $-$ 10,469
and for *FE*, $-$ 19,188 and $-$ 9,547

The total of these moments $= -$ 60,189

$$\therefore\ 4M'_{AB} = +\ 60{,}189$$

$$\therefore\ M'_{AB} = +\ 15{,}047 \text{ lb.-in.}$$

The Shearing Forces in the Members AB and FE.

Shear equation: to find the shears for the end couples $\frac{M_{AB} + M_{BA}}{l_{AB}} + \frac{M_{FE} + M_{EF}}{l_{EF}}$ should be equal to the sum of 2200 lb. acting vertically upwards at *F* and $\frac{1200}{2} = 600$ lb. acting vertically downwards at *A*.

The sum is 1600 lb. acting vertically upwards.

$\frac{M_{AB} + M_{BA}}{l_{AB}} = 750{\cdot}6$ lb. acting upwards at the cut end of the member *AB* at *A*; because both couples are counter-clockwise ones.

$\frac{M_{FE} + M_{EF}}{l_{EF}} = 850{\cdot}6$ lb. acting upwards at the cut end of the member *FE* at *E*, because both couples are counter-clockwise ones. Their sum is 1601·2 lb. acting vertically upwards and thus the shear equation is satisfied.

Total Shearing Force at the Cut Ends ABFE of the Members AB and FE.

AB. The total shearing force at *A* will be 750·6 lb. (upwards) plus $\frac{1}{2}wl = \frac{1200}{2} = 600$ lb., the simple beam shear also acting upwards. The total is therefore 1350·6 lb. (upwards).

The total shearing force at *B* is 750·6 lb. acting downwards due to the end couples plus the simple beam shear of 600 lb. acting upwards. The sum is 150·6 lb. acting downwards.

FE. There are no transverse forces acting on *FE*, therefore the shear at *F* will be 850·6 lb. acting upwards, and at *E* is 850·6 lb. acting downwards, both due to the end couples.

Now 150·6 lb. downwards at *B* plus 850·6 lb. downwards at $E = 1001{\cdot}2$ lb. acting downwards. This balances the external forces acting on the panel *ABEF*, which are 2200 lb. upward at *F* and 1200 lb. downwards on *AB*, giving a resultant of 1000 lb. acting upward.

The sum of the total shear at *A* and *F* must obviously be equal to and of the same sense as the reaction at *F*.

At *A*, the shear is 1350·6 lb. upwards, and at *F* the shear is 850·6 lb. upwards. Their sum is equal to 2201·2 lb. acting upwards and equal in magnitude and sense of action to the reaction of 2200 lb. at *F*.

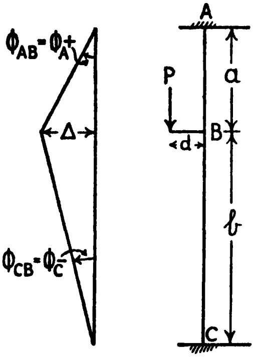


FIG. 252

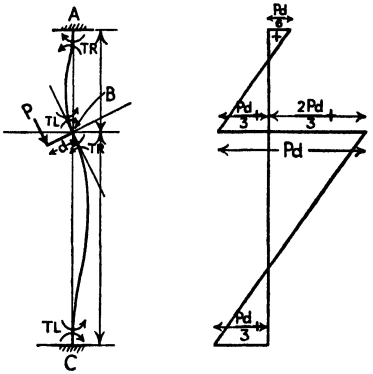


FIG. 253

It can be shown that the direct force in *AF* and *CD* is 1350 lb. (compression) and in *BE* it is 2000 lb. − 2 × 150 lb. = 1700 lb. (compression).

The diagrams of the forces acting on the joints *A* (and *C*), *F* (and *D*), *B* and *E* are given in Fig. 251.

Illustrative Problem 58.

A vertical column *AC*, direction-fixed at *A* and *C*, with an eccentric load *P* applied at an arm *d* from *B*, such that $AB = a$; $BC = b$. Calculate the moments at *A*, *B*, and *C* (Fig. 252).

Referring to Fig. 252, let *B* sway to the left;

then $$\phi_{AB} = \phi_A \text{ and is } +$$

and $$\phi_{CB} = \phi_B \text{ and is } -$$

Also $$a \, . \, \phi_A = - \, b \, . \, \phi_B$$

∴ $$\phi_B = - \frac{a}{b} \, . \, \phi_A$$

$$\theta_A = \theta_C = 0$$

Moment at *B* due to *P* is *Pd*, and is negative.

Slope-Deflection Equations.

$$M_{AB} = 2K_A(\theta_B + 3\phi_A); \qquad M_{BA} = 2K_A(2\theta_B + 3\theta_A)$$

$$M_{BC} = 2K_B\left(2\theta_B - 3\frac{a}{b}\phi_A\right); \; M_{CD} = 2K_B\left(\theta_B - 3\frac{a}{b}\phi_A\right)$$

Joint equation (B): $M_{BA} + M_{BC} + (-Pd) = 0$

$$\therefore \; \theta_B(4K_A + 4K_B) + \phi_A\left(6K_A - 6\frac{a}{b}K_B\right) + (-Pd) = 0 \quad \text{(i)}$$

There is no horizontal force at B.

$$\therefore \text{ Let } \frac{M_{AB} + M_{BA}}{a} = X$$

$$\text{and } \frac{M_{BC} + M_{CB}}{b} = X$$

$$\therefore \frac{M_{AB} + M_{BA}}{a} - \frac{M_{BC} + M_{CB}}{b} = 0$$

$$\therefore \quad 6\theta_B\left(\frac{K_A}{a} - \frac{K_B}{b}\right) + 6\phi_A\left(\frac{K_A}{a} + \frac{K_B}{b} \cdot \frac{a}{b}\right) = 0 \quad \text{(ii)}$$

$\therefore$ if $\frac{K_A}{a} - \frac{K_B}{b} = 0$, then $\phi_A = 0$, and there is no sway;

$$\text{i.e. } \frac{K_A}{K_B} = \frac{a}{b} \text{ or } \frac{I_{AB}}{I_{BC}} = \frac{a^2}{b^2}$$

With no sway, $4\theta_B K_B\left(\frac{a}{b} + 1\right) = +Pd$

$$\therefore \; \theta_B = +\frac{Pd}{4K_B} \cdot \left(\frac{b}{a+b}\right)$$

Then
$$M_{AB} = -2K_B \cdot \frac{a}{b} \cdot \frac{Pd}{4K_B} \cdot \frac{b}{a+b} = +\frac{Pd}{2} \cdot \frac{a}{a+b}$$

$$M_{BA} = +Pd \cdot \frac{a}{a+b}$$

$$M_{BC} = 4K_B \cdot \frac{Pd}{4K_B} \cdot \frac{b}{a+b} = +Pd \cdot \frac{b}{a+b}$$

$$M_{CB} = +\frac{Pd}{2} \cdot \frac{b}{a+b}$$

For other cases take specific values of a, b, d, P, I_{AB}, I_{BC} and solve the two equations (i) and (ii).

Taking the general case, where swaying occurs, find what horizontal force F, applied at B, would prevent sway.

If swaying is prevented, B will come directly under A and directly over C.

Then $M_{AB} = \frac{1}{2}M_{BA}$, and $M_{CB} = \frac{1}{2}M_{BC}$.

Joint equation: $M_{BA} + M_{BC} + (-Pd) = 0$.

Shear equation: $\dfrac{M_{AB} + M_{BA}}{a} - \dfrac{M_{BC} + M_{CB}}{b} = F$.

$$M_{BA} = 2K_A \,.\, 2\theta_B\,;\ M_{BC} = 2K_B \,.\, 2\theta_B$$

$$\therefore\ 4(K_A + K_B)\theta_B = +\,Pd$$

$$\theta_B = \frac{Pd}{4}\ \frac{1}{(K_A + K_B)}$$

$$\therefore\ M_{BA} = +\,Pd \,.\, \frac{K_A}{(K_A + K_B)}\,;\ M_{AB} = +\,\frac{M_{BA}}{2}\,;$$

$$M_{BC} = +\,Pd \,.\, \frac{K_B}{(K_A + K_B)}\,;\ M_{CB} = +\,\frac{M_{BC}}{2}$$

$$\therefore\ F = \frac{3}{2}\frac{Pd}{a} \cdot \frac{K_A}{(K_A + K_B)} - \frac{3}{2} \cdot \frac{Pd}{b} \cdot \frac{K_B}{(K_A + K_B)}$$

Knowing the direction of sway, the direction of F will be known.

$$F = \frac{3}{2} \cdot \frac{Pd}{ab}\,\frac{(bK_A - aK_B)}{(K_A + K_B)}$$

If $\qquad K_A = \dfrac{a}{b} \,.\, K_B,$

then $\qquad F = \dfrac{3}{2} \cdot \dfrac{Pd}{ab} \,.\, b\dfrac{(aK_B - aK_B)}{(aK_B + bK_B)} = 0,$

which confirms the result previously obtained that sway does not occur when $K_A/K_B = a/b$.

No sway. Let $\dfrac{I_{AB}}{I_{BC}} = \dfrac{a^2}{b^2}\quad \therefore\ \dfrac{a}{b} = \sqrt{\dfrac{I_{AB}}{I_{BC}}} = \sqrt{\dfrac{1}{4}} = \dfrac{1}{2},$

i.e. $\qquad a = b/2$

Then $\qquad M_{BA} = +\,Pd \,.\, \dfrac{b/2}{b/2 + b} = +\,Pd \,.\, \dfrac{1}{3}$

$$M_{BC} = +\,Pd \,.\, \frac{b}{b/2 + b} = Pd\frac{2}{3}$$

Illustrative Problem 59. (See Fig. 254.)

A rectangular portal $CABD$ is hinged at the supports C and D.

$$CA = DB = h \text{ (columns)}$$

$$AB = l \text{ (beam)}$$

The columns have a moment of inertia I_1, and the beam that of I. CA and DB are loaded with a uniformly varying load from q per unit length at C and D, to zero at A and B. E is the same for all member and assume $= 1$.

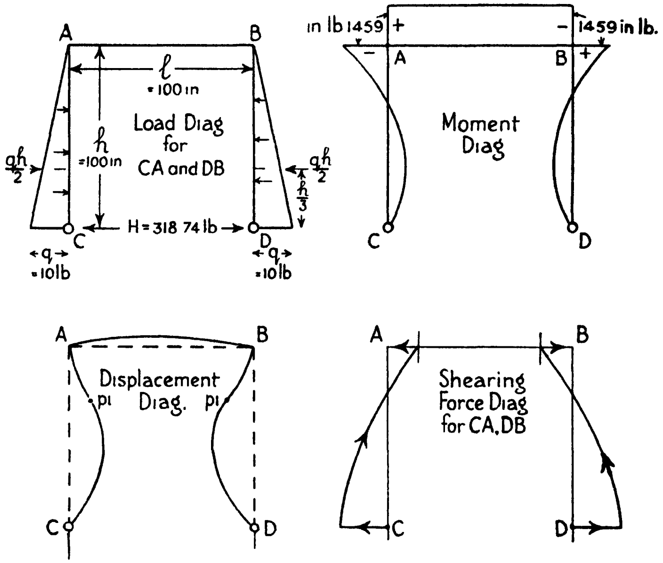


FIG. 254

The fixed bending moments at C and A are equal to $+\dfrac{qh^2}{20}$ and $-\dfrac{qh^2}{30}$ respectively. As the portal is symmetrically loaded, we should consider only the moments at C and A, with the proviso that $\theta_{BA} = \theta_B = -\theta_{AB} = -\theta_A$. Let $\theta_{CA} = \theta_C$.

There will be no sway due to symmetry of loading.

$$K_{CA} = K_C = I_1/h;\ K_{AB} = K_B = I/l$$

It can be shown that

$$M_{AC} = 2K_C(1{\cdot}5\theta_A) - \frac{qh^2}{30} - \frac{1}{2}\cdot\frac{qh^2}{20}$$

$$M_{CA} = 0 = 2K_C(2\theta_C + \theta_A) + \frac{qh^2}{20}$$

$$M_{AB} = 2K_B(2\theta_A + \theta_B) = 2K_B(\theta_A)$$

Joint A equation: $M_{AB} + M_{AC} = 0$

Solving for θ_A; $\theta_A = \frac{7}{120} \cdot \left(\frac{1}{2K_B + 3K_C)}\right) \cdot qh^2$

Then $M_{AB} = -M_{AC} = +\frac{7mqh^2}{60(2m + 3)}$, where $m = K_B/K_C$

It can also be shown that $\theta_C = -\frac{qh^2}{120}\left(\frac{8K_C + 3K_B}{K_C\, 2K_B + 3K_C}\right)$

Solution by Moment-Distribution Method.

Let $h = 100$ in., $I_{AC} = I_{AB} = 100$ in.^{4}, $l = 100$ in. and $I_{AB} = 50$ in.^{4} M_{FAC} (for C hinged on CA)

$$= \left(-\frac{qh^2}{30}\right) + \left(-\frac{1}{2} \cdot \frac{qh^2}{20}\right) = \frac{7}{120} \cdot qh^2 = -5833 \text{ lb.-in.},$$

when $q = 10$ lb. per in.

$M_{AB} = -M_{AC} = +1458$ lb.-in. (Slope-deflection Method).

Member	*CA*	*AC*	*AB*	*BA*	*BD*	*DB*
K	$\frac{3}{4} \times \frac{100}{100}$		$\frac{50}{100}$		$\frac{3}{4} \times \frac{100}{100}$	
Rel. K	3		2		3	
Bal.	0	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	0
C.O.	0		$\frac{1}{2}$		0	
F.B.M.	0	− 5833	0	0	+ 5833	0
Bal. M.	0	+ 3499	+ 2334	− 2334	− 3499	0
C.O.	0	0	− 1167	+ 1167	0	0
Bal. M.	0	+ 700	+ 467	− 467	+ 700	0
C.O.	0	0	− 234	+ 234	0	0
Bal. M.	0	+ 140	+ 94	− 94	− 140	0
C.O.	0	0	− 47	+ 47	0	0
Bal. M.	0	+ 28	+ 19	− 19	+ 28	0
C.O.	0	0	− 10	+ 10	0	0
Bal. M.	0	+ 6	+ 4	− 4	− 6	0
C.O.	0	0	− 2	+ 2	0	0
Bal. M.	0	+ 1	+ 1	− 1	− 1	0
ΣM (lb.-in.)	0	− 1459 (M_{AC})	+ 1459 (M_{AB})	− 1459	+ 1459	0

Bent equation: $-1459 + 1459 = 0$.

It can be shown that the horizontal reaction at the hinge C is 318·74 lb. (acting right to left), and at D it is 318·74 lb. (acting left to right).

The shearing force at the top of column *CA* is 181·26 lb., acting right to left, and at *B*, top of column *DB* it is also 181·26 lb., acting left to right. For *CA* $\frac{qh}{2} = 500\ (\rightarrow) = 318{\cdot}74 + 181{\cdot}26 = 500$ lb. (←).

The total shear at the cut end *C* of the member *CA* is equal to $\frac{1459}{100}$ lb. acting from left to right due to the end couple, plus $\frac{2}{3}\frac{hq}{2}$ simple beam shear acting from right to left.

∴ Total shear at *C* is 14·59 lb. (→) + 333.33 lb. (←)
= 318·74 lb. acting right to left. This is also the reaction at the hinge *C*.

The reaction at the hinge *D* must be also 318·74 lb., but acting left to right.

The shear at the top *A* of the member *CA* is = 14·59 lb. acting from right to left plus the simple beam shear $\frac{1}{3}\cdot\frac{qh}{2}$ acting from right to left. The total shear here is therefore 181·26 lb. acting from right to left. The total shear at *B* for member *BD* must be 181·26 lb., but acting from left to right.

The diagrams of moment, shearing force, and displacement are given in Fig. 254.

Illustrative Problem 60. (See Fig. 255, page 476.)

$$M_{FAC}\ (C \text{ hinged}) = -\ 5833 \text{ lb.-in.} = -\frac{qh^2}{30} - \frac{1}{2}\frac{qh^2}{20}$$

Bent equation:

$$M_{AC} + M_{BD} = \frac{qh}{2} \times \frac{h}{3} = \frac{qh^2}{6} = +\ 16{,}667 \text{ lb.-in.}$$

From the joint equations for *A* and *B* and the bent equation, and using the slope-deflection equations for M_{AC}, M_{AB}, M_{BA} and M_{BD}, it can be shown that

$\theta_B = -\ 3507$; $\theta_A = -\ 2049$; and ϕ, the sway angle for *CA* and *DB* = + 6528.

Therefore,

$$M_{AB} = -\ 7605 \text{ lb.-in.};\quad M_{AC} = +\ 7605 \text{ lb.-in.};$$
$$M_{BA} = -\ 9063 \text{ lb.-in.};\quad M_{BD} = +\ 9063 \text{ lb-in..}$$
$$M_{AC} + M_{BD} = 16{,}668 \text{ lb.-in.} \simeq 16{,}667 \text{ lb.-in.}$$

The horizontal reaction at *C* due to the couple of —7600 lb.-in. at *A* is 76 lb. acting from right to left: the reaction at *D* due to the couple of — 9060 lb.-in. at *B* is 90·60 lb. acting from right to left. The corresponding shears acting at the tops of the columns act in the direction left to right and their sum is equal to 166·6 lb. The proportionate shear due to the load on *CA* and acting at the tops of the columns is 166·66 lb. acting in the direction right to left. Thus the total shearing force acting in the top of the column *CA* at *A* is 166·66 lb. acting from right to left plus 76·0 lb. acting from left to right. The resultant is 90·6 lb. acting from right to left and this balances the 90·6 lb. acting on the top of the column *DB* at *B* in the left to right direction. The total horizontal reaction at *C* is 333·33 lb. acting from right to left plus 76 lb. acting from right to left, giving a total of 409·33 lb. acting in the right to left direction.

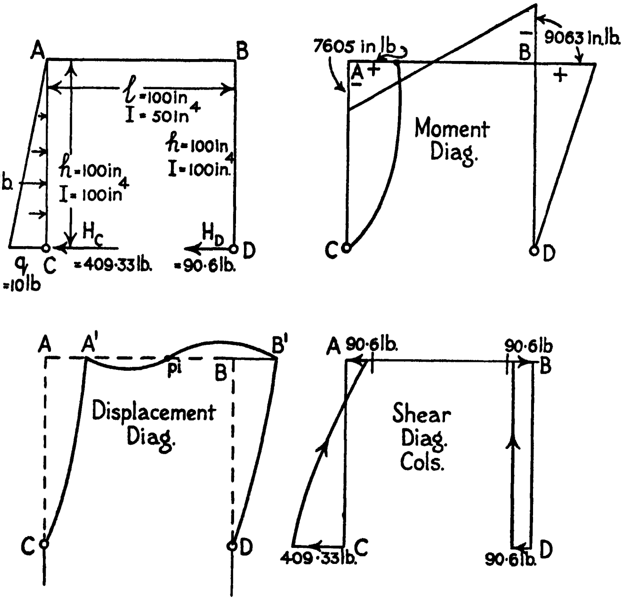


FIG. 255

Thus the sum of the horizontal reactions at *C* and *D* is equal to 500 lb. acting from right to left, and this sum balances the dead load of 500 lb. on *CA* which acts from left to right.

The moment, column shearing force, and displacement diagrams are given in Fig. 255.

Moment-Distribution Method.

Member	*CA*	*AC*	*AB*	*BA*	*BD*	*DB*
Rel. K	3		2		3	
Bal.	0	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	0
C.O.	0		$\frac{1}{2}$		0	
Rel. K' } Rel. K'' }	1		0		1	
F.B.M.	0	− 5833	0	0	0	0
Bal. M.		+ 3499	+ 2334	0	0	0
C.O.	0	0	0	+ 1167	0	0
Bal. S.	0	+ 9500	0	0	+ 9500	0
Bal. M.	0	− 5700	− 3800	− 4267	− 6400	0
C.O.	0	0	− 2134	− 1900	0	0
Bal. S.	0	+ 6050	0	0	+ 6050	0
Bal. M.	0	− 2350	− 1566	− 1660	− 2490	0
C.O.	0	0	− 830	− 783	0	0
Bal. S.	0	+ 2420	0		+ 2420	0
Bal. M.	0	− 954	− 636	− 655	− 982	0
C.O.	0	0	− 328	− 318	0	0
Bal. S.		+ 968	0	0	+ 968	0
Bal. M.	0	− 384	− 256	− 260	− 390	0
C.O.	0	0	− 130	− 128	0	0
Bal. S.	0	+ 387	0	0	+ 387	0
Bal. M.	0	− 154	− 103	− 104	− 155	0
C.O.	0	0	− 52	− 52	0	0
Bal. S.	0	+ 155	0	0	+ 154	0
Bal. M.	0	− 62	− 41	− 41	− 61	0
C.O.	0	0	− 21	− 21	0	0
Bal. S.	0	+ 62	0	0	+ 62	0
Bal. M.	0	− 24	− 17	− 17	− 24	0
C.O.	0	0	− 9	− 9	0	0
Bal. S.	0	+ 24	0	0	+ 24	0
Bal. M.	0	− 9	− 6	− 6	− 9	0
C.O.	0	0	− 3	− 3	0	0
Bal. S.	0	+ 9	0	0	+ 9	0
Bal. M.	0	− 4	− 2	− 2	− 4	0
C.O.	0	0	− 1	− 1	0	0
Bal. S.	0	+ 4	0	0	+ 4	0
Bal. M.	0	− 2	− 1	− 1	− 2	0
ΣM (lb.-in.)	0	+ 7602 (M_{AC})	− 7602	− 9060	+ 9060 (M_{BD})	0

Bent equation: $+ 7602 + 9060 = +16{,}662 \bumpeq 16{,}667.$

The examples given have been in connection with simple continuous frames having horizontal and vertical prismatic members. For the cases in which sway occurred, the relative movement of the panel points could be determined by geometry, and where there were only one or two degrees of sway freedom. For more complex frames, frames with non-prismatic members, with sloping members, and with a number of degrees of sway freedom, the student is referred to other works, a few of which are listed in the references given on page 420.

The Calculation of the Axial Forces in Members of a Frame. In the determination of the axial forces in the members of a frame, the shearing forces acting at the ends of the members must be considered. These can be determined by the method given in para. 52, page 95, and summarized in equation (26), page 95. It will be noticed that the shearing force will consist in part of that due to the lateral loading and in part that due to the end couples. The forces equal and opposite to these shearing forces represent the shear action of the bent members of a frame on its joints. Considering these actions as external forces applied at the joints, we can calculate the axial forces in all members, and, if a frame has been restrained against lateral movement, the forces restraining this movement.

In a structure such as that given in Problem 9, Fig. 264, page 481, a force will act on the joint at A due to the member AD: this will act in the members CA and AE, CA being extended and AE compressed. This is a statically indeterminate problem. The magnitudes of the forces depend on the cross-sectional areas of the members and their lengths. In the problem, the bars are of the same length, and if we assume the cross sections are equal, we conclude that the load at the joint A is equally divided between the two bars. We shall have a tensile force in CA of the same magnitude as the compressive force in AE. The horizontal bars BA and AD can be treated similarly.

The axial forces in the bars can be calculated without any difficulty, provided that the end moments are known. It has been left as an exercise for the student to determine the axial forces in the members of the frames, which have been analysed in the Illustrative Problems.

EXAMPLES

1 to 12. Construct the shear and moment diagrams for each of the structures shown in Figs. 256 to 267 for the loads and end conditions given. In problems 5 and 6 (Figs. 260 and 261), the frames are made of slabs making up culvert sections and a 1 ft. strip of the culvert is considered.

13. (See Fig. 260.) The culvert is just filled with water weighing 62·5 lb. per cubic foot. Again, considering a strip of 1 ft., calculate the moments at the ends of the slabs due to the water load only.

14. (See Fig. 265.)

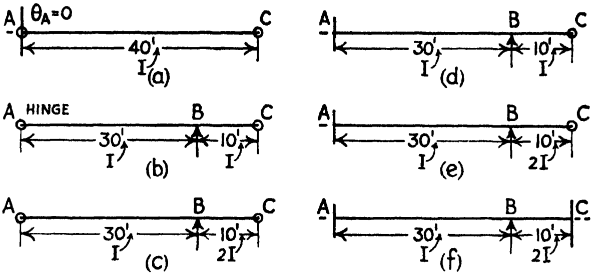
(*a*) Remove the horizontal load, and re-analyse the frame when *BC* is loaded uniformly at the rate of 10 lb. per in.

(*b*) Re-analyse the structure when a load *P* of 2400 lb. acts from left to right at the mid-span of *AB*. There are no other loads on the structure.

(*c*) If $E = 30 \times 10^6$ lb. per sq. in., find the horizontal movement of *BC* in Problems 10 and 14 (*b*).

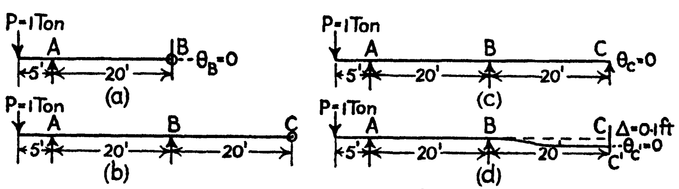
15. Re-analyse the structure in Fig. 266, if *D* is displaced horizontally to the right by 1·0 in., no rotation of the support taking place. $E = 30 \times 10^6$ lb. per sq. in.

16. Re-analyse the structure in Fig. 256 (*b*) if only *BC* is loaded uniformly at the rate of 100 lb. per in.



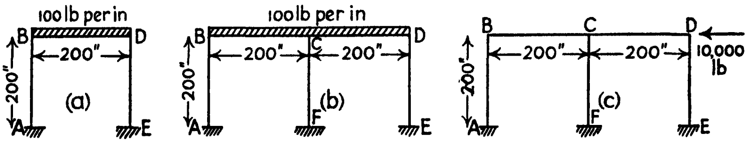
Loading for all beams is 1Ton/ft. over the whole beam length.
E = Constant. No displacement of supports or hinges

FIG. 256



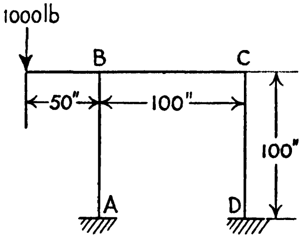
All Members EI = Const. = 40,000(ft.)^2-ton units
No displacement of supports or hinges
in (a), (b), and (c)

FIG. 257



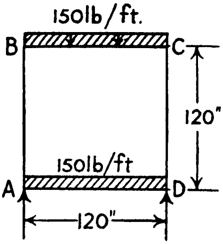
I - All Horizontal members 200 in units
I - Members AB, DE. – 100 in units
I - Members CF –200 in units
E Constant

FIG. 258



E.I. - Constant
All Members

FIG. 259



E I -Constant
All Members

FIG. 260

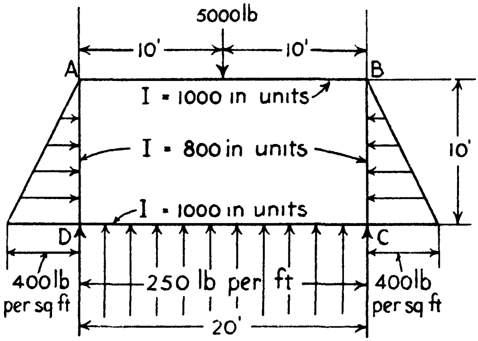
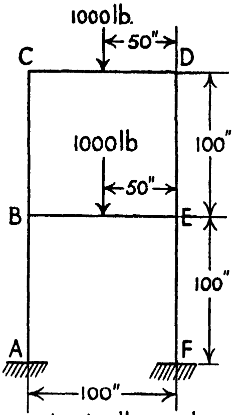
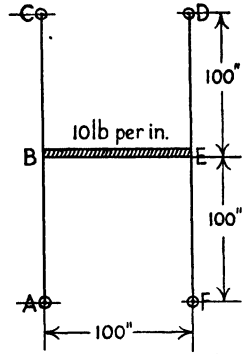


FIG. 261



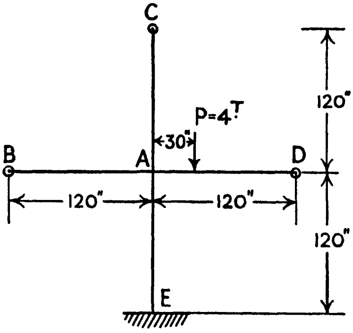
E I constant all members

FIG. 262



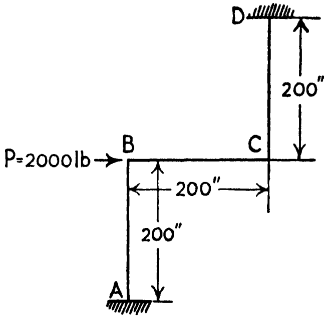
I-verticals = 100 in units
I-beam = 150 in units

FIG. 263



I-AB, AD, AE, = 120 in units
I-AC = 90 in units

FIG. 264



I-AB, CD, = 200 in.-units
I-BC = 100 in-units

FIG. 265

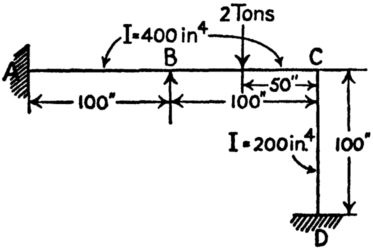
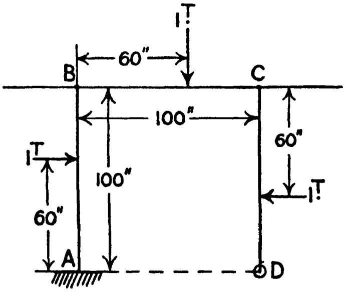


FIG. 266



E.K. - Constant
All Members

FIG. 267

CHAPTER XVI

THE RECIPROCAL THEOREM AND THE MECHANICAL SOLUTION OF STATICALLY INDETERMINATE STRUCTURES BASED ON THIS THEOREM

219. **The Reciprocal Theorem.** This theorem comprises two different states of stress of an elastic system. Suppose any weightless elastic body either solid or a framework—statically determinate or statically indeterminate—is supported in such a way that the reactive forces do no work when applied to the body. This will occur (*a*) when the points of support are fixed in space, for then no movement of the points will occur, and (*b*) when the body is supported on frictionless bearings for here, supposing a rigid support, any movement is at right angles to the reactive forces brought into play. For an elastic body, the relation between load and displacement is linear.

In the first state of stress imagine a single force P_a applied at a point a on the body, and in the second state a single force P_b applied at a point b on the body. The displacements of the points of application in the directions of the applied forces are $P_a \cdot \delta_{aa}$ and $P_a \cdot \delta_{ba}$ in the first state, and $P_b \cdot \delta_{bb}$ and $P_b \cdot \delta_{ab}$ in the second state, where

δ_{aa} = displacement at a due to unit load at a in the direction of P_a
δ_{ba} = ,, ,, b ,, ,, a ,, ,, P_b
δ_{bb} = ,, ,, b ,, ,, b ,, ,, P_b
δ_{ab} = ,, ,, a ,, ,, b ,, ,, P_a

The reciprocal theorem states: the work done by the forces in the first state on the corresponding displacements of the second state is equal to the work done by the forces of the second state on the corresponding displacements of the first state. This means for the simple case taken:

$$P_a \cdot (P_b \cdot \delta_{ab}) = P_b \cdot (P_a \cdot \delta_{ba}) \quad . \quad . \quad . \quad (1)$$

To prove this theorem, let us consider the strain energy of the body when the forces P_a and P_b are acting together, and use the fact that the amount of strain energy does not depend upon the order in which the forces are applied but only upon the final values of the forces.

In the first manner of loading, assume that the force P_a is applied first and later the force P_b. The strain energy stored during the application of P_a is $\frac{1}{2}P_a \, . \, (P_a \, . \, \delta_{aa})$.

Apply now the force P_b at b keeping the force P_a on at a. The work done by P_b is $\frac{1}{2}P_b \, . \, (P_b \, . \, \delta_{bb})$. It will be noted that during the application of P_b, the point of application of the previously applied load P_a has been displaced by an amount $P_b \, . \, \delta_{ab}$. Then P_a does work equal to $P_a \, . \, (P_b \, . \, \delta_{ab})$. This is not divided by 2 because the force P_a remains constant during the time in which the point of application 'a' undergoes the displacement $P_b \, . \, (\delta_{ab})$. Hence the total strain energy stored in the body due to the two operations in the first manner of loading is

$$U = \tfrac{1}{2}P_a \, . \, (P_a \, . \, \delta_{aa}) + \tfrac{1}{2}P_b \, . \, (P_b \, . \, \delta_{bb}) + P_a \, . \, (P_b \, . \, \delta_{ab}) \qquad (2)$$

In the second manner of loading, apply first the force P_b and later the force P_a. Then, repeating the same reasoning as for the first manner of loading, we obtain

$$U_1 = \tfrac{1}{2}P_b \, . \, (P_b \, . \, \delta_{bb}) + \tfrac{1}{2}P_a \, . \, (P_a \, . \, \delta_{aa}) + P_b \, . \, (P_a \, . \, \delta_{ba}) \qquad (3)$$

U must be equal to U_1, therefore equating equations (2) and (3).

$$\therefore P_a \, . \, (P_b \, . \, \delta_{ab}) = P_b \, . \, (P_a \, . \, \delta_{ba})$$

$$\text{i.e. } \delta_{ab} = \delta_{ba} \quad . \quad . \quad . \quad . \quad (4)$$

or, the deflection of 'a' in the direction of P_a when unit load acts at b in the direction of P_b is the same as the deflection at 'b' when unit load is applied at 'a' in the direction of P_a. This is Maxwell's Law (1864).

If $P_a = P_b = P$, then $P^2 \, . \, \delta_{ab} = P^2 \, . \, \delta_{ba}$

$$\text{or } (\delta_{ab}) = (\delta_{ba}) \quad . \quad . \quad . \quad . \quad (5)$$

and a verification of this conclusion for a particular case is given in the Illustrative Problem 9, page 63.

Equation (1) is a simple form of Betti's Law (1872) which extends Maxwell's Law. This theorem can be proved for any number of forces, and also for couples, and for forces and couples.

As an example, consider the bending of a simply-supported prismatic beam of length l. In the first state, it is bent by a load P_c applied at the middle 'c', and in the second state by a bending couple M_d applied at one end 'd'. Unit load in the direction of P_c produces the slope $\theta_{dc} = l^2/16EI$ at 'd,' and a unit couple applied at 'd' in the direction of M_d produces the deflection $l^2/16EI$ at 'c.' Equation (1) gives

$$P_c\left(M_d \cdot \frac{l^2}{16EI}\right) = M_d\left(P_c \cdot \frac{l^2}{16EI}\right) \quad . \quad . \quad (6)$$

or P_c(deflection at c due to M_d) $= M_d$(slope at d due to P_c)

This example shows that a reciprocal relationship exists between a rotation and a deflection: such a relationship may exist between rotations, but care must be taken in selecting the correct relationship. (A rotation must be accompanied by a moment.) The application of Betti's Equation will help.

Considering equation (1),

let $\quad P_b \cdot \delta_{ab} = y_{ab}$ and $P_a \cdot \delta_{ba} = y_{ba}$

Then Betti's equation will read

$$P_a(y_{ab}) = P_b(y_{ba}) \quad . \quad . \quad . \quad (7)$$

Again, considering equation (6),

let $$P_c \cdot \frac{l^2}{16EI} = P_c\theta_{dc} = i_{dc}$$

and $$M_d \cdot \frac{l^2}{16EI} = M_d \cdot \delta_{cd} = y_{cd}$$

Then equation (6) will read

$$P_c \cdot y_{cd} = M_d \cdot i_{dc} \quad . \quad . \quad . \quad (8)$$

If, in equation (7), $P_a = P_b = P$,

then $\quad y_{ab} = y_{ba}$

and if, in equation (8), P_c is of the same magnitude as M_d,

then $\quad y_{cd} = i_{dc}$ in magnitude.

These are particular cases of the reciprocal theorem when considering the total displacements or rotations, for equal values of the forces and couples greater than unity.

Now, let P_a and M_d be a force and a couple in the first state applied at a and d, and let P_b and M_c be a force and a couple in the second state applied at b and c. Let $y_{b(ad)}$ and $i_{c(ad)}$ be the total displacement at b in the direction of P_c and the total rotation at c due to P_a and M_d acting together in the first state. Let $y_{a(bc)}$ and $i_{d(bc)}$ be the displacement at a and the rotation at d in the direction of P_a and M_d due to the load P_b and the couple M_c acting together in the second state.

Then it can be shown that

$$P_a \cdot y_{a(bc)} + M_d \cdot i_{d(bc)} = P_b \cdot y_{b(ad)} + M_c \cdot i_{c(ad)} \quad . \quad (9)$$

Thus, continuing, if a number of forces and couples represented by P_a and M_a act together at representative points a in the first state, and a number of forces and couples represented by P_b and M_b act at representative points b in the second state, then the general form of Betti's equation is

$$\Sigma P_a \cdot y_{ab} + \Sigma M_a \cdot i_{ab} = \Sigma P_b \cdot y_{ba} + \Sigma M_b \cdot i_{ba} \quad . \quad . \quad (10)$$

where y_{ab} and i_{ab} are the displacements and rotations at the points a in the first state due to the forces and couples in the second state, and y_{ba} and i_{ba} are the displacements and rotations at the points b in the second state. The displacements and rotations are in the directions of the respective forces and couples.

EXERCISES.

(1) Show that the slope of a beam at A caused by a moment applied at B is equal to the slope at B when the same moment is applied at A.

(2) Show that the slope of a beam at A caused by a load at B is *not* equal to the slope at B when the same load is applied at A.

(3) Show that the deflection of a beam at A caused by a moment at B is *not* equal to the deflection at B when the moment is applied at A.

Illustrative Problem 61.

A beam ab of uniform section is built-in at a and it is supported on a rigid prop at b. b is at the same elevation as at a. This system is once statically indeterminate. Let R_b be the external redundant. Draw the influence line for the prop reaction R_b, when a load P crosses the span of length l. Neglect displacements due to shear. (See Fig. 268.)

In Fig. 268 (a) is given the actual state of loading; in Fig. 268 (b) is given a second state of loading (imaginary), P having been taken off and R_b replaced by unit load. No work is done by the reactions at the support a. Considering the second stage of stress, it can be shown that the equation of the deflected cantilever is

$$y = \delta_{rb} = \frac{n^3 l^3}{6EI}\left(\frac{3}{n} - 1\right)$$

$$\delta_{bb} = y_{max} = \frac{l^3}{3EI}$$

neglecting any displacement due to shear.

Therefore the work done by the forces of the first state of stress on the displacements of the second state of stress is $P \cdot \delta_{rb} - R_b \cdot \delta_{bb}$

$$\text{or } P \cdot \frac{n^3l^3}{6EI}\left(\frac{3}{n} - 1\right) - R_b\left(\frac{l^3}{3EI}\right) = Y.$$

The work done by the forces of the second state of stress on the displacement of the first state of stress is zero. Thus, using Betti's equation

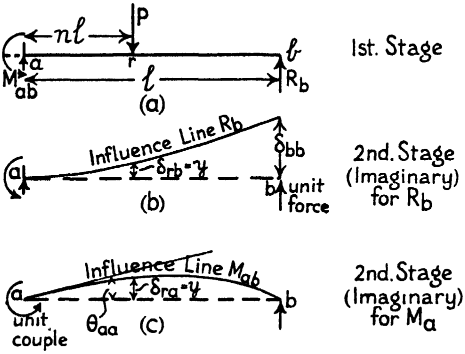


FIG. 268

$$Y = 0$$

$$\therefore R_b = P \cdot \frac{n^2}{2}(3 - n) \qquad . \quad . \quad . \quad (11)$$

and this is the equation of the influence line of R_b, and the deflected beam line in Fig. 268 (*b*) is the influence line of R_b.

Illustrative Problem 62. (See Fig. 268 (*c*).)

Considering the previous problem, let M_{ab} be the external redundant, construct the influence line of the fixing couple M_{ab} at *a* (see Fig. 268 (*c*)). Neglect displacement due to shear. In the second state of stress, the loading is a unit couple applied at *a* and deflecting the beam as shown, imagining a hinge at *b*.

The equation of the displaced beam line can be shown to be

$$y = \delta_{ra} = \frac{nl^2}{6EI}(2 - 3n + n^2)$$

$$\text{and } \theta_{ab} = \frac{l}{3EI}$$

Using the reciprocal theorem and reasoning as before,

$$M_{ab} \cdot \frac{l}{3EI} - P \cdot \frac{nl^2}{6EI}(2 - 3n + n^2) = 0$$

$$\therefore \quad M_{ab} = P \,.\, l \,.\, \frac{n}{2}(2 - 3n + n^2) \,. \qquad . \qquad . \quad (12)$$

in both cases due regard being paid to the sense of the reactions.

∴ The deflected beam line in Fig. 268 (*c*) is the influence line of M_{ab}. The examples considered have been for the determination of external redundants. Considering the first part of the problem:

$$\text{let } n = \frac{1}{2}, \text{ then } R_b = P \cdot \frac{1}{2 \times 4}\left(3 - \frac{1}{2}\right) = \frac{5}{16}P.$$

$$\text{Then } M_{ab} = \frac{5}{16}Pl - \frac{Pl}{2} = \frac{3}{16}Pl$$

From the equation of M_{ab}, for $n = \frac{1}{2}$,

$$M_{ab} = Pl \cdot \frac{1}{2 \times 2}\left(2 - \frac{3}{2} + \frac{1}{4}\right) = \frac{3}{16}Pl$$

Illustrative Problem 63. (See Fig. 268 (*a*) and (*b*).)

Draw the influence line for the prop reaction R_b when a load P crosses the span of length l. Allow for displacement due to shear as well as that due to bending. EI is constant. GA is constant.

Considering Fig. 268 (*b*) for the second state of stress for R_b, let δ_{bbs} and δ_{rbs} be the displacements due to shear at b and r respectively. It was shown in Chapter V, pages 126–128, that the approximate deflection due to shear at a section in a cantilever is equal to $\frac{m}{AG}$ times the area of the shearing force diagram between the origin (support) and the section, where m is the shear coefficient of the section, A is the cross-sectional area, and G is the elastic modulus of the material in shear.

Thus for the case of loading considered, a cantilever with unit load at the free end,

$$\delta_{rbs} = \frac{m}{AG} \times 1 \times nl \quad \text{and} \quad \delta_{bbs} = \frac{m}{AG} \times 1 \times l$$

Then the total deflection at r due to shear and bending is

$$\frac{n^3l^3}{6EI}\left(\frac{3}{n}-1\right)+\frac{m\,.\,nl}{AG}$$

$$\text{and at } b \text{ it is } \frac{l^3}{3EI}+\frac{ml}{AG}$$

Therefore, by the reciprocal theorem,

$$R_b\left(\frac{l^3}{3EI}+\frac{ml}{AG}\right)=P\left\{\frac{n^3l^3}{6EI}\left(\frac{3}{n}-1\right)+\frac{m\,.\,nl}{AG}\right\}$$

Let $X = m\,.\,\frac{E}{G}\cdot\frac{k^2}{l^2}$ where k is the centroidal radius of gyration;

then it can be shown that

$$\frac{R_b}{P}=\frac{n^2(3-n)+6Xn}{2(1+3X)} \quad . \quad . \quad . \quad (13)$$

From this relation, it can further be shown that

$$\frac{M_{ab}}{Pl}=\frac{n(n-1)(n-2)}{2(1+3X)} \quad . \quad . \quad . \quad (14)$$

NOTE. It can be shown for a prismatic beam AB, direction-fixed at both ends, if allowance is made for displacement due to shear, that the end-fixing moments M_{FAB} and M_{FBA}, when a load P is at a distance of nl from the support A, are

$$M_{\text{FAB}} = Pl_{\text{AB}}\{n(1-n)(1-n+6X)\}/(1+12X) \quad . \quad (15)$$

$$M_{\text{FBA}} = Pl_{\text{AB}}\{n(1-n)(n+6X)\}/(1+12X) \quad . \quad (16)$$

where $X = m\,.\,\frac{E}{G}\cdot\frac{k^2_{\text{AB}}}{l^2_{\text{AB}}}$.

It can however be shown, for rectangular sections, if l/d is > 20, where l is the length of a fixed-ended beam and d is the depth of the section, that the effect of the shearing force strain on the end-fixing couples is negligible compared with that due to flexure.

The effect of shearing forces on the angles of rotation at the ends of beams is discussed by Timoshenko and Young,* and their effects on the fixed-end moments by Professor Misé, and A. Amirikian in his book *Analysis of Rigid Frames.*

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* Timoshenko and Young: *Theory of Structures.* (McGraw-Hill Book Co.)

Illustrative Problem 64. (Fig. 269.)

A continuous beam ACB is simply-supported at A and B and continuous over an intermediate rigid support at C. All supports are at the same elevation. A load P can travel over the span CB (see Fig. 269 (a)). EI is constant. Neglect displacements due to shear.

This structure is once statically indeterminate. Construct the influence lines (a) of the reactions R_C and R_B of the supports C and B, and (b) the bending moment M_C at C. See Figs. 269 (b), (c), and (d).

Influence Line of* R_C (see Fig. 269 (a) and (b)). Remove P and replace the support C by unit force acting upwards carrying a displacement δ_{cc} at C and δ_{rc} at r. Then by the reasoning of the previous problem

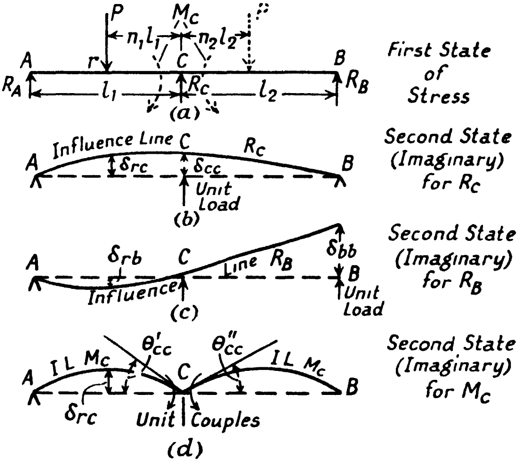

FIG. 269

$$R_C \,.\, \delta_{cc} - P \,.\, \delta_{rc} = 0$$

$$\therefore \quad R_C = P \,.\, \frac{\delta_{rc}}{\delta_{cc}}$$

$\therefore$ the displaced beam line in Fig. (b) is the influence line of R_C.

Influence Line of R_B (see Fig. 269 (a) and (c)). As before,

$$R_B \,.\, \delta_{bb} = P \,.\, \delta_{rb}$$

$$\therefore R_B = P \,.\, \frac{\delta_{rb}}{\delta_{bb}}$$

$\therefore$ the displaced beam line in Fig. (c) is the influence line of R_B.

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* The lines given in Fig. 269 (b), (c), and (d) are those which would be obtained by the mechanical solution of the problem. R_C and R_B are external redundants. M_C is an internal redundant. See para. 220.

Note the sense of action of R_B changes sign when the load P leaves the span AC and enters the span CB. In this part of the problem, R_B and R_C have been external redundants.

Influence Line of M_C. An example of the determination of an internal redundant. (See Fig. 269 (*a*) and (*d*).) The first state of stress is the actual state. There is a bending moment M_C at the support C acting in both spans AC and CB. For the second state of stress remove the load P and cut the beam to the left and right of C, and replace M_C by two equal and opposite unit couples, one acting on the span AC and the other on the span CB. This case is now statically determinate and we have two beams displaced similarly as in the illustrative problem 62 p. 486. The deflected beam lines are as indicated in Fig. 269 (*d*), and they are the influence lines for M_C.

In the first state of stress M_C does no external work as there is no cut and there is no displacement of C.

Using the reciprocal theorem and Betti's equation,

$$M_C(\theta'_{cc} + \theta''_{cc}) - P \,.\, \delta_{rc} = 0$$

$$\therefore \quad M_C = P \,.\, \frac{\delta_{rc}}{\theta'_{cc} + \theta''_{cc}}$$

$$\text{But } \theta'_{cc} + \theta''_{cc} = 1 \,.\, \frac{(l_1 + l_2)}{3EI}$$

$$\therefore \quad M_C = \frac{P \,.\, 3\delta_{rc}}{1 \,.\, (l_1 + l_2)} \,.\, EI$$

$\therefore$ M_C is proportional to δ_{rc} and the deflected beam lines in the second state (Fig. 269 (*d*)) are the influence line for M_C.

Taking C as origin, if P when on span AC is distant $n_1 l_1$ from C, $\delta_{rc} = \frac{n_1 l_1^2}{6EI}(2 - 3n_1 + n_1^2)$ for M_C = unity.

Taking C as origin, if P when on span CB is distant $n_2 l_2$ from C, $\delta_{rc} = \frac{n_2 l_2^2}{6EI}(2 - 3n_2 + n_2^2)$ for M_C = unity.

EXERCISE ON THE CONTINUOUS BEAM PROBLEM (Fig. 269).

If $l_1 = l_2 = 100$ ft. and P is 4 tons, (*a*) calculate the value and sense of action of R_B when P is at the middle of the first span, and (*b*) calculate the value of M_C when P is at the middle of the first span.

Ans. (*a*) 3/8 ton; (*b*) 37·5 tons-ft.

PROBLEMS

1. A beam AB is direction-fixed at both ends: show by the reciprocal theorem that $M_{AB} = Pl \, . \, n(1 - n)^2$, where P is a transverse load at a distance nl from A. EI is a constant. Sketch the influence line of M_{AB}.

2. Referring to Fig. 117, Illustrative Problem 28, page 183, show by the reciprocal theorem that if a load of 50 tons is applied at the joint A_1, then the deflection at A is the same as that at A_1 when the 50 tons was applied at A.

Use the formula $y = \sum \dfrac{F \, . \, u \, . \, l}{A \, . \, E}$.

3. A beam ACB is continuous over a rigid support C and simply-supported at A and B. $AC = CB = 2l$. EI is constant. Show by the reciprocal theorem that when a load P is at a distant l from A the reaction R_c at C is equal to $11/16P$. Sketch the influence line of R_c.

220. An example showing the use of the reciprocal theorem for the analysis of a structure in which there are two (external) redundants to be found.

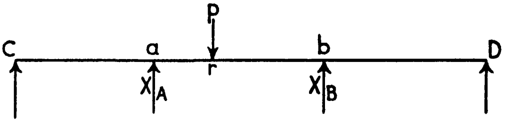


FIG. 270

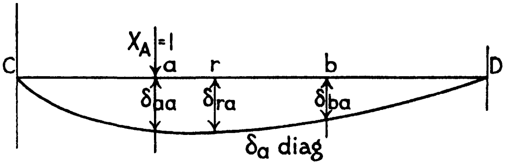


FIG. 271

Consider a beam $CabD$ simply-supported at the ends C and D, and continuous over two intermediate rigid supports at a and b (Fig. 270).

Let a load P be at any position r; this system is twice statically indeterminate. Let the two redundant quantities be the support reactions X_a and X_b acting at a and b respectively.

The principal *statically determinate system* is the simply-supported beam CD. Remove the load P and the two unknowns X_a and X_b.

(1) Apply at 'a' a load X_a = unity to the simply-supported beam CD and we get the displacement diagram given in Fig. 271.

The displacements are—

at a, δ_{aa} = displacement at a due to unit load at a;
at r, δ_{ra} = displacement at r due to unit load at a;
at b, δ_{ba} = displacement at b due to unit load at a.

By Maxwell's reciprocal theorem—

$\delta_{ra} = \delta_{ar}$ = displacement at a due to unit load at r;
$\delta_{rb} = \delta_{br}$ = displacement at b due to unit load at r;
$\delta_{ba} = \delta_{ab}$ = displacement at a due to unit load at b.

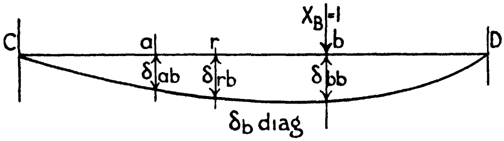


FIG. 272

(2) Similarly consider the simply-supported beam CD with X_b = unity acting at b, and we get the displacement diagram Fig. 272.

The displacements are—

at b, δ_{bb} = displacement at b due to unit load at b;
at r, δ_{rb} = displacement at r due to unit load at b;
at a, δ_{ab} = displacement at a due to unit load at b.
Also $\delta_{rb} = \delta_{br}$ and $\delta_{ab} = \delta_{ba}$.

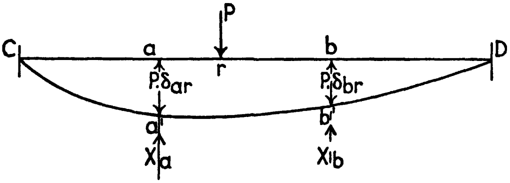


FIG. 273

To solve for X_a and X_b, consider the simply-supported beam CD with the load P acting on it at r.

Then the displacement at a will be $P \cdot \delta_{ar}$ which, by Maxwell's reciprocal theorem, is equal to $P \cdot \delta_{ra}$.

Now place on the beam at a', the displacement position of a, the reaction X_a (vertically upwards). Now the total displacement upwards due to X_a will be $X_a \cdot \delta_{aa}$.

Now place on the beam at b the reaction X_b acting vertically upwards; then the displacement at a due to X_b will be $X_b \, . \, \delta_{ab}$ or $X_b \, . \, \delta_{ba}$. If there is no displacement of the support at a, then

$$P\delta_{ar} = X_a \, . \, \delta_{aa} + X_b \, . \, \delta_{ab}$$

or

$$P\delta_{ra} = X_a \, . \, \delta_{aa} + X_b \, . \, \delta_{ba} \qquad . \quad . \quad (17)$$

Similarly we find

$$P \, . \, \delta_{br} = X_a \, . \, \delta_{ba} + X_b \, . \, \delta_{bb}$$

or

$$P \, . \, \delta_{rb} = X_a \, . \, \delta_{ab} + X_b \, . \, \delta_{bb} \qquad . \quad . \quad (18)$$

Considering the equations (17) and (18) and solving for X_a and X_b,

$$P_{ra} = X_a \, . \, \delta_{aa} + X_b \, . \, \delta_{ba}$$

$$P_{rb} = X_a \, . \, \delta_{ab} + X_b \, . \, \delta_{bb}$$

Noting that $\delta_{ab} = \delta_{ba}$,

$$\text{we find } X_a = \frac{\left(\delta_{ra} - \delta_{rb} \cdot \dfrac{\delta_{ba}}{\delta_{bb}}\right)}{\left(\delta_{aa} - \delta_{ab} \cdot \dfrac{\delta_{ba}}{\delta_{bb}}\right)} \, . \, P = \frac{\Delta_{ra}}{\Delta_{aa}} \, . \, P \qquad . \quad (19)$$

$$\text{and } X_b = \frac{\left(\delta_{rb} - \delta_{ra} \cdot \dfrac{\delta_{ab}}{\delta_{aa}}\right)}{\left(\delta_{bb} - \delta_{ba} \cdot \dfrac{\delta_{ab}}{\delta_{aa}}\right)} \, . \, P = \frac{\Delta_{rb}}{\Delta_{bb}} \, . \, P \qquad . \quad (20)$$

Interpretations of Equation (19). Considering the denominator of equation (19),

δ_{aa} is reduced by $\delta_{ab} \times \dfrac{\delta_{ba}}{\delta_{bb}}$,

i.e. the displacement at a due to unit load at b ($= \delta_{ab}$) multiplied by the ratio

$$\frac{\delta_{ba}}{\delta_{bb}} = \frac{\text{displacement at } b \text{ due to unit load at } a}{\text{displacement at } b \text{ due to unit load at } b}$$

Considering the numerator of equation (19),

δ_{ra} the displacement at r due to unit load at a is reduced by

$$\delta_{rb} \times \frac{\delta_{ba}}{\delta_{bb}},$$

i.e. the displacement at r due to unit load at b multiplied by

$$\frac{\text{displacement at } b \text{ due to unit load at } a}{\text{displacement at } b \text{ due to unit load at } b}$$

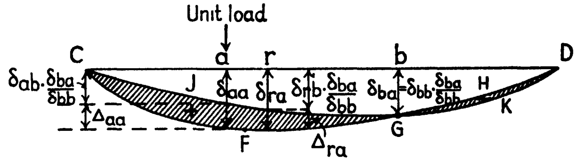
If the load P is at $r = b$, then

$$\delta_{rb} \times \frac{\delta_{ba}}{\delta_{bb}} = \delta_{bb} \times \frac{\delta_{ba}}{\delta_{bb}} = \delta_{ba}$$

also δ_{ra} becomes $= \delta_{ba}$

and the numerator of equation (19) is equal to zero.

Therefore, if all the ordinates of the displacement diagram


Unit load
C
a
r
b
D
$\delta_{ab} \cdot \frac{\delta_{ba}}{\delta_{bb}}$
J
δ_{aa}
δ_{ra}
$\delta_{rb} \cdot \frac{\delta_{ba}}{\delta_{bb}}$
$\delta_{ba} = \delta_{bb} \cdot \frac{\delta_{ba}}{\delta_{bb}}$
H
K
Δ_{aa}
F
Δ_{ra}
G

CFGHD is the δ_a diag. CJGKD is the δ_b diag reduced by δ_{ba}/δ_{bb}

FIG. 274

for the principal system (the simply-supported beam CD) with unit load at b are reduced in the ratio of $\frac{\delta_{ba}}{\delta_{bb}}$ and the new diagram superimposed on the displacement diagram for the simply-supported beam CD with unit load at a, then we get the above displacement diagram, Fig. 274, which is the diagram for the beam $CabD$ with only the support a and therefore the unknown Xa removed from the system.

Equation (20) is interpreted similarly.

The Previous Discussion by the Reciprocal Theorem Method. Fig. 270 shows the actual load conditions of the beam; consider it as the first state of stress. A second, and imaginary second, state of stress is shown in Fig. 274. The external load and one redundant X_a have been removed. The support at b has been kept on: the reaction here does no work. A unit load is applied at a, causing a displacement of Δ_{aa} at a, and a displacement Δ_{ra} at r. Then by previous reasoning and using Betti's equation,

$$X_a \cdot \Delta_{aa} - P \cdot \Delta_{ra} = 0$$

$$\therefore \quad X_a = P \cdot \frac{\Delta_{ra}}{\Delta_{aa}}$$

Similarly for X_b.

The second, or imaginary, state of stress is for the condition of unit load applied at b after removing the support at b and the load P and retaining the support at a. A displacement of Δ_{bb} at b occurs and Δ_{rb} at b. Then

$$X_b \,.\, \Delta_{bb} - P \,.\, \Delta_{rb} = 0$$

$$\therefore \quad X_b = P \,.\, \frac{\Delta_{rb}}{\Delta_{bb}}$$

The influence lines for X_a and X_b are therefore the respective displaced beam lines: for X_a when the support at a is removed and that at b retained, and for X_b when the support at b is removed and the support at a retained.

In the mechanical solution to be discussed later, it is the shaded diagram which is obtained. There is no displacement at b for unit load at r, that is the support at b remains on the structure and the only one removed is that at a. Here a known displacement Δ_{aa} is caused at a, and the corresponding displacements Δ_{ra} are measured at points r.

$$\text{Then} \quad X_a = P \,.\, \frac{\Delta_{ra}}{\Delta_{aa}} \qquad \text{. . . . . (21)}$$

Similarly for X_b. There is no displacement of the beam at a, that is, the support remains on the structure. At b, a known displacement Δ_{bb} placed on the beam and the corresponding displacements at points r equal to Δ_{rb} are measured.

$$\text{Then} \quad X_b = P \,.\, \frac{\Delta_{rb}}{\Delta_{bb}} \qquad \text{. . . . . (22)}$$

In Fig. 274 the ordinates of the shaded diagram are those of the influence line of X_a to scale. In practice they would be plotted on a horizontal base.

If there was a third redundant X_c, then X_c would be obtained from the equation

$$X_c = P \times \frac{\Delta_{rc}}{\Delta_{cc}} \qquad \text{. . . . . (23)}$$

If X_c is a couple, then Δ_{cc} represents an angular movement in radian measure: see later discussion on p. 497.

This is the mechanical solution; the method of attack is as follows: keep all the redundants on the structure, except the unknown which is required. Release this one, and at the place at which it acts cause a known displacement of, say, Δ_{aa}, then by some means measure the displacement Δ_{ra} at the point of application of the load P at the position r. Then the unknown required (say) $X_a = P \cdot \frac{\Delta_{ra}}{\Delta_{aa}}$.

Vary the position r, then Δ_{ra} will vary and then we are able to plot the corresponding value of X_a against the position r, so that we are able to obtain an influence line for X_a.

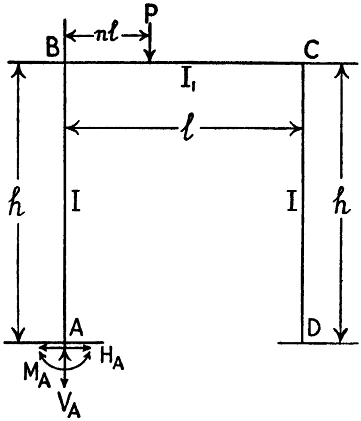


Fig. 275

221. Before proceeding to a discussion of the mechanical solution of statically indeterminate structures based on the reciprocal theorem, consider the solution of a simple portal *ABCD* (Fig. 275) fixed at the bases *A* and *D*. Let the lengths of the columns be h and the length of the beam l: let the moment of inertia of the two columns be I and that of the beam I_1. Let a load P be applied at r distant nl from B. The frame is three times statically indeterminate: let the three redundants be the components of reaction at the base A. These are the vertical force V_A, the horizontal force H_A, and the couple M_A.

The student should prove the following relations by the slope-deflection method. Assuming that E for all members is the same and neglecting axial and shearing force effects:

$$\frac{M_A}{Pl} = -n(1-n)\left\{\frac{1}{2(2+gi)} - \frac{1-2n}{2(1+6gi)}\right\} \quad . \quad (24)$$

$$\frac{H_A}{P} = n(1-n) \cdot \frac{g}{\frac{2}{3}(2+gi)} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$\frac{V_A}{P} = (1-n)\left\{1 + \frac{n(1-2n)}{(1+6gi)}\right\} \quad . \quad . \quad . \quad (26)$$

where $g = h/l$ and $i = I_1/I$.

If $g = 1$ and $i = 1$,

$$\frac{M_A}{Pl} = -n(1-n)\left\{\frac{1}{6} - \frac{1-2n}{14}\right\} \quad . \quad . \quad . \quad (27)$$

$$\frac{H_A}{P} = n(1-n)\,.\,\tfrac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$\frac{V_A}{P} = (1-n)\left\{1 + \frac{n(1-2n)}{7}\right\} \quad . \quad . \quad . \quad (29)$$

If the members have different values of E, say E for the columns and E_1 for the beam, and letting $E_1/E = e$, then the equations become

$$\frac{M_A}{Pl} = -n(1-n)\left\{\frac{1}{1+2lgi} - \frac{1-2n}{2(1+6egi)}\right\} \quad . \quad (30)$$

$$\frac{H_A}{P} = n(1-n)\left\{\frac{g}{\frac{2}{3}(2+egi)}\right\} \quad . \quad . \quad . \quad . \quad (31)$$

$$\frac{V_A}{P} = (1-n)\left\{1 + \frac{n(1-2n)}{1+6egi}\right\} \quad . \quad . \quad . \quad (32)$$

EXERCISE.

Develop expressions for the sway angle ϕ of the columns, the rotations θ_B and θ_C of the joints B and C respectively in the problem, and draw the influence lines of ϕ, θ_C, θ_B, M_A/Pl, H_A/P, and V_A/P for the case when $g = 1$ and $i = 1$, and $e = 1$.

Let M_A correspond to X_C, H_A to X_B, and V_A to X_A.

By the reciprocal theorem, we have

$$\frac{X_a}{P} = \frac{\Delta_{ra}}{\Delta_{aa}}; \quad \frac{X_b}{P} = \frac{\Delta_{rb}}{\Delta_{bb}};$$

and these displacement ratios can replace the force ratios in the corresponding previously developed equations (31) and (32).

Let X_C correspond to M_A, then $\frac{X_c}{P} = \frac{\Delta_{rc}}{\theta^c{}_{cc}}$, where $\theta^c{}_{cc}$ = angular displacement in radians at C: let $\theta^c{}_{cc} = \frac{\Delta_{cc}}{1} = \frac{\Delta_{c'c'}}{l}$ (see Fig. 276 where Δ_{cc} is the displacement of c at unit distance from the centre of rotation, and $\Delta_{c'c'}$ is that at distance l from the centre of rotation at C.

$$\text{Then} \qquad \frac{X_c}{Pl} = \frac{\Delta_{rc}}{\Delta_{c'c'}} \text{ or } \frac{X_c}{P \times 1} = \frac{\Delta_{rc}}{\Delta_{cc}}$$

The displacement ratio $\frac{\Delta_{rc}}{\Delta_{c'c'}}$ or $\frac{\Delta_{rc}}{\Delta_{cc}}$ can replace the moment ratio in equation (30), assuming again that shear and axial force displacements are negligible.

Conclusion. We have now seen that a force or moment ratio is equal to a displacement ratio which is equal to a dimensionless expression incorporating the position of the load, the properties of the material, the lengths and moments of inertia of the members.

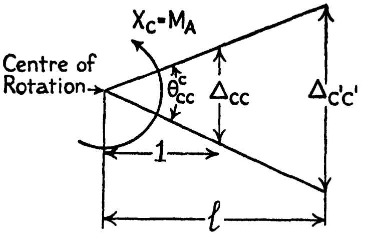


FIG. 276

222. The Mechanical Solution of Statically Indeterminate Structures Based on the Reciprocal Theorem. It has been shown that the deflection curve for any continuous structure due to a unit load or couple replacing any redundant is to some scale the influence line of the redundant. This fact makes possible the mechanical analysis of redundant structures by the use of models. Methods of analysis based on this principle have been developed. Various methods range from crude paper models or timber splines to others which make use of precise instruments to create and measure deflections of accurately constructed models of celluloid or other uniform material.

When a mathematical theory is used, ideal conditions are assumed. Loads are assumed to act at points: materials are assumed to be perfectly elastic; centre line dimensions are used neglecting the thickness of members; the effects of brackets, fillets, gusset plates, etc., are neglected. With a model, one deals with a fabricated structure; loads not point loads: no material is perfectly elastic, all members have thickness: and dimensions are subject to human errors. The accuracy of the results obtained from the use of a model will entirely depend on how accurately the model represents its full scale structure or prototype. The same is true of mathematical theories. The object of any mathematical analysis is to learn as much as possible about the probable distribution of stresses in an imperfect fabricated structure.

It has also been shown that considering two points on the deflection curve of a continuous structure for unit load applied at one point, that the ratio of displacements of these points is equal to a force ratio at these points and that this force ratio by means of a mathematical analysis is equal to a dimensionless expression or quantity incorporating the position of the load, the properties of the material, the lengths and properties (moments of inertia, radius of gyration) of the members. In fact, there have been developed a number of dimensionless equations, which are derived equations of similitude. Thus we can have any number of systems, having the same displacement ratio, the same force ratio for the same dimensionless structural quantity: i.e. having the same geometrical form of displacement curve.

Let us consider the dimensionless equations for the simple portal with the column bases fixed. Considering the cases where E is the same for all members and neglecting axial and shearing force effects, it will be seen that the force and moment ratios depend upon n, and the ratios g and i, where $g = h/l$ and $i = I_1/I$; h and l represent the centre line lengths of their respective members. This is a case of designing a model to check the analytical analysis and to design the model with respect to assumptions made in the theoretical analysis. Now for the two systems: we shall have the same force or moment ratio when n is the same in the two systems, g is the same and i is the same. Thus when we make a reduced scale model of the prototype, the corresponding g ratios will all be the same: i.e. all centre line dimensions will be set out to the same scale. To fix the i ratios: as shear does not play a part, we can build the model with all members having rectangular sections—although in the prototype the sections of the corresponding members will usually be of other sections, I, channel, angle, etc., and these sections can all be of the same breadth b. Thus when considering "planar" structures we can fabricate our model out of a sheet of uniform material and of uniform thickness. Then if I_1 and I are the moments of inertia of the two members in the prototype and I_{1m} and I_m are the corresponding moments of inertia in the model, then

$$\frac{I_{1m}}{I_m} = \frac{I_1}{I} = \frac{bd_1{}^3{}_m/12}{bd_m{}^3/12} = \frac{d_1{}^3{}_m}{d^3{}_m}$$

where d_{1m} and d are the depths of the rectangles equal to the model: or

$$\frac{d_{1m}}{d_m} = \sqrt[3]{\frac{I_1}{I}}$$

The model is therefore a distorted structure when compared with the prototype.

We can now briefly describe Professor Beggs' method of the mechanical analysis of statically indeterminate structures. The line diagram of the model is geometrically similar to that of the prototype, and the depths of the rectangular sections of the model are proportional to the cube root of the moments of inertia. The breadths of the sections in the model are constant. Celluloid is chiefly used in the construction of the model, and, as celluloid which has been kept in stock for some months is less subject to shrinking than new celluloid, it is recommended that celluloid be seasoned. The model is cut out of the sheet of celluloid. Professor Beggs' apparatus consists of a number of gauges which, in the case of an external redundant, are fixed one part to a drawing board and one part to the model at the section at which it is required to determine the redundant. The gauge consists of two parallel steel bars, with pairs of opposing V-notches, held together by springs to allow a small relative motion between the bars. Very precise gauge plugs of accurately specified sizes (tolerance plus or minus 0·0002 in.) are provided for introduction between the bars, for the purpose of determining amounts of thrust, shear, and rotative displacements. The thrust and shear displacements employed in practice equal about 0·050 in. It is necessary to use small deflections with the aid of elastic models because the ratio of deflection is theoretically and practically correct only when the model is deformed a small microscopically measurable distance from its geometric shape. In an external redundant determination, the so-called "fixed" bar of the gauge is secured to a drawing board. By means of a clamping plate, the model is attached to the movable bar of the gauge if the support is assumed fixed, or by a needle point if the support is assumed hinged. The model is supported at intervals on one-eighth-inch steel balls. In the unstrained position of the model, two "normal" gauge plugs remain in the two pairs of opposing V notches of the gauge. A filar micrometer microscope is set up over the corresponding assumed position of the load in the prototype, and the scale of the microscope is set in the direction of the applied load. The different sets of plugs are calibrated in units of the micrometer head. Suppose the vertical thrust (or pull) component of an external reaction at the support is required. The normal plugs are removed from the gauge and a

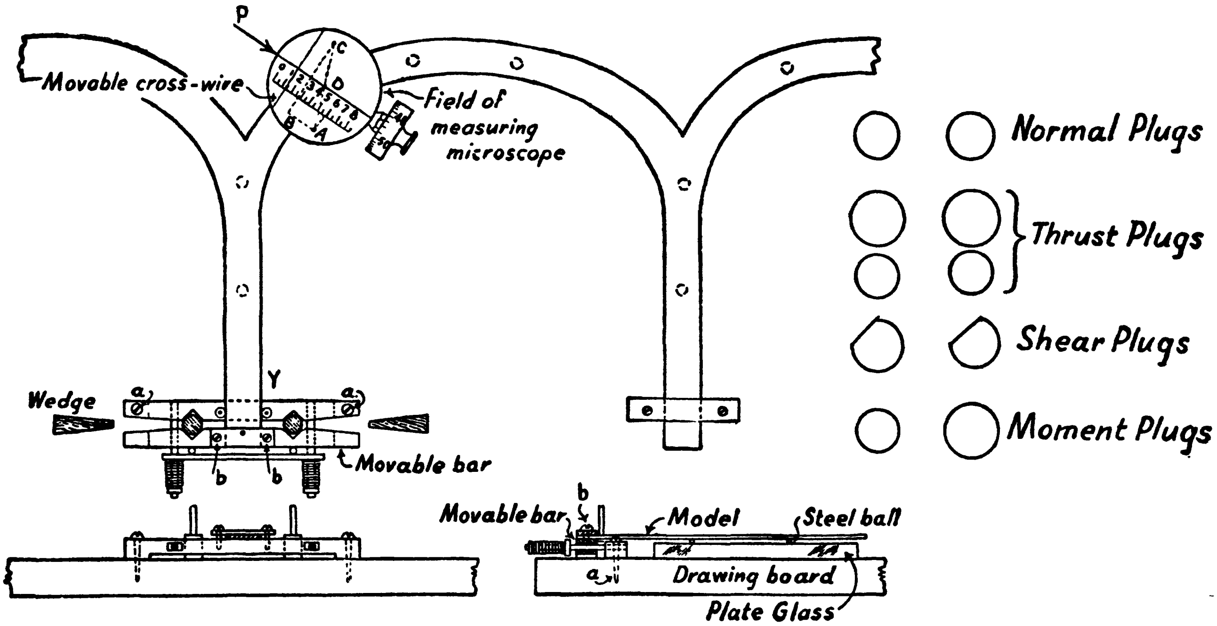


FIG. 277. BEGGS' APPARATUS

(*Institution of Structural Engineers*)

pair of smaller "thrust" plugs inserted, so causing a vertical displacement of the support. The moving cross-wire of the micrometer is now brought tangential to a reference mark on the model at the assumed load point and an initial reading of the micrometer taken. The small thrust plugs are now removed and larger thrust plugs are inserted in the gauge. The reference mark will move across the field of view, and again the moving cross-hair is brought tangential to the reference mark. A second reading is taken. The difference between the two micrometer readings represents the displacement in the direction of the applied load. The required component of the reaction now equals the applied load multiplied by the ratio of two known displacements, namely, the measured one Δ_{ra} at point r and the gauge displacement Δ_{aa}. Then $X_a = P \cdot \frac{\Delta_{ra}}{\Delta_{aa}}$. (NOTE. There has been no rotation or horizontal displacement of the support.) To find the horizontal component of the reaction, the movable bar is caused to move parallel to the fixed bar by a pair of "shear" plugs, the displacements of the point of the applied being measured as before. Let Δ_{bb} be the gauge movement and Δ_{rb} be the tangent movement. Then $X_b = P \cdot \frac{\Delta_{rb}}{\Delta_{bb}}$.

In this case no vertical or rotative movement of the support has been allowed. To determine the rotative, or moment, component of the reaction, the support is rotated through a very small angle determined by the rotative movement of the movable bar with reference to the fixed bar. The movable bar is caused to rotate by a pair of 'moment' plugs: one plug is slightly larger than the other. These are inserted in the V-notches and a reading of the micrometer at the load point taken. The plugs are reversed and another micrometer reading taken. The gauges are so arranged then, that the centre of rotation remains fixed and there is no 'vertical' or 'horizontal' movement of the bars. The gauges and plugs have been calibrated so that the movement at unit distance $abc(\Delta_{cc})$ along the bar from the centre of rotation is known. Now let one inch of the model correspond to N feet of the prototype. Let Δ_{rc} be the displacement of the reference point in the direction of the applied load P lb. (of the prototype).

Let X_c be the moment component of the redundant reaction for the prototype in lb.-ft. units; then it can be shown that

$X_c = P$ (lb.) . $\dfrac{\Delta_{rc}}{\Delta_{cc}/1 \text{ inch (model)}}$. N (ft.), where $\Delta_{cc}/1$ inch corresponds to θ_{cc} radians, and 1 inch of model corresponds to N feet of the prototype.

It is only in connection with moment determinations that the scale ratio of the model and prototype is taken account of. This has been noted in the theoretical analysis, for a moment ratio contains a length dimension of the structure.

Thus the operations can be repeated for a number of points corresponding to different n values and thus the influence line for the unknown can be obtained.

SIGNS. The microscopes employed in measuring deflections are optically inverting, so that the image in the microscope moves in an opposite direction from the observed load point. Accordingly, the following general rule for determining the sense of any reaction or stress is formulated. If the image of the load point in the microscope moves in the direction of the assumed load, the reaction component acts in the same direction as the corresponding gauge displacement of the support: if the load point appears to move in the opposite direction to the assumed load, the reaction component acts in the opposite direction as the corresponding gauge displacement of the support. When carrying out experiments, notes are made of the sense and direction of these displacements.

Care must be taken in operating the gauges and in the positioning of the microscope scale. Temperature may affect the readings and if a large number of microscopes are used, then the temperature of the room in which the experiment takes place should remain constant. Using only a single microscope and performing the gauge operations for every target point, the author has found that temperature changes are not of serious consequence.

For internal redundant determinations, a member is cut in two; the cut ends are fixed to the gauge bars and these are mounted on a frictionless bearing consisting of two glass plates in between which are steel balls. The operations are carried out as described for external redundants.

A detail drawing of the deformeter apparatus is shown in Fig. 277, p. 501. For axial force and shearing force strain effects on the moment distribution to be minimized as much as possible, the ratio l_m/d_m should, say, be greater than 20 for any member in the model. The effect of the size of the connections will then also be made as small as possible. Further work, however, is required in the matter of the design of distorted models for the probable time stress determination of their prototypes. There is a second consideration in the design of models, the design of a model to give the time stress analysis of its prototype. This can be obtained by having a model which is an exact replica of the prototype as regards material, sections, connections, etc., but the further problem arises, is it feasible to experiment on such a model? Further research work is required in this connection.

SUMMARY. To design a planar rigid frame model of uniform thickness on the basis that the moment distribution will be due to flexure only, the sections of the members can be rectangular, and the depths of pairs of members to have the ratio $\frac{d}{d_1} = \sqrt[3]{\frac{I}{I_1}}$, where I and I_1 are the moments of inertia of the corresponding members in the prototype. The ratio of the length of a model member (centre line length) to the depth of its rectangular section should be greater than 20. If the prototype members have the same value of E, then the model can be cut out of a sheet of isotropic material, such as celluloid.

Experimental analysis of the portal for which the equations for V_A, H_A *and* M_A *are given in paragraph* 221, *by means of Beggs' Deformeter Apparatus.* The theoretical results have been obtained from equations (27), (28), and (29): i.e. for members having the same centre line dimensions and the same section, and the ratio $e = 1$.

The following table gives the experimental and theoretical results for a plane celluloid frame (fixed base portal) having

n	V_A (lb.)		H_A (lb.)		M_A (lb.-in.)	
	Experimentally	Theoretically	Experimentally	Theoretically	Experimentally	Theoretically
0	1·00	1·00	0	0	0	0
0·1	0·92	0·91	0·05	0·05	0·02	0·10
0·2	0·84	0·82	0·10	0·08	0·16	0·20
0·3	0·73	0·71	0·11	0·11	0·27	0·29
0·4	0·63	0·61	0·133	0·12	0·33	0·37
0·5	0·51	0·50	0·131	0·13	0·37	0·42
0·6	0·42	0·39	0·130	0·12	0·38	0·43
0·7	0·30	0·29	0·12	0·11	0·35	0·41
0·8	0·18	0·19	0·07	0·08	0·26	0·34
0·9	0·08	0·09	0·04	0·05	0·13	0·20

$h = l = 10$ in.; sections 0·08 in. thick by 0·6 in. deep. The beam length BC was divided into inches, so that values of n varied from 0 by 0·1 to 1·0. P assumed = 1 lb., V_A always acts upwards: H_A from left to right and M_A in an anti-clockwise direction. The model has been treated as a full-scale structure.

The difference between the experimental and theoretical values, especially for M_A, is probably mainly due to the fact that in the celluloid frame the joints B and C have size, whilst in the theoretical analysis they are points at B and C. Shear and axial force strains have, also, slightly affected the moment-distribution for the l/d ratios are equal to $10/0{\cdot}6 = 16{\cdot}66$.

In the experimental frame the beam length between the columns is 9·40 in.; the centre line length is 10·0 in., thus we introduce another linear ratio $9{\cdot}40/10{\cdot}0 = 0{\cdot}94$. The corresponding ratio for the columns is $9{\cdot}70/10{\cdot}0 = 0{\cdot}97$. These ratios are unity in the theoretical analysis and it will be left to experiment to determine the precise effects of these ratios.

The results of V_A, H_A and M_A can be plotted against n, to give influence lines for these quantities.

A complete analysis of a multi-storey structure is given in the *Structural Engineer*, October 1930, in a paper by the author and Mr. H. V. Lawton (see References).

A second type of device is the Continostat-Gottschalk. A model frame can be built up of steel-splines of about $\frac{3}{8}$ in. in width and of variable thickness. On a straight edge are mounted knife edge supports for the splines. These supports can move along the straight edge about right angles to it. The displacements are large and the displaced elastic lines of the splines give the influence line required. The necessary displacements can be scaled off. No microscopes are required and this apparatus is therefore cheaper than the Beggs' Deformeter apparatus. There are other methods of analysis of structures by means of models, and a description and a large number of references are given in *The Fundamentals of Indeterminate Structures*, by F. L. Plummer (Pitman Publishing Corporation).

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CHAPTER XVII

PRE-STRESSED CONCRETE

223. Many building materials have distinct limitations in their resistance to internal forces in that they generally can withstand only low-tensile stresses as compared with their compressive strengths. Concrete is one of these materials. The invention of reinforced concrete was prompted by the intention of overcoming this weakness. Since the theoretical principles of the basic properties of what is now called normal reinforced concrete had been discovered, practical work developed on these lines. However, for some time the idea of imparting an initial compressive stress to the concrete had been mooted; but the true reason for doing this was not recognized, i.e. the basic purpose of initially compressing the concrete is to eliminate or reduce tensile stresses in the concrete. Serious progress towards making the initially compressed or pre-stressed concrete an economical and competitive structural engineering process was not made until E. Freyssinet* discovered the *sine qua non* of success was the use of high-quality concrete and high-tensile steel capable of resisting very high stresses, so as to allow preliminary stresses of such magnitude that adequate pre-compression would be retained after all losses of stress due to various factors had taken place. Once this basic principle was understood, the engineers' task was limited to the development of suitable methods of pre-stressing; the development of suitable steel could be left to the metallurgist, and the development of suitable concrete to the concrete technologist.

In the fundamentals of design, and especially for the simple case of beams, to which a large proportion of pre-stressed concrete work is related, no special theory is necessary for the stress analysis for the conditions arising at the initial pre-compression of the concrete; nor, in most cases, for the estimation of the stresses, when the beam is under full working load.

The usual basic design principle is that for these conditions there should be no cracking of the member, hence the stresses may be reasonably calculated according to the classic theory of elasticity. It has been said that the only type of reinforced concrete which will eventually be used will be pre-stressed concrete. A complete study of the subject requires recourse to many textbooks and publications, references to a number of which are given at the end of this chapter. Some give information on the design of concrete mixes of high strength and quality.

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* E. Freyssinet, "New Ideas and Methods," *Science et Ind.* (*Travaux*), Vol. 17 (1933), No. 1, pp. 1–17.

224. Definitions.

Shrinkage of Concrete is the volume change due to the drying out of the cement gel between the aggregate particles. There may be also some shrinkage of the aggregate itself. The unit shrinkage strain is determined by tests on unloaded specimens of plain or reinforced concrete.

Creep or Plastic Flow of Concrete is regarded as the additional volume change of loaded specimens obtained by subtracting from the total strain (at time $= t$) the elastic strain due to the application of the load (at time $= 0$) and the skrinkage strain as defined above.

For Steel, if the total strain at a particular stress is greater than the strain at the limit of proportionality, then the steel will continue to deform under this stress; that is *creep* or *plastic flow* will take place.

Permanent Set. If, during the elastic stage of say a tension test on a bar of high-tensile steel, the experiment be stopped and the load removed, the extension that had occurred will be found to have disappeared. If the bar had passed the proportional limit (i.e. the stress at which the strain ceases to be proportional to the stress) and the load be wholly removed, it will be found that the bar has been permanently lengthened. This permanent extension is called *permanent set* or sometimes *set.* This argument will also apply to materials subjected to compression and torsion tests.

Proof Stress or Yield Strength. This is the stress at which a material exhibits a specified limiting permanent set. It is determined from the stress-strain diagram of, e.g., a mild or high-tensile steel. A previously specified offset is laid off along the strain axis from the point corresponding to zero stress and zero strain, and a line is drawn from the end of the offset and parallel to the initial tangent to the stress-strain curve.

The stress corresponding to the intersection of this line with the stress-strain curve is taken as the proof stress or yield strength of the material. The stress that represents the proof stress or yield strength of the material will, in general, vary with the value selected for the offset. (See also *B.S. Specification No.* 18*: Tensile Testing of Materials*, and Chapter II, p. 19.) *The proportional limit* is the proof stress or yield strength for 0·00 per cent offset. The value of the offset must be stated in reporting the proof stress or yield strength. For general use the offset recommended for steel is 0·20 per cent. The value is expressed as a unit strain and given as a percentage for convenience in writing; thus 0·20 per cent or $\varepsilon =$ unit strain $= 0{\cdot}002$. For example, the 0·2 per cent proof stress for a particular steel is 94·9 tons/sq. in.: the 0·1 per cent proof stress is 92·0 tons/sq. in.

Pre-cast Concrete. The term "pre-cast concrete product" is used to describe the many types and varieties of concrete units which are cast in moulds, either in a factory or on the site, and are not built into the structure or used until they are fully hardened. They include such items as cast stone, concrete lintels, beams, posts, piles, railway sleepers, blocks, etc. Some of these are pre-stressed and they can be mass produced.

225. **Pre-stressed Concrete.** The present chapter treats very briefly with this modern development known as pre-stressed concrete. *Pre-stressing means the creation of stresses in the whole or part of a structure before it is loaded.* These stresses are artificially induced so as to counteract those occurring in the structure under loading. The preliminary stresses induced in the concrete are essentially pre-compressions, which may be uniform or non-uniform, with the primary object of avoiding cracking under working conditions.

The pre-stressing process is specially suitable in its application to reinforced concrete, as the economy and properties of this medium are improved by the process. In a pre-stressed reinforced concrete member, the reinforcement consists of wires, or rods, or a combination of the two. It is the concrete that is pre-stressed, usually by tensioning the reinforcement and transferring the tensile force as compression to the concrete. Thus in reinforced concrete beams, by the application of eccentric forces, or pre-stress compression, acting upon the beam, a counter bending is produced.

This will be accompanied by an axial pre-stress force, and it will be seen that the consistent presence of this force precludes the case of simple bending. The counter bending of a reinforced concrete member is obtained by an eccentric compression force; it is produced by external means, preferably by tensioning the reinforcement and transferring the tensile force as compression to the concrete. At present, tensioning the reinforcement appears to be the only practical means available for prestressing. In this connection two different processes have to be distinguished; these are *pre-tensioning* and *post-tensioning*, terms which indicate whether the tensioning of the steel is carried out *before* or *after* the hardening of the concrete.

226. **Pre-tensioning.** With pre-tensioning, the reinforcement usually consists of wires. These are attached to anchorages at the ends of the moulds or shuttering, and then tensioned generally by mechanical means. The concrete is poured round the tensioned wires, and subsequently, by severing them at the anchorages, the force causing tension is released, imparting compression to the concrete. This last step is carried out only when the concrete has attained sufficient strength and bond resistance. Thus the steel is deemed to be pre-tensioned if it is stretched before the concrete has set.

The initial tensioning causes a small reduction in the diameter of the wires throughout the tensioned length. Severing at the anchorages permits the free ends to return to their original diameter, but where the concrete has set round the wires this will to some extent be prevented. The resultant wedge action on the concrete at the ends effects a permanent anchorage of the wires. The higher the tension stress applied and the smaller the diameter of the wires, the more efficient this anchorage will be. In this method the steel is bonded to the concrete. Transmission of the force from the wires to the concrete may be effected partly by friction and radial compression, but is mainly by bond resistance. A definite bond length is required. When transfer is effected by bond only, the diameter of the tensioned wire should be small: wires used in practice are of the order of 0·276 in., 0·200 in., 0·157 in., 0·128 in., 0·104 in., and 0·08 in. Independent of the concrete stress required, the concrete should be of such high strength and quality as to ensure sufficient bond resistance.

227. **Post-tensioning.** This is carried out against either the *hardened* concrete or an assembly of pre-cast blocks; compression is transmitted by such means as bearing or distribution

plates. In this case, the reinforcement to be tensioned is prevented from adhering to the hardened concrete, and may be located either *inside* or *outside* the concrete member. Thus the reinforcement is not bonded to the concrete, for the steel is deemed to be post-tensioned when it is stretched after the concrete has hardened.

With post-tensioning there are no limitations to the diameter or size of the tension steel. The wires are usually of 0·276 in. and 0·200 in. diameter, and rods are used in some systems of ½ in. to 1⅛ in. diameter. The concrete need not be of super high strength unless high concrete stresses occur. For the mechanical and other processes of pre-tensioning and post-tensioning, the student is referred to the many textbooks, publications and trade pamphlets dealing with the pre-stressing of concrete.

LOSSES OF STRESS. There are certain losses which may occur in the initial pre-stress of the reinforcement. "Creep" of steel commences when the tensioning stress exceeds the limit of elasticity which is much below the tensioning stress. However, it is possible to offset this loss of stress due to the creep of the steel either by a temporary or by a permanent increase in the value of the initial pre-stress. Provided that the loss of the initial pre-stress due to the creep of the steel is offset, it is possible, with post-tensioning, to transfer the entire nominal initial pre-stress. With pre-tensioning, however, first an instantaneous shortening due to the elastic deformation of the concrete occurs: also shrinkage of the concrete between production and transfer causes a shortening with the consequence that the pre-stress and transfer is much less than the original pre-stress. After transfer, further reductions in the length and thus further losses of the pre-stress occur, owing to that part of the shrinkage which takes place after transfer and owing to the creep in both concrete and steel. This happens both for pre-tensioning and post-tensioning. It is impossible to calculate exactly the losses in pre-stress. It is better to base the losses on special tests on specimens of the materials in question, or in lieu of such tests to introduce simple reduction factors. These are brought out in the analysis of pre-stressed beams as shown on p. 517.

228. Notes on Concrete and Steel for Pre-stressed Concrete.

CONCRETE. The essence of pre-stressing concrete for beams is to counteract bending. Low-tensile stresses up to 300 lb./sq.

in. were considered admissible for a long time, but not stresses of the order of 600 lb./sq. in. or higher. Full exploitation of the strength property of concrete is possible, and therefore high-strength concrete is preferable. High-quality concrete is used with minimum water content, careful grading of the aggregates for maximum density, and thorough compaction by vibration. Vibration of the concrete mix after placing (and the mix must therefore be suitably designed) is necessary to obtain a high degree of adhesion for pre-tensioned members; for these members, on the whole, the permissible compressive stresses may be taken as a third* of the cube strength with a maximum of about 3000 lb./sq. in. at a working load, and a transfer not exceeding say 2500 lb./sq. in. In general for post-tensioned members, an initial maximum compression stress of 2500 lb./sq. in. may be used. This is based on a minimum 28 days, 6 in. cube strength of 6250 lb./sq. in. Where the design permits a lower pre-stress, the cube strength may be reduced, but should never be less than 5000 lb./sq. in. Supervision and control is imperative to maintain the quality of the concrete during manufacture. For all reinforced concrete construction, concrete of high quality is necessary.

NOTE. In mass production, when manufacture can be done in a factory under constant supervision, there is no difficulty in specifying higher strengths. In such conditions, strengths of 8000 to 9000 lb./sq. in. can with appropriate means be readily produced. The use of certain forms of vibration, of compaction by compression and of heat curing, together with the choice of good aggregates, permit of the production of concrete of 11,000 to 13,000 lb./sq. in. in crushing strength. The full employment of these strengths has not yet entered into normal practice.

With increase in compressive strength, progress has been made in tensile strength. This is important because just as soon as a minimum tensile strength can be counted on it will be illogical not to use it. It is essential to know the tensile strength of the concrete, whether it be from the point of view of tensile stresses due to shear or anchorage, or of factors of safety against cracking, from the point of view of the possibility of allowing tensile stresses in bending. Tests should therefore be carried out to determine the tensile strength of concrete as well as its compressive strength. Concrete has a tensile strength of about one-tenth to one-half of its compressive strength, giving an average value of about 600 lb./sq. in. Nevertheless as the dispersion is always large, perhaps it is wise to count upon not more than half of this value in bending, say 300 lb./sq. in., although stresses higher than this have been used.

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* British regulations limit the permissible compressive strength to 33 per cent of the compressive strength at 28 days.

STEEL REINFORCEMENT. Early experimenters used mild-steel rods for applying the pre-stress. They were stressed to about 16,000 lb./sq. in., which produces a strain of approximately 0·0005. Due to the creep of concrete, which after a few months was about as much as the strain in the steel, the whole pre-stress was lost. Freyssinet was the first to realize that only by the use of high-tensile steel could this difficulty be overcome. If the stress in the steel which produces the pre-stress is 120,000 lb./sq. in. then the strain is about 0·004, and, whereas the creep in the concrete will remove, say, one-eighth of this, a considerable pre-stress is maintained in the concrete. It is the use of high-tensile steel which has made pre-stressed concrete possible. Conversely, it can be stated that it is pre-stressing which has rendered the use of high-tensioned steel possible. The steel stress in normal reinforced concrete work has to be limited to about 27,000 lb./sq. in. because of the excessive cracking which would occur at higher stresses. For wires used in pre-stressing work both for pre-tensioning and post-tensioning, development has mainly centred on the two sizes, 0·200 in. and 0·276 in. in diameter, giving an ultimate stress of 225,000 to 250,000 lb./sq. in. for the 0·276 in. diameter wire and for both a 0·1 per cent proof stress of about 157,000 lb./sq. in. The average design stress for these wires is about 145,000 lb./sq. in. In the pre-tensioning system, other sizes of wires are used, e.g. wires of diameter 0·157 in., 0·128 in., 0·104 in., and 0·08 in., for which the design tension is of the order of 200,000 lb./sq. in. The ultimate stress of high-tensile steel wires of 0·104 in. diameter is of the order of 270,000 to 300,000 lb./sq. in.

In post-tensioning work, some systems use high-tensile steel rods of the order of $\frac{1}{2}$ in. to $1\frac{1}{8}$ in. diameter, having an ultimate strength of 145,000 to 160,000 lb./sq. in. and an 0·2 per cent proof stress of 130,000 to 143,000 lb./sq. in., and for which an initial pre-stress is recommended at 94,000 to 100,000 lb./sq. in. It is stated that if an initial steel stress of 94,000 lb./sq. in. is used the residual stress in the steel, after creep and shrinkage of the concrete (there being no significant creep of the steel at working stresses) are complete, is of the order of 80,000 lb./sq. in. On the same basis the eventual concrete stress on a fibre having an initial pre-stress of 2500 lb./sq. in. is about 2100 lb./sq. in. Unless the design load is greatly exceeded the maximum steel stress will generally not again increase to its initial value.

229. **Advantages of Pre-stressing Concrete.** Normal reinforced concrete can seldom be employed with complete efficiency, because of the fact that concrete itself has a low tensile strength. Pre-stressed concrete, however, due to the inclusions of induced or built-in pre-stressing forces which prevent the development of harmful tensile stresses, is a fully efficient material.

Pre-stressed concrete makes possible the erection of a wide range of structures which could not be satisfactorily attempted in normal reinforced concrete. It is a material of construction which up to any desired multiple of the full design working load is completely free from cracks, and it is admirably suited to water retaining structures and those subjected to corrosive atmospheres.

Owing to the pre-compression caused by the pre-stressing force, the principal tensile stresses are reduced and high resistance to shear is obtained with very thin sections containing little or no diagonal tension steel reinforcement. Because little reinforcement is required for bending or diagonal tension, there is practically no obstruction to the placing and compacting of a much higher grade of concrete than is generally used in normal reinforced concrete members. Higher working stresses in the concrete can therefore be permitted with consequent reduction in the size and weight of the members. A saving occurs in the supporting members and foundations which in consequence have to carry smaller dead loads. With post-tensioned members a further gain in the lightness of section is achieved, because the dead weight of the member has little or no effect in determining the size of the cross-section required. This enables much longer spans to be constructed than would be possible with normal reinforced concrete.

When compared with structural steel, pre-stressed concrete members of the same depth and span have smaller deflections under working loads. This is important where long bridges or beams in building are concerned.

The pre-stressing of beams is advantageous because properly located pre-stressed steel will produce compressive stresses at the bottom of the beam which will offset the tensile stresses caused by the loads. The pre-stressing causes a deflection upwards which will reduce the resultant deflection when the loads are applied.

For pre-cast pre-fabricated construction the reduction in weight when pre-stressed concrete members are used leads to considerable economy in handling, transport and erection. In combination with *in-situ* concrete it permits great reductions in the amounts of shuttering required. The weight of steel required in pre-stressed concrete is only about 25 per cent of that required in normal reinforced concrete and as little as 10 per cent of that used in comparable structural steel designs.

230. **Summary.** There are two ways of pre-stressing concrete—*pre-tensioning* in which wires are stretched and held whilst the concrete is cast and hardens round them, so that when released they are anchored by bond; and *post-tensioning*, in which the tension is applied to the reinforcement after the concrete has hardened. The reinforcement can be set either inside or outside the concrete and anchored by various methods.

Essential to this method of construction are carefully detailed designs, and materials with stable high-performance characteristics with which to carry out the designs. The concrete should be of very high quality, and the high-tensile steel wires and rods must show only negligible creep at normal design tensions.

231. **Analysis: Beams, Steel on the Dead Load Tension Side Only ; Sections Symmetrical about a Vertical Axis ; Eccentrically Pre-stressed.** The analysis of relatively slender pre-stressed concrete members is simplified by the assumption of a straight-line stress distribution. Quick loading tests as reported by a number of observers reveal a nearly proportional relation between instantaneous loads and deflections at least up to the appearance of cracks. The straight-line stress relation seems to be justified for all normal loads.

Considering only beams—and the analysis can be extended to members subject to internal or external axial forces—the element of a pre-stressed beam consists of a short prism subjected to internal and external bending moments, and axial forces due to the pre-stress force. In the case of the internal and the external moments balancing each other, there will result a uniformly-stressed beam due to the pre-stress force. For the purpose of analysis, it is convenient to consider three typical stages. The ultimate stress condition will not be discussed, but reference can be made to it for analysis and design. The analysis investigates the original stress system under dead load and finally under normal live loads.

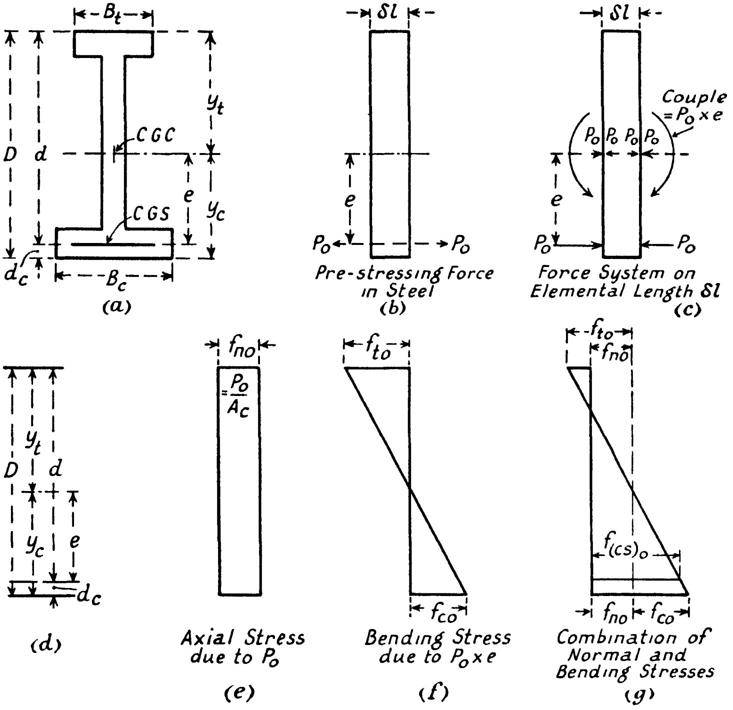
The first loading stage comprises the stresses caused by the original pre-stress force P_0; the second stage by sustained loads such as the dead load moment. The resulting effective stress system will be subject to time variations caused by gradual volume changes of the concrete (shrinkage or swelling, plastic flow or creep). The magnitude and rate of change of these time deformations are influenced by numerous variables, such as relative dimensions, humidity conditions and chemical changes.

The designer is mainly interested in the initial and ultimate values. The third stage of loading includes the temporary stress changes due to live load moments. (In contrast to the substained loads, the temporary stress changes due to live load moments are expected to produce negligible time deformations.) The above loading conditions are assumed not to produce cracks, so that truly monolithic action prevails over the entire concrete sections. This condition requires the elimination of tensile cracks under normal load conditions. In order to obtain a definitely crack-free concrete, it may be specified that the tensile stresses due to given dead and live loads shall not exceed the flexural or tensile strength of the concrete in beams or tension members respectively.

The elementary design and analytical principles of prestressed members should be applicable to all types of structures independent of pre-stress methods and load transfer. The embedding of stressed steel wires or rods in concrete was suggested by a number of experimenters and further developed by Freyssinet and others.

The methods of pre-stressing are based on mechanical steel-stretching devices. In general it may be said that a relatively high proof stress steel and a high-strength concrete are adapted to produce a high concrete pre-stress combined with a minimum error in the effective stress determinations. The combinations of high-strength materials also lead to desirable simplifications in the analytical expressions.

232. Analysis of Pre-stressed Beams: Steel on the Dead Load Tension Side Only ; Eccentrically Pre-stressed. For the general analytical discussion an I-sectioned beam has been taken (Fig. 278). Fig. 278 (*e*), (*f*) and (*g*) are stress diagrams for the concrete.



(*a*)

Pre-stressing Force in Steel (*b*)

Force System on Elemental Length δl (*c*)

(*d*)

Axial Stress due to P_o (*e*)

Bending Stress due to $P_o \times e$ (*f*)

Combination of Normal and Bending Stresses (*g*)

FIG. 278

NOTATION 1—CONCRETE BEAM SECTIONS

D = total depth of section.
d = distance from top fibre to centre of gravity of steel (C.G.S.).
C.G.C. = centre of gravity of concrete section.
C.G.S. = centre of gravity of steel.
y_t = distance from C.G.C. to top fibre.
y_c = distance from C.G.C. to bottom fibre.
d_c = distance from bottom fibre to C.G.S.
e = distance from C.G.C. to C.G.S.

ε = ratio of $\frac{e}{D/2}$.
B = breadth of beam, i.e. B_t or B_c (suffix to denote top or bottom).
A_c = area of concrete section.
A_s = total area of steel.
k = radius of gyration about C.G.C. = $\sqrt{I_c/A_c}$.

NOTATION 2—STRESSES IN POST-TENSIONED BEAMS

P_o = original steel pre-stress force transferred to concrete.
f_{so} = original steel pre-stress due to $P_o = P_o/A_s$.
f_{no} = axial stress over concrete section due to $P_o = P_o/A_c$.
f_{to} = bending stress in top fibre due to P_o at eccentricity e.
f_{co} = bending stress in bottom fibres due to P_o at eccentricity e.
$f_{(cs)o}$ = bending stress in concrete at level of C.G.S. due to P_o at eccentricity e.

NOTATION 3—STRESSES AFTER SHRINKAGE AND PLASTIC FLOW IN POST-TENSIONED BEAMS

P' = remaining pre-stress transferred to concrete after shrinkage and plastic flow.
f_s' = remaining steel stress after shrinkage and plastic flow.
f_n' = remaining axial stress after shrinkage and plastic flow.
f_t' = remaining bending stress in top fibres after shrinkage and plastic flow.
f_c' = remaining bending stress in bottom fibres after shrinkage and plastic flow.
$f'_{(cs)}$ = remaining bending stress in concrete at level of C.G.S. after shrinkage and plastic flow.
Δ_s = steel stress reduction losses due to shrinkage and plastic flow.
β = ratio of Δ_s/f_{so}.
m = modular ratio = E_s/E_c.

NOTATION 4—STRESSES DUE TO TOTAL DEAD LOAD IN POST-TENSIONED BEAMS

f_{tD} = bending stress in top fibre due to total dead load.
f_{cD} = bending stress in bottom fibre due to total dead load.

NOTATION 5—STRESSES DUE TO COMBINED EFFECT OF PRE-STRESS AND TOTAL DEAD LOAD AFTER SHRINKAGE AND PLASTIC FLOW

f_s'' = steel stress due to pre-stress plus total dead load after shrinkage and plastic flow.
f_t'' = stress in top fibres due to pre-stress plus total dead load after shrinkage and plastic flow.
f_c'' = stress in bottom fibres due to pre-stress plus total dead load after shrinkage and plastic flow.

NOTATION 6—STRESSES IN PRE-TENSIONED BEAMS

P_o = original pre-stress force applied to the steel wires.
f_{so} = original steel pre-stress due to $P_o = P_o/A_s$.

NOTATION 7—STRESSES AFTER RELEASE OF PRE-STRESSING MECHANISM IN PRE-TENSIONED BEAMS

f_s' = steel stress after release of pre-stressing mechanism.
ϕ = ratio of f_s' to $f_{so} = f_s'/f_{so}$ (see also eq. 24).
$f'_{(cs)}$ = concrete stress at level of C.G.S. after release of pre-stressing mechanism.

All other notations are the same as for post-tensioning.

Signs of Stress for the Concrete. A plus sign indicates a compressive stress and a minus sign a tensile stress. On the stress diagrams for concrete, stresses to the left of the vertical are tensile and to the right of the vertical are compressive. See also p. 524.

Signs of Stress for the Steel. The stresses are always tensile.

233. **Dimensional Ratios**

Let $g_c = \dfrac{y_c}{D/2}$; $g_t = \dfrac{y_t}{D/2}$; $\varepsilon = \dfrac{e}{D/2}$; $C = \left(\dfrac{D/2}{k}\right)^2 = \dfrac{D^2}{4k^2}$

where $k^2 = \dfrac{I_c}{A_c}$

$$= \frac{\text{Moment of inertia of gross concrete section about C.G.C.}}{\text{Area of gross concrete section}}$$

$$p = \frac{A_s}{A_c};\ z = \frac{d}{D}$$

From above

$$ey_t = \frac{\varepsilon}{4} g_t D^2;\ ey_c = \frac{\varepsilon}{4} g_c D^2;\ \frac{1}{k^2} = \frac{4C}{D^2}$$

$$\therefore \frac{ey_t}{k^2} = C\varepsilon g_t;\ \frac{ey_c}{k^2} = C\varepsilon g_c$$

234. **Case 1—Post-tensioning, Steel not Bonded to Concrete.** Let P_o = original steel pre-stress force transferred to the concrete element.

The concrete stresses developed are*—

$$f_{no} = \frac{P_o}{A_c} = \frac{A_s f_{so}}{A_c} = pf_{so} \quad . \quad . \quad . \quad (1)$$

$$f_{to} = \frac{P_o}{A_c} - \frac{P_o e y_t}{A_c k^2} = f_{no}\left(1 - \frac{ey_t}{k^2}\right) \quad . \quad (2)$$

$$f_{co} = \frac{P_o}{A_c} + \frac{P_o e y_c}{A_c k^2} = f_{no}\left(1 + \frac{ey_c}{k^2}\right) \quad . \quad (3)$$

From (2)

$$\frac{f_{to}}{f_{no}} = 1 - \frac{ey_t}{k^2} = 1 - C\varepsilon g_t \quad . \quad . \quad . \quad (4)$$

From (3)

$$\frac{f_{co}}{f_{no}} = 1 + \frac{ey_c}{k^2} = 1 + C\varepsilon g_c \quad . \quad . \quad . \quad (5)$$

Now $$\frac{f_{(cs)o}}{f_{no}} = \frac{\text{Concrete stress at C.G.S. due to } P_o}{\text{Normal stress due to } P_o}$$

$$= 1 + \frac{e^2}{k^2} = 1 + C\varepsilon^2 \quad . \quad . \quad . \quad . \quad (6)$$

$$\therefore \quad f_{(cs)o} = pf_{so}(1 + C\varepsilon^2) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Let $m = E_s/E_c$

n = steel stress factor due to plastic flow

Δ_s = steel stress reduction losses due to shrinkage and plastic flow

f_{so} = original steel pre-stress due to P_o.

Now due to shrinkage and plastic flow (creep), the concrete, and therefore the steel, will shorten. Let the reduction in the steel stress be Δ_s, and let it be equal to βf_{so}.

The section must remain in equilibrium under the reduced stresses.

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* See paras. 78 and 79, p. 145.

Now,

f_{so} becomes $f_s' = f_{so}(1 - \beta)$ . . . . . . (8)

P_o ,, $P' = A_s f_s' = A_s f_{so}(1 - \beta)$ . . . (9)

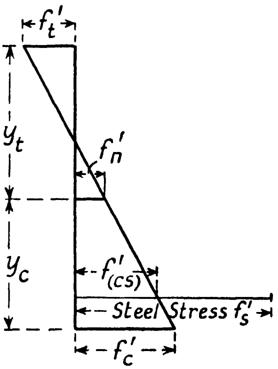
f_{no} ,, $f_n' = pf_s' = pf_{so}(1 - \beta)$ . . . (10)

f_{to} ,, $f_t' = pf_s'(1 - C\varepsilon g_t) = pf_{so}(1 - C\varepsilon g_t)(1 - \beta)$ (11)

f_{co} ,, $f_c' = pf_s'(1 + C\varepsilon g_c) = pf_{so}(1 + C\varepsilon g_c)(1 - \beta)$ (12)

$f_{(cs)o}$,, $f'_{(cs)} = pf_s'(1 + C\varepsilon^2) = pf_{so}(1 + C\varepsilon^2)(1 - \beta)$ (13)

The stress distribution is shown in Fig. 279.



Original Pre-stress
+ Shrinkage + Flow

FIG. 279

Due to dead load, assuming no further pre-stress in the steel due to the deflection of the beam—

$$f_{cD} = -\frac{M_D y_c}{I_c} = -\frac{M_D y_c}{A_c k^2} \quad . \quad . \quad (14)$$

$$f_{tD} = +\frac{M_D y_t}{I_c} = +\frac{M_D y_t}{A_c k^2} \quad . \quad . \quad (15)$$

The steel stress will remain at—

$$f_s'' = f_s' = f_{so}(1 - \beta)$$

then $f_t'' = f_t' + f_{tD} = pf_{so}(1 - C\varepsilon g_t)(1 - \beta) + \dfrac{M_D y_t}{A_c k^2}$. (16)

and $f_c'' = f_c' + f_{cD} = pf_{so}(1 + C\varepsilon g_c)(1 - \beta) - \dfrac{M_D y_c}{A_c k^2}$. (17)

At the time the live load is applied, additional shrinkage and flow may have occurred. If so, as previously done for flow, etc., after the application of the pre-stressing force, one can apply a percentage reduction of the stresses obtained for pre-stress and dead load. The live load is regarded as intermittently applied; therefore it will produce only elastic stresses. If the relation of the live load moment to the dead load moment is known, then the stresses produced can be easily calculated from those of the dead load when it was initially applied.

235. **Case 2—Pre-tensioning, Steel Bonded to Concrete.** The steel is installed in place and then pre-stressed to f_{so}. The concrete is then poured around the steel. After the concrete has hardened sufficiently to withstand the required pre-stress, the pre-stressing mechanism is released with the result that the steel reinforcement induces compressive pre-stress forces in the concrete due to the bond between the steel and the concrete.

The induced stresses in the concrete will cause a shortening of the member and consequently a reduction of the original steel pre-stress f_{so}.

Let the original steel pre-stress f_{so} be reduced to some value

$$f_s' = \phi f_{so} \quad . \quad . \quad . \quad . \quad (18)$$

The concrete at the level of the steel (C.G.S.) will receive a pre-stress compression of $f'_{(cs)}$, and, if the two materials are bonded, the strains at this level must be equal.

$$\therefore \frac{f_{so} - f_s'}{E_s} = \frac{f'_{(cs)}}{E_c}$$

$$\therefore mf'_{(cs)} = f_{so} - f_s' = f_{so}(1 - \phi)$$

$$\therefore f'_{(cs)} = \frac{f_{so}(1 - \phi)}{m} \quad . \quad . \quad . \quad . \quad (19)$$

The force exerted by the steel on the concrete will equal

$$f_s' \cdot p \cdot A_c = \phi f_{so} \cdot p \cdot A_c \quad . \quad . \quad (20)$$

The analysis proceeds as previously for the case where no pre-stress is lost (post-tensioning), for ascertaining the stresses in the concrete due to the pre-stressing force. The effect of shrinkage and flow can be taken into account prior to the application of the total dead load.

$$f_t' = pf_s'(1 - C\varepsilon g_t) = p\phi f_{so}(1 - C\varepsilon g_t) \quad . \quad (21)$$

$$f_c' = pf_s'(1 + C\varepsilon g_c) = p\phi f_{so}(1 + C\varepsilon g_c) \quad . \quad (22)$$

$$f'_{(cs)} = \frac{f_{so}(1 - \phi)}{m} = p\phi f_{so}(1 + C\varepsilon^2) \quad . \quad . \quad (23)$$

Also

$$\frac{1 - \phi}{m} = p\phi(1 + C\varepsilon^2)$$

$$1 - \phi = mp \, . \, \phi(1 + C\varepsilon^2)$$

Dividing by ϕ,

$$\frac{1}{\phi} - 1 = m \, . \, p(1 + C\varepsilon^2)$$

$$\frac{1}{\phi} = mp(1 + C\varepsilon^2) + 1$$

$$\therefore \phi = \frac{1}{[mp(1 + C\varepsilon^2) + 1]} \quad . \quad . \quad . \quad (24)$$

Stresses due to Loads. The solution for stresses due to dead or live loads can be made by the use of the transformed area, if the resultant tensile stresses are assumed to be low enough so that the whole concrete section can be considered in the calculations.

If the steel area is very small compared with the whole concrete area, then simply the concrete sectional area only need be considered.

236. **Analyses of a Rectangular Sectioned Pre-stressed Beam.** The details given below apply to Illustrative Problems 65 and 66 on pp. 524–7.

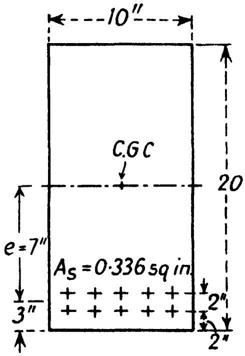


FIG. 280

The beam shown in Fig. 280 is reinforced with 0·168 per cent of steel. The wires comprising the steel reinforcement are pre-stressed to 150,000 lb./sq. in. Analyse the beam for a maximum dead load moment of 200,000 lb.-in. and a maximum live load moment of 400,000 lb.-in., (*a*) when the steel is not bonded to the concrete (post-tensioning, illustrative problem 65), and (*b*) when the steel is bonded to the concrete (pre-tensioning, illustrative problem 66).

Beam Details.

$D = 20$ in. $\qquad p = 0{\cdot}00168$

$d = 17$ in. $\qquad \therefore A_s = 0{\cdot}336$ sq. in.

$b = 10$ in. $\qquad m = 15$

$e = 7$ in.

Max. dead load moment = 200,000 lb.-in.

Max. live load moment = 400,000 lb.-in.

$$g_c = \frac{y_c}{D/2} = \frac{10}{10} = 1, \text{ also } g_t = 1;\ \varepsilon = \frac{e}{D/2} = \frac{7}{10} = 0{\cdot}7.$$

$$I_c = \frac{10 \times 20^3}{12};\ A_c = 10 \times 20 \qquad \therefore k^2 = \frac{10 \times 20^3}{12 \times 10 \times 20} = \frac{400}{12}$$

$$\therefore C = \left(\frac{D/2}{k}\right)^2 = \frac{100 \times 12}{400} = 3.$$

Illustrative Problem 65.

Analyse the beam when the steel is not bonded to the concrete.

Wires pre-stressed to $f_{so} = 150{,}000$ lb./sq. in.

Assume that due to shrinkage and flow there is a 10 per cent reduction in steel stress, i.e. $\beta = 0{\cdot}1$.

Due to pre-stressing only; the stress in the top fibres of the concrete

$$f_{to} = pf_{so}(1 - C\varepsilon g_t) = \frac{1{\cdot}68}{10^3} \times 150{,}000\ (1 - 3 \times 0{\cdot}7 \times 1{\cdot}0)$$
$$= -\ 278 \text{ lb./sq. in.}$$

Due to pre-stressing only; the stress in the bottom fibres of the concrete

$$f_{co} = pf_{so}(1 + C\varepsilon g_c) = \frac{1{\cdot}68}{10^3} \times 150{,}000\ (1 + 3 \times 0{\cdot}7 \times 1{\cdot}0)$$
$$= +\ 780 \text{ lb./sq. in.}$$

The minus sign indicates a tensile stress, and the plus sign a compressive stress. The stress diagram is given in Fig. 281 (*a*).

Assume that, due to shrinkage and flow, there is a 10 per cent reduction in the steel stress f_{so}, so that

$f_{so} = 150{,}000$ becomes $f_s' = 150{,}000(1 - \beta) = 135{,}000$ lb./sq. in.

where $\beta = 0{\cdot}1$.

f_{to} becomes $-\ 278 \times 0{\cdot}9 = f_t' = -\ 250$ lb./ sq. in.

and $\quad f_{co}$ becomes $780 \times 0{\cdot}9 = f_c' = 702$ lb./sq. in.

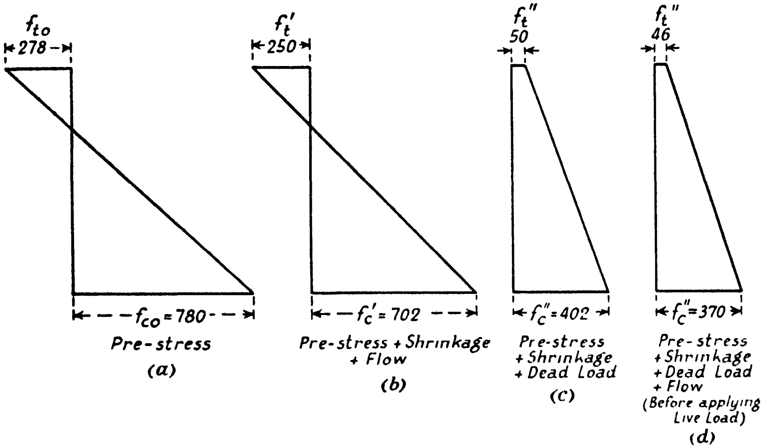
Dead Load. Assuming no further pre-stress in the steel due to the deflection of the beam,

$$f_{cD} = -\frac{200{,}000 \times 10 \times 12}{10 \times 20^3} = -300 \text{ lb./sq. in.}$$

$$f_{tD} = +300 \text{ lb./ sq. in.}$$

Allowing for beam deflection, the values would be

$$f_{cD} = -297{\cdot}5 \text{ lb./sq. in. and } f_{tD} = +293 \text{ lb./sq. in.}$$



Pre-stress (a)

Pre-stress + Shrinkage + Flow (b)

Pre-stress + Shrinkage + Dead Load (c)

Pre-stress + Shrinkage + Dead Load + Flow (Before applying Live Load) (d)

FIG. 281

The combined effect is the sum of the calculated stresses. The combined effect of pre-stress + dead load is

$$f_t'' = -250 + 300 = +50 \text{ lb./sq. in.}$$
$$f_c'' = +702 - 300 = +402 \text{ lb./sq. in.}$$

Before the live load is applied, assume for shrinkage and flow a further reduction of stress for $\beta = 0{\cdot}08$. Then

f_t'' becomes $50 \times 0{\cdot}92 = +46$ lb./sq. in. compression
f_c'' becomes $402 \times 0{\cdot}92 = +370$ lb./sq. in. compression
f_s'' becomes $135{,}000 \times 0{\cdot}92 = 124{,}000$ lb./sq. in. tension.

Stress diagrams for the concrete are given in Fig. 281 (*b*), (*c*), and (*d*).

Live Load Stresses. The maximum live load moment is twice that of the dead load moment.

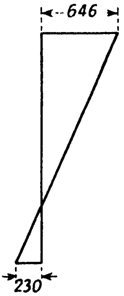
$\therefore$ The increased compressive stress in the top fibres is $+ 600$ lb./sq. in.

The increased tensile stress in the top fibres is $- 600$ lb./sq. in.

$$\therefore\ f_t'' \text{ becomes } + 46 + 600 = + 646 \text{ lb./sq. in.}$$

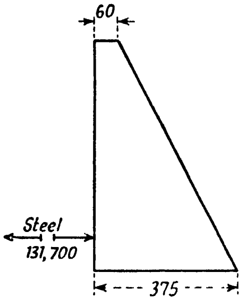
$$f_c'' \text{ becomes } + 370 - 600 = - 230 \text{ lb./sq. in.}$$

The resultant stress diagram is given in Fig. 282.



(Pre-stress + Flow)
+ (Dead Load + Flow)
+ (Live Load)

FIG. 282



Pre-stress + Flow
+ Dead Load

FIG. 283

Illustrative Problem 66.

Analyse the beam section of the previous problem when the steel is bonded to the concrete (pre-tensioning).

Take $m = 10$.

From eq. (24)
$$\phi = \frac{1}{\left[10 \times \frac{1{\cdot}68}{10^3}(1 + 3 \times 0{\cdot}7^2) + 1\right]}$$
$$= \frac{1}{1{\cdot}0415} = 0{\cdot}96$$

On release of the pre-stressing mechanism

$$f_{so} \text{ becomes } 150{,}000 \times 0{\cdot}96 = 144{,}000 \text{ lb./sq. in} = f_s'$$

Then due to pre-stressing the top fibres will have a stress of

$$f_t' = - 278 \times 0{\cdot}96 = - 267 \text{ lb./sq. in. (tension)}$$

and the stress in the bottom fibres will have a value

$$f_c' = + 780 \times 0{\cdot}96 = + 750 \text{ lb./sq. in. (compression).}$$

Shrinkage and Flow. Assume again that $\beta = 0{\cdot}10$. Then the steel stress becomes

$$144{,}000 \times 0{\cdot}90 = 129{,}600 \text{ lb./sq. in.}$$

Stresses in top fibres become $-\,267 \times 0{\cdot}90 = -\,240$ lb./sq. in. (tension).

Stresses in bottom fibres become $+\,750 \times 0{\cdot}90 = +\,675$ lb./sq. in. (compression).

Dead Load Stresses. As the steel area is very small compared with the concrete area, we will use the whole of the concrete section. (*Note.* As the steel is bonded to the concrete, the transformed area of concrete and steel in terms of concrete could be used.)

$\therefore$ Concrete stress, top fibres, $= f_{tD} = +\,300$ lb./sq. in. (compression)

Concrete stress, bottom fibres, $= f_{cD} = -\,300$ lb./sq. in (tension).

There will be an increase of the pre-stress in the steel $= 10 \times 0{\cdot}70 \times 300 = 2100$ lb./sq. in.

The resultant stresses after pre-stressing, flow, and dead load application will be—

Concrete, top fibres, $= +\,60$ lb./sq. in. (compression).

Concrete, bottom fibres, $= +\,375$ lb./sq. in. (compression).

The steel stress will be $129{,}600 + 2100 = 131{,}700$ lb./sq. in.

The stress diagram is shown in Fig. 283.

For live loads, the analysis can be continued as previously.

237. **Note.** The carrying capacity of a pre-stressed concrete beam, like that of any other homogeneous material, depends on the section modulus. The size of the sectional area affects the magnitude of the pre-stressing forces. Generally, an I-section or box section will be more suitable than a rectangle. The analysis of these sections (see Fig. 278) proceeds on the same lines as that indicated for the rectangular section; also for those sections which have reinforcement on the compression side, as well as on the tension side.

The problems of design and the positioning of the reinforcement due to the variation of the magnitude and sign of the bending moment along a beam must be left to other textbooks and publications, references to some of which are given at the end of this chapter.

238. **Shearing and Diagonal Tension Stresses.** The problem of shearing stresses in pre-stressed concrete members has been treated by a number of writers. It is reduced to a determination of the principal stresses. During the dead and live loadings it is desirable to avoid principal tensile stresses in the concrete. For further theoretical work in connection with this subject, and consequently the analysis and design of diagonal tension reinforcement, the student should consult textbooks, papers, etc., which treat fully with the subject.

239. **Bond: Pre-tensioned Members and Bond Stresses between the Steel and Concrete.** High-strength wires used have a large perimeter-to-area ratio, giving a greater perimeter over which to distribute a given force. Also the drop from the original pre-stress f_0 to the final pre-stress f_0' tends to cause the wires to increase slightly in diameter and thus add friction to the surface. Shrinkage and flow tend to offset the increase of steel stresses due to the load, and the resultant stresses do not exceed the original pre-stress f_0.

The pre-stresses are constant at all sections, so there is no need to transfer varying steel forces to the concrete except at the end of the wire. A properly designed bearing cone at the ends will safely transfer the steel stresses to the concrete. Subsequent bond stresses are due to the loads only. The usual bond relation in normal reinforced concrete can be restated as $u = \frac{qb}{\Sigma o}$ for rectangular-sectioned beams. The shear stress at the level of the steel for the normal reinforced concrete beam is $q = \frac{S}{b \, . \, jd}$ and is constant below the neutral axis.

However, the pre-stressed beam includes the concrete on the tension side in calculations, and the shear stress then varies as for a homogeneous beam section, i.e. for a rectangular section, a parabolic distribution with the maximum q at the neutral axis. Therefore, at the level of the steel in a pre-stressed beam, the shear stress is very low compared with that obtained from the above formula, and by the equation for u, the bond stress is correspondingly low.

240. **Note on the Design of Uniformly Stressed Members.*** The advantage of pre-stressed columns lies in the elimination of steel compressive stresses due to shrinkage and plastic flow. Any buckling tendencies in the longitudinal steel reinforcement are thereby excluded.

REFERENCES

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    No. 18: *Tensile Testing of Materials.*
2. *Code of Practice, C.P.* 114. The structural use of normal reinforced concrete in buildings.
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Textbooks—

4. *The Fundamental Principles of Reinforced Concrete Design,* W. T. Marshall. (Blackie & Son.)
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6. *Pre-stressed Concrete,* Y. Gugon. (Contractors' Records.)
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11. "Pre-stressed Concrete, Design Principles and Reinforcing Units," H. Schorer. (*American Concrete Institute Journal,* June, 1943.)
12. "Analysis and Design of Elementary Pre-stressed Concrete Members," H. Schorer. (*A.C.I.J.*, Sept., 1946.)
13. *Three Ways of Pre-stressing, and How to Make Pre-stressed Cables.* (Cement and Concrete Association, London, S.W.1.)
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20. "New Ideas and Methods," E. Freyssinet. *Science et Ind.* (*Travaux*), Vol. 17, (1933), No. 1, pp. 1–17.

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* Herman Schorer, "Design Principles and Reinforcing Units." *A.C.T.I. Journal,* Vol. 39, p. 493, June, 1943.

EXAMPLES

1. A tensile stress of 240 lb./sq. in. is produced in the outside fibre *AB* of the beam shown in cross section in Fig. 284. By applying a force *P* on the line *EF*, determine *P*, and the compressive stress in the outer fibre *CD*.

How many 0·2 in. diameter high-tensile steel wires are required to produce this force, if the stress in the wires, making due allowance for shrinkage and creep, is 100,000 lb./sq. in.?

What external moment will produce zero stress at *CD*?

2. The cross section of a pre-cast, pre-stressed beam of reinforced concrete is shown in Fig. 285. The beam is simply supported and carries a dead load of 300 lb./ft., and a live load of 220 lb./ft. on a span of 27 ft.

It is pre-stressed by means of eighteen 0·2 in. diameter wires, in which, allowing for creep, etc., the effective stress is 120,000 lb./sq. in.

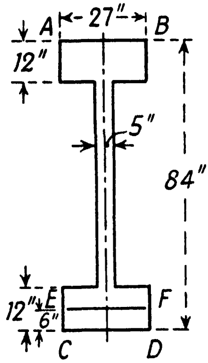

Fig. 284

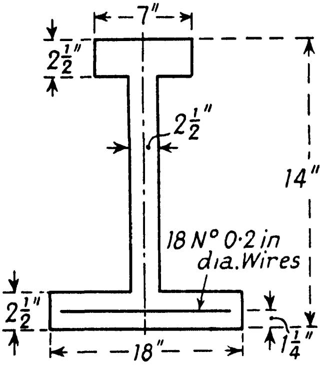

Fig. 285

Determine the outside fibre stresses due to—

(*a*) pre-stressing only,
(*b*) pre-stressing and the dead load,
(*c*) pre-stressing, the dead load, and the live load.

3. What is the object of pre-stressing a concrete beam? What is pre-tensioning and post-tensioning of the steel? How is the operation of pre-stressing carried out for both these conditions? A 10 in. × 5 in. concrete sleeper, 9 ft. long, is designed to withstand a bending moment of 9·25 tons-ft. It is pre-stressed by means of high-tensile steel wires 0·2 in. in diameter, placed 1½ in. from the outside edge so that all tensile stresses are eliminated when the member is fully loaded.

How many wires with an effective stress of 120,000 lb./sq. in. are required, and what is the maximum resultant compressive stress in the section? (U. of B.)

4. The cross section of a concrete beam is shown in Fig. 286. A pre-stressing force of 1460 tons is applied along a horizontal plane 9·25 in. below the axis of the beam.

Determine the maximum and minimum fibre stresses in lb./sq. in., due to this force, and the resultant stresses when an external moment of $54{\cdot}0 \times 10^{6}$ lb.-ft. is applied to the beam.

The areas of the pre-stressing wires may be omitted when calculating the area and the moment of inertia of the beam. (U. of B.)

5. The beam shown in Fig. 287 is reinforced with 0·8 per cent of steel, which has an effective pre-stress force of 80,000 lb./sq. in.

Calculate the stresses when it has to carry a dead load of 150 lb./ft., and a live load of 300 lb./ft. on a freely supported span of 50 ft. Take $m = 18$. Neglect all losses.

(By effective pre-stress is meant the pre-stress after losses due to creep, shrinkage and transfer have been deducted.)

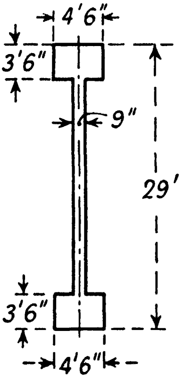

FIG. 286

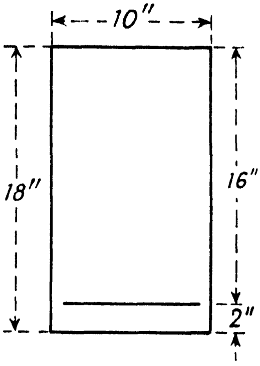

FIG. 287

ANSWERS TO EXAMPLES

CHAPTER I

1. Shear force $\begin{cases}\text{Max.} & 26{\cdot}25\ (-). \\ \text{Min.} & 18{\cdot}75\ (+).\end{cases}$

   Moments $\begin{cases}\text{Max.} & 271{\cdot}9 \text{ ton-ft.} \\ \text{Min.} & \text{zero.}\end{cases}$

2. Reactions at supports, $3\frac{3}{14}$ and $1\frac{11}{14}$ tons.
   Moments at supports + 10 ton-ft.
   + 6 ton-ft.

   At central 2-ton load pt. + ·5 ton-ft.

4. Support reaction 8·13 tons ; at pier + shear 11·87 tons.
   Pier reaction 21·87 tons ; at pier − shear 10·00 tons.
   Moment 10′ from left support = 33·75 ton-ft.

5. 12·5 ton-ft. ; ·5 ton-ft.

6. Reaction—Left support, 25 tons.
   Right support, 35 tons.

7. Moments at pier + 50 ton-ft.
   + 112·5 ton-ft.
   − 59 ton-ft. at 10 tons load point.

8. Max. : 42,900 lb.-ft. at 11·7 ft. below surface.

9. Produce 10 tons to *N.A.* and then resolve vertically and horizontally. Moment at fixed point = vertical component × arm along *N.A.* to fixed point.

10. Reaction at *A*, ·33 tons ; max. moment at pier = 7·2 ton-ft.
    Reaction at 12 ft. pt. 3·17 tons.

CHAPTER II

1. Load on *a*, *c*, $\frac{1}{40}$ ton. Load on *b*, $\frac{1}{5}$ ton.
2. 3·472 ton-in. 10·4 tons/sq. in.
3. 37·5 sq. in.
4. f_c = 4·05 tons/sq. in. ; f_t = 1·15 tons/sq. in.
5. 10,300 lb./sq. in. tension. 25,000 lb./sq. in. compression.
6. 1000 ft. 144,000 lb.-in.
7. 6·75 tons/sq. in.
8. 8 tons ; $\frac{2}{3}$ ton per foot run.
10. 1200 lb./sq. in. at right support due to max. positive moment ; 344 lb./sq. in. at left support due to max. negative moment.
11. 440 in.^{4}
12. $I_{xx} = I_{yy}$; d = 5·1 in.
13. I_{xx} = ·0126 in.^{4}
14. $\frac{1}{16}$ in.

CHAPTER III

1. ·316 in.
2. Max. moment = 1,832 lb.-ft. Central deflection, 2 in.
3. h = 5·5 in.—(neglecting beam deflection).
   = 5·1 in.—(allowing beam deflection).
4. ·30 in.
5. E = 12,000 tons/sq. in. (nearly).
7. f_t = 1·25 tons/sq. in. $y = \frac{1}{26}$ in.
9. $y_c = 0$.
10. y_c = ·39 in.; y_{max} = ·396 in. at 13·2 ft. from right-hand support.
11. M_x = – 45·1 ton-ft. at 10·8 ft. from 5 cwt. end.
    y_c = ·63 in. Slope, ·000058.
12. ·07 tons/foot run. y = ·375 in.
13. E = 1,125,000 lb./sq. in. f_c = 3750 lb./sq. in.
14. y = ·062 in.
15. 7 tons/sq. in.
16. (a) ·152 tons/foot; (b) ·17 in.; (c) ·19 tons/foot;
    (d) y_4 = ·157 in. (e) y_2 = ·115 in. (E = 13,000 tons/sq. in.)
17. y_c = ·181 ft.; y_{15} = ·127 ft.; y_{25} = ·176 ft.

CHAPTER IV

1. R_A = 60 tons; R_B = 200 tons; R_C = 60 tons.
   M_B = 400 ton-ft.; f centre support 4·82 tons/sq. in.
2. R_A = 9·0 tons; R_B = 29·3 tons; R_C = 20·25 tons; R_D = 1·45 tons.
3. Max. stress = 3·52 tons/sq. in. Moments. Ends 42,000 lb.-in.
   Centre 30,000 ,,
4. (a) $M_B = M_C = +\frac{wl^2}{12}$ (b) $M_B = \frac{wl_1^2}{8}$
   Max. neg. $= \frac{wl^2}{24}$ Max. neg. $= \frac{9wl_1^2}{128}$
6. I = 173 in.^{4}
7. R_A (left support) = 11·95 tons; R_B = 29·11 tons;
   R_C = 22·62 tons; R_D = 31·32 tons.
8. 14·2 tons.
9. Tension in tie 5·32 tons. Resultant reaction at A = 8·2 tons;
   M_A = + 25 ton-ft.

CHAPTER V

1. $C = \frac{2{\cdot}39}{A}$ y = ·00368 in. (varying q method).
   Max. shear = ·97 ton/sq. in. (nearly).
2. Max. shear stress = ·148 ton/sq. in.
   E = 6000 tons/sq. in. N = 2500 ton/sq. in.
   y_s = ·00144 in. y_b = ·108 in.

$$\left[C \text{ for ellipse } = \frac{4}{3A} \text{ as for circle.}\right]$$

5. $\cdot 129/\cdot 071 = 1\cdot 82$ tons/sq. in.

6. y_s (concentrated load) $\left(C = \frac{1\cdot 5}{A}\right) = \cdot 0166$ in.

   y_s (uniform load) „ $= \cdot 0185$ in.

7. $L/D = 9$.

CHAPTER VI

1. $\left(a = \frac{1}{7500};\ f = 21 \text{ tons/sq. in.}\right)$ $I_{min} = 108$ in.^{4}; $E = 13{,}000$ tons/sq. in.

   Rankine, 396 tons; Euler, 1530 tons.

3. $x = \frac{(D^2 + d^2)}{8D}$ Short column.

4. $I_{min} = 89\cdot 1$ in.^{4} 28 tons.

5. Max. $f_c = 7\cdot 07$ tons/sq. in. $f_t = 6\cdot 53$ tons/sq. in.

6. 4·77 tons/sq. in.

7. Max. compressive stress $= \cdot 59$ ton/sq. in.
   Min. compressive stress $= \cdot 12$ ton/sq. in.

9. Factor of safety, 4. 88·8 tons.

10. Factor of safety, 4. 12·9 tons.

11. Max. stress, 3·18 tons/sq. in. Min. zero.

12. $a = \frac{1}{1800}$ $E = 10{,}500$ tons/sq. in. 81·4 tons, Rankine.
    2,260 tons, Euler (factor of safety = 4)

13. R.S.J., 215·4 tons. Cast-iron, 250·1 tons.

14. Euler, 20 ft.; ·434 ton. (*b*) 36 in. by Rankine; 12·6 tons.

    $a = \frac{1}{7500}$; $E = 13{,}000$ tons/sq. in.

    $f_c = 21$ tons/sq. in.

15. External diameter $= 4\cdot 54$ in. $t = \cdot 454$ in.

16. $f_{max} = 6\cdot 78$ tons/sq. in. ($y_c = 4$ in.).

17. 48·3 tons.

18. 271 tons (Rankine: $f_c = 21$ tons/sq. in.; $a = 1/7500$).

19. 3·7 in. by Rankine; 2·55 in. approx. by Euler (obtained from formulae by trial and error method).

20. 5·5 tons.

21. Rankine, $a = \frac{1}{7500}$ $y_c = 4$ in.; 115 tons.

    Euler. $E = 13{,}000$ tons/sq. in. 471·0 tons $\left(y_{max} = \frac{Pel^2}{8EI}\right)$

22. Hinged ends; $a = \frac{1}{7500}$ Rankine, 26·4 tons.

    Euler, 36·58 tons.

CHAPTER VII

2. 2·57 tons (compression) in *EP*.

3. ·91 ton (comp.) in *EP* ; 1·15 tons in *FQ* (comp.)

4. *af* = 67·5 tons (tensile) ; *ce* = 25·0 tons (compression).
be = 74·0 tons (compress.) ; *ef* = 16·25 tons (tensile).

5. *DJ*, *EL*, 10 tons compression ; *KA*, 6 tons tension.

6. 1·02 tons, *DJ* compression ; ·11 ton, *KA* compression.

7. Load in *FL*, 3 tons compression ; Load in *NM*, ·05 ton compression.

8. Wind load only, 1·125 tons in *FL* (compression)
·176 tons in *NM* (tension).

9. *CF*— 29 tons (comp.).
EC— 6·5 tons („).
DE—33·5 tons („).

10. 2·89 tons comp. in *CE*.

11.

Member.		*Member.*		*Member.*	
1·2	1·0 ton (C.)	4·7	1·5 tons(C.)	7·10	1·0 ton (C.)
2·3	0	4·5	0·5 ton (C.)	10·8	0·7 ton (T.)
1·4	2 tons(C.)	5·6	0·5 ton (T.)	8·11	0·7 ton (C.)
4·2	0·7 ton (T.)	5·9	0·7 ton (C.)	9·11	1·0 ton (C.)
2·6	0·7 ton (C.)	5·7	0·7 ton (T.)	11·10	0·5 ton (T.)
3·6	0	6·9	0·5 ton (C.)		
4·5	0·5 ton (C.)	7.8	0·5 ton (C.)		
5·6	0·5 ton (T.)	8·9	0·5 ton (T.)		

CHAPTER VIII

2. $\frac{2{\cdot}86}{A}$ in.

3. y = ·405 W in. (W is in tons).

5. y at 6T. load point = 1·41 in.
y at 10T. load point = ·76 in.

6. Centre joint, top boom: y = ·04 in. approx. (l/A = 40 all members).
Lower joints: y = ·033 in. approx. (l/A = 40 all members).

7. y = 171 in.

8. ·88 in. under load ; ·695″ – 6′ from left support.

9. ·212 in. ; y_c = ·125 in.

10. H.D. of B to right = 0·2554 in.; of C to right = 0·138 in.

CHAPTER IX

1. Fixing couples, beam and columns.

$$M_B = \frac{wl^3}{4}\left(\frac{I_c}{2hI_b + 3lI_c}\right) = \frac{wl^2}{4}\cdot\frac{1}{2\frac{h}{I_c}\cdot\frac{I_B}{l} + 3}$$

Max. neg. for beam $= \frac{wl^2}{8} - M_B$.

2. Fixing couples + 1·87 ton/ft. end of 8-ft. column.
+ 2·81 „ „ 12 ft. „
Max. neg. moment beam, 7·5 − 2·34 = 5·16 ton/ft.
$H = 0{\cdot}228$ tons.

3. *CG* 11·8 tons by Lt. Wk. (comp.) 11·8 tons by super-position.
HD 11·8 tons „ (tens.) 11·8 tons „ „

4. (Same frame as in No. 1.)

Col. base fixing couples $= -\frac{\text{Beam fixing couples}}{2}$

Beam couples $+ \frac{wl^3}{6}\left(\frac{I_c}{2I_cl + lI_b}\right)$

(Frame of No. 2.)
Fixing couples; col. bases. Short, − 1·5 ton/ft.; beam, 3·0 ton/ft.
„ „ „ Long, − 2·25 „ „ 4·5 „

5. *CA* by Lt. Wk. − 3·11 tons (compression). *CA* by supn. − 3·55 tons.
CE „ − 3·47 „ („) *CE* „ − 3·85 „
BF „ = + 4·0 (tension) *BF* „ = + 3·55 tons.
FD „ = + 4·23 („) *FD* „ = + 3·85 „

6. + 27·75 ton/ft. M_B.

7. M_B, + 21·84 ton/ft. M_C = 19·70 ton/ft. (+).

8. $M_A = Wnl(1-n)^2$. $M_B = Wn^2l(1-n)$.

CHAPTER X

3.

	3rd Bay. Tons.	4th Bay. Tons.
Top member . .	145·4 (Comp.)	155·4 (C.)
Bottom member.	114·5 (Tension)	145·4 (T.)
Lt. vertical . .	45·4 (C.)	26·0 (C.)
Right vertical .	26·0 (C.)	12·5 (C.)
Diagonal . .	64·2 (T.)	37·65 (T.)

4th and 5th Bays (counterbrace)

4.

1st Bay	121 tons	[No counterbracings]
2nd „	95 „	
3rd „	69·5 „	
4th „	45 „	
5th „	17 „	

6. Height frame = 20 ft. (1) 160 tons. (2) Comp. 5·65 tons.
Tensile 90·5 „

7.

	Tons.		Tons.
40 ft. vertical	50·8 (C.)	Top member	180 (C.)
60 ft. „	27·9 (C.) 7·8 (T.)	Diagonal	65 (C.)
		Bottom member	144 (T.)

8

	Tons.		Tons.
Top member	92·5 (C.)	Diagonal	30·9 (Tensile). 7·7 (Compress.).
Bottom member	81·0 (T.)		

9. Centre →15 tons leading $+ 10·5, - 9·5$ tons.
←15 „ „ $+ 9·5, - 10·5$ „
Ends →15 „ „ Right support, $+ 15$ tons ; left, $- 23$ tons.
←15 „ „ „ „ $+ 23$ „ ; „ $- 15$ „

10. 571·5 ft.-tons. $R_B + 39·17$ tons ; $R_A - 42·1$ tons.
Loads only → (10 tons at B) (14 tons at R_B)

12.

	Tons.
Top member	114·2 (compression)
Diagonal	62·1 (tensile).
Bottom member	86·4 (tensile).

13. $AC - 23·3$ tons (compression). $AB - 18\frac{5}{8}$ tons (tensile).

CHAPTER XI

1. (1) 15 tons. (2) 25·8 tons and 17·5 tons.
(3) Max. neg. at load point = 240 ton-ft.
Max. pos., 75 ft. from left support = 135 ton-ft.

2 (a) Shear, 3·39 tons. Thrust, 3·07 tons.
(b) 75 ton-ft (max. neg.) 3·7 ton-ft. (max. +).

4. (1) 12·5 tons.
(2) Max. B.M. 50 ft. from the abutment ; $-$ 110 ton-ft., load at section point.
$+$ 72 ton-ft., load at crown.

	Load at Section Point.	Load at Crown
Resultant thrust at left abut.	$\sqrt{8·34^2 + 6·67^2} = 10·65$ tons ;	13·5 tons.
„ „ right „	$\sqrt{8·34^2 + 3·33^2} = 9$ „ ;	13·5 „

5. 3·6 tons ; 2·9 tons ; 2·1 tons ; 1·83 tons ; 2·1 tons ; 2·9 tons ; 3·6 tons.

6. 4·4 sq. in.

7. 36·3 ton-ft. at 28·1 ft. from each support ; 65 tons.

8.

Max. + and – shears each support	6,000 lb.
„ $\frac{1}{4}$ and $\frac{3}{4}l$	2,250 „
Centre	±4,500 „

9. Horizontal thrust, 50 tons.
Max. B.M. $+ 125$ at 15 ft. from left support.
$+ 450$ at 15 ft. „ right „
$- 150$ at 45 ft. „ left „

10. Max. B.M. at 28 ft. from left support = 228 ton-ft.
at 92 ft. „ „ = 228 ton-ft.
$H_0 = 26·15$ tons.

11. H_0, max. 29·1 tons.
Max. positive moment = + 141 ton-ft., 10 tons at centre.
Max. negative moment = − 159 ton-ft., 12 tons at 50 ft. section.
Resultant thrust, 12 tons at 50 ft.; 31 tons.

12. Max. negative moment at 45 ft. = 400 ton-ft.
Max. positive moment at 30 ft. from right support = 520 ton-ft.
Normal thrust at 30 ft. from right support = 128 tons.
Max. stress in rib, 7·7 tons sq. in. (compress.); 4·27 tons (comp.) (Min.).

13. Max. negative moment, 20·4 ton-ft. at section 25′ and 95′ ⎫ two loads on
⎬ smaller part
Max. positive moment 14·2 „ „ 30′ and 90′ ⎭ of beam
Max. shear negative and positive 1·85 tons, when both loads on the beam and one is at the ends.

14. T_{max} = 300 tons; Vertical pressure = 3·12 tons.

15. H = 1·80 lb. when unit load at C; 53,800 lb.

CHAPTER XII

2 5·7 tons/sq. in. (tension). 19° 20′.

3. p_n = 4·76 tons/sq. in. p_t = 2·76 tons/sq. in.

4. R_{max} 36·2 tons/sq. in. tensile
R_{min} 1·38 „ tensile.
Angle 31° 43′ and 121° 43′ with 3 tons plane.

5. *See* Para. 166, equations 37, 40.

6. p_n = 2·61 tons/sq. in. p_t = 0·87 tons/sq. in.

7. *See* Question 4.

9. R_{max} = + 6 tons/sq. in. (tens.). R_{min} = − 6·0 tons/sq. in. (comp.).
a = 13° 15′ and 103° 15′ to vertical.

10. (*a*) 2·5 tons at 38° to normal. p_x = 3·82 tons or ·78 tons ⎫
p_y = ·78 „ or 3·82 „ ⎭

CHAPTER XIII

1. 6·6 ft. f_{max} 4,200 lb./sq. ft. (Rankine).
5·15 „ f_{max} 4,200 „ (wedge) or (friction).

2. 3·6 ft. from back of wall.

3. Base 6 ft.

4. *See* text.

5. b = 10·0 ft.

6. Normal stress at dry face. Water face.
10,000 lb./sq. ft.; (C.) 6,300 lb. sq. ft.
Shear (dry face), 5,000 lb./sq. ft. (water face) zero.

7. Resultant thrust = 52·5 tons.
Normal stress (dry face), 3·7 tons/sq. ft. water face, zero.

8. 4,800 lb.

9. 32 lb./sq. ft.

10. Depth, 3·12 ft. breadth, 4·8 ft.

11. 5 ft.

12. Rankine, 5,000 lb. Fric., 5,080 lb.

13. Dry face, 1·5 tons/sq. ft. water face, zero.
{ Normal stress, vertical surfaces.
{ Dry face, nil ; Water face, 1·17 tons/sq. ft.

14. (*a*) $1\frac{1}{3}$ tons/sq. ft. (const.).
(*b*) $2\frac{2}{3}$ tons/sq. ft. to zero
(*c*) 3·55 tons/sq. ft. (comp.) to ·89 ton/sq. ft. (tension).
(*d*) 4 tons/sq. ft. to zero at 6 ft. from dry face.

CHAPTER XIV

1. Width, 7·8 in. Effective depth, 23·4 in. (overall, say, 25·0 in.).
Area steel (tension only), 1 sq. in. say, 5, $\frac{1}{2}$ in. bars.
$\frac{1}{4}$ in. dia. stirrups ; 6-in. pitch near abutments.

2. (*a*) $1{\cdot}2 \times 10^{-4}$ in. per inch length. (*b*) 2·7 in. lb. per inch length.

3. Effective depth steel, 17 in. (steel in tension 3, $\frac{1}{2}$-in. bars, or 5, $\frac{3}{8}$-in. bars).
Area steel, ·56 sq. in. ($c = 600$ lb./sq. in.; $t = 16{,}000$ lb./sq. in.; $m = 15$).

4. 8 and 9 ft. 1·43 sq. in. (area of bar = ·715 sq. in.).
9 and 10 ft. 1·59 „ („ „ = ·795 „).

5. Weight of cubic feet concrete, 150 lb. $W = 1800$ lb.
$= {\cdot}8$ ton.

6. 168 lb./sq. ft. per foot run

7. 455 in.^{4} ($n = {\cdot}36d$). 605 in.^{4} if $m = 15$.

8. $t = 7700$ lb /sq. in. $c = 526$ lb./sq. in. (neglecting weight of beam).
($m = 15$, $d = 7$ in.).

9. 416 lb /ft. run : $f = 5{\cdot}95$ tons/sq. in.

10. Short column, 107·5 tons.

11. $x = \frac{3}{8}d$; $M = 98{\cdot}5bd^2$; $A_s = 0{\cdot}007bd$.

12. (*a*) $x = 12$ in.; (*b*) 1,872,000 lb.-in.

13. $f_c = 750$ lb. per sq. in.; $f_s = 18{,}000$ lb. per sq. in.; $b = \frac{d}{2}$; $b = 10{\cdot}3$ in. and $d = 20{\cdot}6$ in.

14. $f_s = 10{,}000$ lb. per sq. in.

15. $f_s = 13{,}500$ lb. per sq. in.; $f_c = 475$ lb. per sq. in.; $f_{cs} = 4740$ lb. per sq. in.

16. Load = 19,875 lb.

17. 20 in. × 20 in. overall.

18. $f_{c\ max} = 440$ lb. per sq. in.; $f_{c\ min} = 80$ lb. per sq. in.
$f_{s\ max} = 4848$ lb. per sq. in.; $f_{s\ min} = 1392$ lb. per sq. in.

20. $u = 69{\cdot}5$ lb. per sq. in.; $f_s = 18{,}000$ lb. per sq. in.; $l = 40{\cdot}8$ in.

21. Flange only taking tension load; N.A. depth = 4 in.; $f_s = 13{,}200$ lb. per sq. in.; $M = 902{,}500$ lb.-in.; Stirrups—stress = 10,000 lb. per sq. in.; load = 2200 lb.

22. (*a*) 323,000 lb.-in.; $A_s = 1{\cdot}12$ sq. in.
(*b*) $A_{sc} = 1{\cdot}22$ sq. in.; $A_s = 1{\cdot}49$ sq. in. (tension).

CHAPTER XV

1. (*a*) $M_{\mathrm{AC}} = +\ 200$ ton-ft. $= M_{\mathrm{CA}} = 0$.
(*b*) $M_{\mathrm{BA}} = -\ 87{\cdot}5$ ton-ft.
(*c*) $M_{\mathrm{BA}} = -\ 98{\cdot}5$ ton-ft.
(*d*) $M_{\mathrm{BA}} = -\ 63{\cdot}3$ ton-ft.

(*e*) $M_{BA} = -63{\cdot}6$ ton-ft.
$M_{AB} = +80{\cdot}68$ ton-ft.
(*f*) $M_{AB} = +79{\cdot}76$ ton-ft.; $M_{BC} = 65{\cdot}53$ ton-ft.;
$M_{CB} = +19{\cdot}27$ ton-ft.

2. (*a*) $M_{AB} = -25$ ton-ft.; $M_{BA} = -12{\cdot}5$ ton-ft.
(*b*) $M_{AB} = -25$ ton-ft.; $M_{BA} = +6{\cdot}25$ ton-ft.; $M_{CB} = 0$.
(*c*) $M_{AB} = -25$ ton-ft.; $M_{BA} = -7{\cdot}14$ ton-ft.; $M_{CB} = +3{\cdot}57$ ton-ft.
(*d*) $M_{AB} = -25$ ton-ft.; $M_{BA} = -7{\cdot}15$ ton-ft.; $M_{CB} = +3{\cdot}57$ ton-ft.

3. (*a*) $M_{AB} = -8333$ lb.-in.; $M_{BA} = -16{,}667$ lb.-in.; $M_{BC} = +16{,}667$ lb.-in.
(*b*) $M_{AB} = -5560$ lb.-in.; $M_{BA} = -11{,}120$ lb.-in.;
$M_{CB} = -44{,}480$ lb.-in.; $M_{CF} = M_{FC} = 0$.
(*c*) $M_{ED} = M_{AB} = -10{\cdot}5 \times 10^6/39$ lb.-in.
$M_{DE} = M_{BA} = -9 \times 10^6/39$ lb.-in.
$M_{BC} = M_{CB} = +9 \times 10^6/39$ lb.-in.
$M_{CF} = -18 \times 10^6/39$ lb.-in.
$M_{DC} = M_{CD} = +9 \times 10^6/39$ lb.-in.
$M_{FC} = -21 \times 10^6/39$ lb.-in.

4. $M_{AB} = +4760$ lb.-in.; $M_{BA} = +20{,}230$ lb.-in.
$M_{BC} = +29{,}750$ lb.-in.; $M_{CB} = +13{,}090$ lb.-in.
$M_{CD} = -13{,}090$ lb.-in.; $M_{DC} = -11{,}900$ lb.-in.

5. $M_{BC} = +135{,}000$ lb.-in.; $M_{CB} = -135{,}000$ lb.-in.
$M_{BA} = -135{,}000$ lb.-in.; $M_{CD} = +135{,}000$ lb.-in.

6. $M_{AB} = +104{,}760$ lb.-in.; $M_{DC} = -64{,}700$ lb.-in.

7. $M_{AB} = -1786$ lb.-in.; $M_{BA} = -3572$ lb.-in.
$M_{AE} = +10{,}714$ lb.-in.; $M_{BC} = -7144$ lb.-in.
$M_{CB} = -8930$ lb.-in.; $M_{CD} = +8930$ lb.-in.

8. $M_{BA} = -2778$ lb.-in.; $M_{BC} = -2778$ lb.-in.
$M_{BF} = +5555$ lb.-in.

9. $M_{AD} = +59{\cdot}46$ ton-in.; $M_{AE} = -25{\cdot}72$ ton-in.;
$M_{AB} = -19{\cdot}29$ ton-in.; $M_{AC} = -14{\cdot}47$ ton-in.;
$M_{EA} = -12{\cdot}86$ ton-in.

10. $M_{AB} = +180{,}000$ lb.-in.; $M_{BA} = +60{,}000$ lb.-in.;
$M_{CD} = -60{,}000$ lb.-in.; $M_{DC} = -180{,}000$ lb.-in.;
$\Delta_{BC} = 1/3$ in.

11. $M_{AB} = -9{\cdot}10$ ton-in.; $M_{BA} = -18{\cdot}20$ ton-in.;
$M_{CB} = +11{\cdot}36$ ton-in.; $M_{BC} = +5{\cdot}68$ ton-in.

12. $M_{AB} = +109{\cdot}7$ ton-in.; $M_{BA} = -115{\cdot}7$ ton-in.;
$M_{CD} = +150{\cdot}5$ ton-in.; $M_{DC} = 0$.

13. $M_{AB} = -4688$ lb.-ft.; $M_{BA} = +520$ lb.-ft.

14. (*a*) $M_{AB} = -16{,}000$ lb.-in.; $M_{BA} = -32{,}000$ lb.-in.;
$M_{CD} = +3200$ lb.-in.; $M_{DC} = +16{,}000$ lb.-in.
(*b*) $M_{AB} = +143{,}572$ lb.-in.; $M_{BA} = +2144$ lb.-in.;
$M_{CD} = -27{,}856$ lb.-in.; $M_{DC} = -66{,}428$ lb.-in.
(*c*) $\Delta_{BC} = 0{\cdot}12$ in.

15. $M_{AB} = -\ 50{,}000$ lb.-in.; $M_{BA} = -\ 100{,}000$ lb.-in.;
$M_{BC} = +\ 116{,}000$ lb.-in.; $M_{CB} = +\ 204{,}000$ lb.-in.;
$M_{DC} = -\ 281{,}000$ lb.-in.

16. $\phi_{AB} = \phi_{FC} = \phi_{ED} = 0$ (by analysis);
$M_{AB} = -\ 70{,}000$ lb.-in.; $M_{BA} = -\ 140{,}000$ lb.-in.;
$M_{CB} = -\ 306{,}600$ lb.-in.; $M_{CD} = +\ 140{,}000$ lb.-in.;
$M_{CF} = +\ 166{,}600$ lb.-in.; $M_{FC} = +\ 83{,}300$ lb.-in.;
$M_{DE} = -\ 28{,}000$ lb.-in.; $M_{ED} = -\ 14{,}000$ lb.-in.

CHAPTER XVII

1. 137; 25,600,000 lb.-in.

2. (*a*) 417 lb./sq. in. (tension); 1504 lb./sq. in. (comp.).
(*b*) 1093 lb./sq. in. (comp.); 628 lb./sq. in. (comp.).
(*c*) 2193 lb./sq. in. (comp.); 14 lb./sq. in. (tension).

3. 9; 1320 lb./sq. in.

4. (*a*) 235 lb./sq. in. (tension); 1301 lb./sq. in. (comp.).
(*b*) 1125 lb./sq. in. (comp.); 60 lb./sq. in. (tension).

5. Concrete: 1350 lb./sq. in. (comp.); 220 lb./sq. in. (comp.). Steel: 103,660 lb./sq. in. (tension).

SUMMARY OF FORMULAE

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